Homework 13 – due 11/28/07

Math 600

66. Let F be any field, and $E \supseteq F$ any extension field. Let $A, B \in M_n(F)$. Suppose that A and B are similar when viewed as elements of $M_n(E)$. Show that A and B are similar as elements of $M_n(F)$.

67. Let F be any field, and $A, B \in M_n(F)$. Show that A and B are similar if and only if the matrices $XI - A$ and $XI - B$ have the same Smith forms in $M_n(F[X])$.

68. Prove that any matrix $A \in M_n(F)$ is similar to its transpose $A^t \in M_n(F)$.

69. (a) Let $A \in M_n(F)$. Let $\overline{F}$ be an algebraic closure of F (you may assume this exists). Show that the minimal polynomial $\min(A)$ and the characteristic polynomial $\mathrm{char}(A)$ have the same roots in $\overline{F}$ (neglecting multiplicities).

(b) Suppose that A is a 2×2 or a 3×3 matrix over a field F. Show that the invariant factors of A (hence the rational canonical form of A) can be computed once one knows $\mathrm{char}(A)$ and $\min(A)$.

70 . We say $A \in M_n(F)$ is *diagonalizable over* F if there exists $P \in \mathrm{GL}_n(F)$ such that $P^{-1}AP$ is a diagonal matrix.

(a) Let $\overline{F}$ denote an algebraic closure of F. Show that $A \in M_n(\overline{F})$ is diagonalizable over $\overline{F}$ if and only if its Jordan form in $M_n(\overline{F})$ is diagonal.

(b) Let $A \in M_n(F)$. Show that A is diagonalizable over F if and only $\min(A)$ factors in $F[X]$ as a product of pairwise distinct linear factors.

71. Use the "minors algorithm" discussed in class to find the invariant factors and the RCF for the matrix

$$\begin{bmatrix} -1 & -2 & 6 \\ -1 & 0 & 3 \\ -1 & -1 & 4 \end{bmatrix}$$

(To check your answer compare it with what you can derive just using #69 (b) above.)