The solutions are only sketched below for problems which present no difficulty.

30 (10 points) Dummit-Foote, 7.1 #25
All parts are straight-forward computations. Integral quaternions of norm 1 are the units:

\[ I^\times = \{ a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1 \} = \{ \pm 1, \pm i, \pm j, \pm k \} \]

This group is the quaternion group of order 8, by definition.

31 (10 points) Dummit-Foote, 7.1 #26
a) \( R = \{ x \in K^\times \mid \nu(x) \geq 0 \} \cup \{ 0 \} \) is a subring of \( K \): \( \nu(ab) = \nu(a) + \nu(b) \) ensures \( R \) is closed wrt multiplication, and \( \nu(a + b) \geq \min\{\nu(a), \nu(b)\} \) ensures \( R \) is closed wrt addition.

b) If \( x \in K^\times \) then \( \nu(x) < 0 \Rightarrow \nu(x^{-1}) > 0 \).

c) We have: \( x, x^{-1} \in R \Leftrightarrow \nu(x), -\nu(x) \geq 0 \Leftrightarrow \nu(x) = 0 \)

32 (10 points) Dummit-Foote, 7.1 #27
Here \( R \) is the subring of rational numbers whose denominators are free of \( p \).
\( R^\times \) is the multiplicative group of rational numbers which are free of \( p \).

33 (10 points) Dummit-Foote, 7.2 #2
Follows from the computation mentioned in the problem statement in square brackets. If I find a more elegant proof then I will include it the next solution set.

34 (10 points) Dummit-Foote, 7.3 #33
First suppose \( R \) has no nilpotents. We will show that \( R[x]^\times = R^\times \) where we identify \( R^\times \) with its image under the embedding \( R \hookrightarrow R[x] \). We will also show that \( R[x] \) has no nilpotents except 0.
Let \( p(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x] \) be a unit. Wolog \( a_n \neq 0 \). Let \( q(x) = b_0 + b_1 x + \cdots + b_m x^m \) with \( b_m \neq 0 \) (wolog) be the inverse of \( p(x) \) and observe that \( a_0 \cdot b_0 = 1 \). Thus \( a_0 \in R^\times \).

If \( m = 0 \) then we have \( b_0 \cdot p(x) = 1 \) which when multiplied by \( a_0 \) yields \( p(x) = a_0 \), thus \( p(x) \in R^\times \).

If \( m > 0 \) and assume \( b_m \) is not nilpotent, then we derive a contradiction (saying \( b_m \) is nilpotent) thereby showing the impossibility of the case \( m > 0 \).

Suppose \( b_m \) is not nilpotent, then, we recursively multiply the coefficients of \( x^{n+m-i} \) in \( p(x) \cdot q(x) \) \((0 \leq i \leq n)\) by \( b_m^i \) to obtain the relations \( a_{n-i} \cdot b_m^{i+1} = 0 \). In particular \( a_0 \cdot b_m^{i+1} = 0 \) which (when multiplied by \( b_0 \)) yields that \( b_m \) is a nilpotent. Hence we have shown \( R[x]^\times = R^\times \).

Suppose now that \( p(x) \) is nilpotent and \( p(x) \neq 0 \). Then write \( p(x) = a_0 + a_1 x + \cdots + a_n x^n \) with \( a_n \neq 0 \). We have \( p(x)^N = 0 \) for some \( N > 0 \). We then have \( a_n^N = 0 \) whence \( a_n = 0 \) contradicting \( p(x) \neq 0 \). Thus the zero polynomial is the only nilpotent element of \( R[x] \).

Now, for a general commutative ring \( R \) let \( \mathfrak{n} \) be the nilradical of \( R \) and \( S = R/\mathfrak{n} \). By problems 7.3.29 – 30 of the text, we know that \( S \) has no nilpotents. Consider the surjection \( \Phi : R[x] \to S[x] \) with kernel denoted \( \mathfrak{n}[x] \). Let \( p(x) \in R[x]^\times \), then \( \Phi(p(x)) \in S[x]^\times \), because a ring homomorphism takes units to units. Therefore by our consideration of \( S[x]^\times \), we know \( \Phi(p(x) - a_0) = 0 \) whence \( p(x) \in R^\times + \mathfrak{n}[x] \). Together with the fact \( R^\times + \mathfrak{n} = R^\times \) which is problem 7.1.14d of the text we have established part a) of the present problem (that \( a_i \) must be nilpotent except \( a_0 \) which must be a unit).

If \( p(x) \) is a nilpotent element of \( R[x] \) then \( \Phi(p(x)) \) is nilpotent in \( S[x] \) (problem 7.3.32), and therefore by our consideration of the nilpotents in \( S[x] \) we know \( \Phi(p(x)) = 0 \) which is the same as \( p(x) \in \mathfrak{n}[x] \) as was to be shown for part b) of the problem.

**35 (10 points) Dummit-Foote, 8.1 #3**

Let \( R \) be a Euclidean domain, and \( a \in R \) be of minimum norm, then by the Euclidean algorithm, we have that \( a \) divides every \( b \in R \). In particular \( a \) divides 1 and hence is a unit. Since the norm is nonnegative, any element of norm zero has minimum norm and would be a unit.