

Solutions to Homework 6

Math 600, Fall 2007

29 (10 points) Dummit-Foote, 6.1 #12

Let $K = \{\text{id}, (12)(34), (13)(24), (14)(23)\} \simeq C_2 \times C_2$. Then $K \triangleleft S_4$ and hence $K \triangleleft A_4$. Let $H = \{\text{id}, (123), (132)\} \simeq C_3$. We have $A_4 = K \rtimes H$. We can check that conjugating by any hk is never a trivial automorphism of A_4 , hence $Z(A_4) = \{\text{id}\}$. We already know that $Z(S_4) = \{\text{id}\}$. Thus:

the upper central series of both S_4 and A_4 is: $\{\text{id}\} \leq \{\text{id}\} \leq \{\text{id}\} \leq \dots$

Since $K \triangleleft A_4$, we have $[A_4 : A_4] \leq K$. Now, let $h = (123)$ and let $k_1 = (12)(34)$, $k_2 = (13)(24)$ and $k_3 = (14)(23)$. We can check that $[h : k_i] = k_{i+1}$.

Thus $[H : K] = K$ and, using $K \geq [A_4 : A_4] \geq [A_4 : K] \geq [H : K] = K$, we conclude that the

lower central series of A_4 is: $A_4 \geq K \geq K \geq \dots$

Let $L = \{\text{id}, (12)\} < S_4$, then $S_4 = A_4 \rtimes L$. Since $A_4 \triangleleft S_4$ we have $A_4 \geq [S_4 : A_4] \geq [A_4 : A_4] = K$. Thus we know that $[S_4 : A_4]$ is either K or A_4 . The calculation $[(12) : (123)] = (123)$ shows $[S_4 : A_4] = A_4$. Moreover, for any pair of groups $H \leq G$: G/H is an abelian group $\Leftrightarrow H \geq [G : G]$. (Aside: A subgroup of G which contains $[G : G]$ is always normal). In the present case this shows $[S_4 : S_4] \leq A_4$. Therefore $A_4 \geq [S_4 : S_4] \geq [S_4 : A_4] = A_4$ and thus the

lower central series of S_4 is: $S_4 \geq A_4 \geq A_4 \geq A_4 \geq \dots$

28 (10 points) Dummit-Foote, 5.5 #18

In any semidirect product $G = H \rtimes K$ given by $k \mapsto \phi_k \in \text{Aut}(H)$, we have subgroups $H = \{(h, 1) \mid h \in H\}$ and $K = \{(1, k) \mid k \in K\}$. Conjugation by $k = (1, k)$ when restricted to H is precisely ϕ_k : $(1, k) \cdot (h, 1) \cdot (1, k^{-1}) = (\phi_k h, 1)$. (So, in fact G is the internal product HK). To answer the problem, we let $K = \text{Aut}(H)$. (G is the holomorph of H) \square

27 (10 points) Suppose the semidirect product $G \rtimes \mathbb{Z}$ is such that the action of $1 \in \mathbb{Z}$ is an inner automorphism of G . Show that $G \rtimes \mathbb{Z} \cong G \times \mathbb{Z}$

Let the automorphism of G corresponding to $1 \in \mathbb{Z}$ be conjugation by $g_0 \in G$. Let $H = \{(g_0^{-n}, n) \mid n \in \mathbb{Z}\} < G \rtimes \mathbb{Z}$. Let $G = \{(g, 0) \mid g \in G\} < G \rtimes \mathbb{Z}$. It is easy to check that $gh = hg$ and $G \cap H = \{id\}$, whence $G \rtimes \mathbb{Z} \cong G \times H \cong G \times \mathbb{Z}$

Comment: Problem 5.5 #6 from the text cannot be used to solve the problem above, I think.

26 (10 points) Suppose G is a group with $|G| = 5 \cdot 11 \cdot 17$. Suppose that G has an element of order 55. Show that G is cyclic.

Let H denote a Sylow-17 subgroup. Let n_{17} be the number of conjugates of H . We have $n_{17} = 1$ by the Sylow theorems. Let $K = \langle x \rangle$ where x is an element of order 55 which exists by hypothesis. The conjugation of $H \simeq C_{17}$ by $K \simeq C_{55}$ gives us a homomorphism $\psi : C_{55} \rightarrow \text{Aut}(C_{17})$. We know $|\text{Aut}(C_n)| = \phi(n)$ where ϕ is the Euler totient function. Thus $|\psi(K)|$ divides both 16 and 55 and hence ψ is trivial, therefore $G = H \times K \simeq C_{17} \times C_{55} \simeq C_{17 \cdot 55}$.

25 (5 points) Suppose $1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$ is an exact sequence of groups. Prove that A and C are finite if and only if B is finite, in which case $|B| = |A| \cdot |C|$.

We have $C \simeq B/A$. If B is finite, then so is any subgroup and quotient group (A and C).

Conversely, if A and C are finite, then B is finite because it is the disjoint union of $|C| < \infty$ cosets of A each of which has $|A| < \infty$ elements. In the finite case $C \simeq B/A$ implies $|B| = |A| \cdot |C|$.

The solution posted for Hw5, Problem 23 (see below) overlooked the case $n = 4$. This is rectified here. Moreover, the modified proof below does not use the nontrivial result of Problem 20, and is thus a simpler proof.

23 from HW5: A_n is generated by (123) and $(23 \cdots n)$ if $n \geq 4$ is even

Let $a = (123)$ and $b = (23 \cdots n)$. Let $H = \langle a, b \rangle < A_n$. We must show $H = A_n$. First, we show that $H \triangleleft S_n$. Observe that:

$$S_n = \langle (12), (23), \dots, (n-1 n) \rangle = \langle (12), (123 \cdots n) \rangle = \langle (12), (12)b \rangle = \langle (12), b \rangle$$

Therefore $H \triangleleft S_n \Leftrightarrow (12)H(12) = H$. We have $(12)a(12) = a^{-1}$ and $(12)b(12) = a^{-1}b$, by direct calculation, and therefore $H \triangleleft S_n$. We now observe that S_n/H has order at most 2. However $A_n \geq H$ and so we must also have $S_n/A_n = (S_n/H)/(A_n/H)$. Thus the only solution is $|S_n/H| = 2$ and $|A_n/H| = 1$, i.e $H = A_n$ \square