72. In the following all exact sequences are in the category of $R$-modules.

(a) Let $M' \to M \to M'' \to 0$ be a sequence of $R$-modules and homomorphisms. Prove that this sequence is exact if and only if, for every $R$-module $N$ the induced sequence

$$0 \to \text{Hom}_R(M'', N) \to \text{Hom}_R(M, N) \to \text{Hom}_R(M', N)$$

is exact.

(b) Let $0 \to N' \to N \to N''$ be a sequence of $R$-modules and homomorphisms. Prove that this sequence is exact if and only if, for every $R$-module $M$ the induced sequence

$$0 \to \text{Hom}_R(M, N') \to \text{Hom}_R(M, N) \to \text{Hom}_R(M, N'')$$

is exact.

73. (5 points) Suppose we have a commutative diagram in $R - \text{Mod}$:

$$
\begin{array}{ccc}
A' & \longrightarrow & A \\
\downarrow \cong & & \downarrow \cong \\
B' & \longrightarrow & B \\
\end{array}
$$

such that the vertical arrows are isomorphisms, and the first row is exact. Prove that the second row is exact.

74. Let $R$ and $S$ be rings. Consider functors $F : R - \text{Mod} \to S - \text{Mod}$ and $G : S - \text{Mod} \to R - \text{Mod}$. We say that $F$ is a left adjoint of $G$ (or that $G$ is a right adjoint of $F$) provided that we have natural isomorphisms

$$\text{Hom}_S(FX, Y) \cong \text{Hom}_R(X, GY)$$

for $X$ an $R$-module and for $Y$ an $S$-module. What does natural mean? By definition, it means that given $X' \to X$ in $R$-Mod and $Y \to Y'$ in $S$-Mod, the following diagram (with the arrows having the obvious meanings) is commutative:

$$
\begin{array}{ccc}
\text{Hom}_S(FX, Y) & \longrightarrow & \text{Hom}_R(X, GY) \\
\downarrow & & \downarrow \\
\text{Hom}_S(FX', Y') & \longrightarrow & \text{Hom}_R(X', GY')
\end{array}
$$

Prove the following statement: If $F$ is left adjoint to $G$, then $F$ is right exact and $G$ is left exact.
75. Let $R$ and $S$ be commutative rings. Let $M$ be an $R$-module, $P$ an $S$-module, and $N$ an $(R, S)$-bimodule (that is, simultaneously an $R$-module and an $S$-module and the two structures are compatible in the sense that $r(xs) = (rx)s$ for all $r \in R$, $s \in S$, $x \in N$). Prove that $M \otimes_R N$ is naturally an $S$-module, $N \otimes_S P$ an $R$-module, and that we have 

$$(M \otimes_R N) \otimes_S P \cong M \otimes_R (N \otimes_S P).$$

76. Show that if $m$ and $n$ are coprime integers, then $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} = 0$. 