Exam 1 – Answers

Math 600

1. (a) (5 points) Suppose G is a group and K and H are subgroups, satisfying $K \subset H \subset G$ and $H \lhd G$. Show that if H is cyclic, then $K \lhd G$.

(b) (5 points) Find an example of subgroups $K \lhd H \lhd G$, where K is NOT normal in G.

ANSWER: (a) Since K is the unique subgroup of H having order $|K|$ (since H, begin cyclic, has a unique subgroup of each order allowed by Lagrange), it is *characteristic* in H. In other words, each automorphism of H preserves K. In particular, the automorphism of H induced by conjugation by $g \in G$ preserves K. Hence K is normal in G.

(b) The smallest example is given by $G = D_8$, the dihedral group of order 8. Suppose a is order 2 and b is of order 4, satisfying $aba = b^{-1}$. Then $H := \langle a, b^2 \rangle$ is a subgroup of order 4, hence in normal in G. Also, $K := \langle a \rangle$ is a subgroup of H having order 2, hence is normal in H (in both cases, we used the fact that subgroups of index 2 are always normal). However, K is not normal in G. If it were, then bab^{-1} would belong to $\langle a \rangle$, and would thus be a. But this would imply $ba = ab$, a contradiction.

Here is another example: take G to be the strictly upper triangular matrix group with entries in the finite field $\mathbb{F}_p$. Let $K = \{ \begin{bmatrix} 1 & a & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid a \in \mathbb{F}_p \}$, $H = \{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mid a, b \in \mathbb{F}_p \}$. Then $K \lhd H \lhd G$ since each is of index p in its successor (and p is the smallest prime dividing $|G| = p^3$). But it's easy to see that K is not normal in G.

2. Let P denote a p-Sylow subgroup of a finite group G. Let $N_G(P)$ denote the normalizer of P in G.

(a) (5 points) Show that P is the unique p-Sylow subgroup of $N_G(P)$.

(b) (5 points) Show that $N_G(N_G(P)) = N_G(P)$.

ANSWER: (a) is clear since P is certainly a p-Sylow subgroup of $N_G(P)$, and it is the unique one since it is (by definition) normal in $N_G(P)$. For (b), we use (a). Indeed, if $g \in G$ normalizes $N_G(P)$, then conjugation by g induces an automorphism of $N_G(P)$ which necessarily preserves P (since automorphisms permute p-Sylow subgroups, and P is the unique such in $N_G(P)$, by (a)). Thus g normalizes P, hence belongs to $N_G(P)$, as desired.

3. (10 points) Suppose G is a group of order $380 = 4 \cdot 5 \cdot 19$. Show that G is not simple.

ANSWER: WLOG $s_{19} \neq 1$ (otherwise there is a unique hence normal 19-Sylow subgroup). Since $s_{19} \equiv 1$ modulo 19 and $s_{19}|20$, we must have $s_{19} = 20$. Then we have $20 \cdot (19 - 1) = 360$ distinct elements of order 19. This leaves only 20 elements in the group which do not have order 19. Now if $s_5 \geq 6$, then we'd have at least $6 \cdot (5 - 1) = 24$ distinct elements of order 5, which is impossible. Hence $s_5 = 1$, and there is a normal subgroup of order 5. So in any case, G is not simple (we showed it has either a normal subgroup of order 19 or one of order 5).

4. (a) (10 points) Consider the group S_4. Write down a table containing the following data:

- a list of representatives for the conjugacy classes in S_4;
- for each representative, the cardinality of its conjugacy class and its centralizer.

(b) (10 points) Do the same for the alternating group A_4. Justify your answer (you may use results discussed in class).

(c) (5 points) Describe a normal subgroup in A_4 having order 4. What is the structure of this subgroup?

ANSWER: (a). The reps are e, (12), (123), (12)(34), and (1234). The conjugacy classes have orders, respectively, as follows: 1, $\binom{4}{2} = 6$, $2 \cdot \binom{4}{3} = 8$, 3, and $3! = 6$. The centralizers have orders, respectively, 24,4,3,8, and 4.

(b) Here the reps are e, (123), (132), and (12)(34). The conjugacy classes have orders 1,4,4,and 3, respectively. The centralizers have orders 12, 3,3, and 4, respectively.

The point is that (123) is conjugate to (132) by an odd element (namely (23)), hence these two cannot be conjugate in A_5. Indeed, if they were, say by $\pi \in A_5$, then the odd element $(23)\pi$ would centralize (123); this is impossible, since $C_{S_4}(123)$ is the cyclic group $\langle(123)\rangle$ (note that the above table showed that the centralizer has order 3).

Also, any product of pairwise disjoint 2-cycles $(ij)(lk)$ in A_4 is conjugate in A_4 to (12)(34). This may be seen by a direct computation. Alternatively, one can use a principle we discussed in class. Namely, since $\sigma := (12)(34)$ is centralized by an odd element (namely (12)), then we see that $|C_{A_4}(\sigma)| = \frac{1}{2}|C_{S_4}(\sigma)|$. It follows that the A_4-conjugacy class of σ has as many elements as the S_4-conjugacy class of σ, and thus these two agree. Finally, this means that the A_4-conjugacy class of σ has order 3, from the table in part (a).

(c) The Klein 4-group in A_4 consists of the identity element together with the three elements (12)(34), (13)(24), and (14)(23). It has 3 elements of order 2, so is not cyclic. Hence it is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. By the table in (b), this is a union of two conjugacy classes in A_4, so it is a normal subgroup of A_4 (it's also easy to see this last point directly).