Homework 5 – due 10/03/07

Math 600

20. (5 points) Suppose $n \geq 5$. Show that the only normal subgroups of S_n are $1, A_n$, and S_n. You may use the fact, proved in class, that A_n is simple.

21. (a) (5 points) Let G be a p-group, $G \neq 1$. Let $N \lhd G$, $N \neq 1$. Show that $N \cap Z(G) \neq 1$. .

(b) (5 points) Let G be a non-abelian group of order p^3 (where p is prime). Show that $|Z(G)| = p$ and that $Z(G)$ has no complement in G.

22. (10 points) Show that a group of order $2^2 \cdot 5 \cdot 19$ is not simple.

23. (10 points) If $n \geq 4$ is even, show that A_n is generated by (123) and $(23 \cdots n)$.

24. (15 points) Let G be a finite group of order $p^b n$ (here we do not assume $(p, n) = 1$). Let H be a subgroup of order p^a, where $0 \leq a \leq b$. Let $N(p^b, H)$ denote the number of subgroups of G which contain H and have order p^b. The following steps will prove that $N(p^b, H) \equiv 1 \mod p$. (This gives another proof of a (stronger) version of Sylow's theorems.)

(a) Let Ω denote the set of subsets of G which have order p^b and are stable under left multiplication by elements of H. The group G acts on Ω by $M \mapsto Mg$. Let $\{T_i\}_{i \in I}$ denote the orbits. For $M_i \in T_i$, let G_i denote the stabilizer of M_i. Show that $|G_i|$ is a divisor of p^b.

(b) Show that $|T_i| = n \Leftrightarrow |G_i| = p^b$. Show also that $|T_i| \neq n \Leftrightarrow |G_i| < p^b$, in which case $|T_i| \equiv 0 \ \ (pn)$.

(c) Prove that there is a 1-1 correspondence between orbits T_i having length exactly n and subgroups U of G which contain H and have order exactly p^b.

(d) Deduce that

$$|\Omega| \equiv \sum_{|T_i| = n} |T_i| \equiv nN(p^b, H) \ \ (pn).$$

(e) Show that $|\Omega| = \binom{p^{b-a}n}{p^{b-a}}$, and deduce the equation

$$\binom{p^{b-a}n}{p^{b-a}} \equiv nN(p^b, H) \ \ (pn).$$

(f) By considering the above equation in the case of a cyclic group of order $p^b n$, show that

$$\binom{p^{b-a}n}{p^{b-a}} \equiv n \ \ (pn).$$

(g) Conclude that $N(p^b, H) \equiv 1 \ \ (p)$.