Homework 3 – due 09/19/07

Math 600

11. (a) (5 points) Show that any group of order 33 is cyclic.

(b) (5 points) Show that any group of order 35^2 is abelian. Up to isomorphism, how many groups of order 35^2 are there?

12. (10 points) Let G be a p-group. Show that G possesses a chain of subgroups $1 \lhd H_1 \lhd \cdots \lhd H_n = G$, each normal in its successor, such that each quotient group H_i/H_{i-1} is cyclic.

13. Let p denote a prime number and let $\mathbb{F}_p$ denote the finite field of cardinality p. Let $G = \mathrm{GL}_n(\mathbb{F}_p)$.

(a) (5 points) Show that $|G| = (p^n - 1)(p^n - p) \cdots (p^n - p^{n-1})$. HINT: Let $g \in \mathrm{GL}_n(\mathbb{F}_p)$. The first column of g may be any non-zero vector. The second column may be any vector not in the $\mathbb{F}_p$-span of the first column. Etc.

(b) (5 points) Find a p-Sylow subgroup of G. Hint: For $n = 2$ i.e. for $G = \mathrm{GL}_2(\mathbb{F}_p)$, the group U of elements of the form $\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}$ for $x \in \mathbb{F}_p$ is a p-Sylow subgroup. Check this and try to generalize to general n.

(c) (5 points) For $n = 2$, show that the normalizer of U above contains B, the subgroup consisting of the upper triangular matrices in $\mathrm{GL}_2(\mathbb{F}_p)$.

(d) (5 points) For $n = 2$, show that the number of p-Sylow subgroups in G is $1 + p$ and simultaneously show that the normalizer of U above is precisely the subgroup B.

(e) (BONUS 10 points extra credit) For n general, find the normalizer of your p-Sylow subgroup from part (b), and use this information to prove that the number of p-Sylow subgroups in G is

$$\frac{p^n - 1}{p - 1} \cdot \frac{p^{n-1} - 1}{p - 1} \cdots \frac{p^2 - 1}{p - 1}.$$

(Or, you are welcome to find another way to compute the number of p-Sylow subgroups.)

14. (10 points) Let G be a finite group. Suppose that for every positive integer k, we have

$$|\{x \in G \mid x^k = 1\}| \le k.$$

Prove that G is cyclic. HINT: It is helpful to remember the formula $\sum_{d|n} \phi(d) = n$, which we proved in class. Also, consider the question: How many elements of order d can there be?