Solutions to Homework 7
Math 600, Fall 2007

The solutions are only sketched below for problems which present no difficulty

30 (10 points) Dummit-Foote, 7.1 #25
All parts are straight-forward computations. Integral quaternions of norm 1 are the units :

$$I^\times = \{a + bi + cj + dk \mid a^2 + b^2 + c^2 + d^2 = 1\} = \{\pm 1, \pm i, \pm j, \pm k\}$$

This group is the quaternion group of order 8, by definition.

31 (10 points) Dummit-Foote, 7.1 #26

a) $R = \{x \in K^\times \mid \nu(x) \geq 0\} \cup \{0\}$ is a subring of K: $\nu(ab) = \nu(a) + \nu(b)$ ensures R is closed wrt multiplication, and $\nu(a + b) \geq min\{\nu(a), \nu(b)\}$ ensures R is closed wrt addition.

b) If $x \in K^\times$ then $\nu(x) < 0 \Rightarrow \nu(x^{-1}) > 0$.

c) We have: $x, x^{-1} \in R \Leftrightarrow \nu(x), -\nu(x) \geq 0 \Leftrightarrow \nu(x) = 0$

32 (10 points) Dummit-Foote, 7.1 #27
Here R is the subring of rational numbers whose denominators are free of p.
$R^\times$ is the multiplicative group of rational numbers which are free of p

33 (10 points) Dummit-Foote, 7.2 #2
Follows from the computation mentioned in the problem statement in square brackets. If I find a more elegant proof then I will include it the next solution set.

34 (10 points) Dummit-Foote, 7.3 #33

First suppose R has no nilpotents. We will show that $R[x]^\times = R^\times$ where we identify $R^\times$ with its image under the embedding $R \hookrightarrow R[x]$. We will also show that $R[x]$ has no nilpotents except 0.

Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in R[x]^\times$ be a unit. Wolog $a_n \neq 0$. Let $q(x) = b_0 + b_1 x + \cdots + b_m x^m$ with $b_m \neq 0$ (wolog) be the inverse of $p(x)$ and observe that $a_0 \cdot b_0 = 1$. Thus $a_0 \in R^\times$.
If $m = 0$ then we have $b_0 \cdot p(x) = 1$ which when multiplied by a_0 yields $p(x) = a_0$, thus $p(x) \in R^\times$.
If $m > 0$ and assume b_m is not nilpotent, then we derive a contradiction (saying b_m is nilpotent) thereby showing the impossibility of the case $m > 0$.
Suppose b_m is not nilpotent, then, we recursively multiply the coefficients of x^{n+m-i} in $p(x) \cdot q(x)$ $(0 \leq i \leq n)$ by b_m^i to obtain the relations $a_{n-i} \cdot b_m^{i+1} = 0$. In particular $a_0 \cdot b_m^{n+1} = 0$ which (when multiplied by b_0) yields that b_m is a nilpotent. Hence we have shown $R[x]^\times = R^\times$.

Suppose now that $p(x)$ is nilpotent and $p(x) \neq 0$. Then write $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ with $a_n \neq 0$. We have $p(x)^N = 0$ for some $N > 0$. We then have $a_n^N = 0$ whence $a_n = 0$ contradicting $p(x) \neq 0$. Thus the zero polynomial is the only nilpotent element of $R[x]$.

Now, for a general commutative ring R let $\mathfrak{n}$ be the nilradical of R and $S = R/\mathfrak{n}$. By problems $7.3.29 - 30$ of the text, we know that S has no nilpotents. Consider the surjection $\Phi : R[x] \to S[x]$ with kernel denoted $\mathfrak{n}[x]$. Let $p(x) \in R[x]^\times$, then $\Phi(p(x)) \in S[x]^\times$, because a ring homomorphism takes units to units. Therefore by our consideration of $S[x]^\times$, we know $\Phi(p(x) - a_0) = 0$ whence $p(x) \in R^\times + \mathfrak{n}[x]$. Together with the fact $R^\times + \mathfrak{n} = R^\times$ which is problem $7.1.14d$ of the text we have established part a) of the present problem (that a_i must be nilpotent except a_0 which must be a unit)

If $p(x)$ is a nilpotent element of $R[x]$ then $\Phi(p(x))$ is nilpotent in $S[x]$ (problem 7.3.32), and therefore by our consideration of the nilpotents in $S[x]$ we know $\Phi(p(x)) = 0$ which is the same as $p(x) \in \mathfrak{n}[x]$ as was to be shown for part b) of the problem.

35 (10 points) Dummit-Foote, 8.1 #3

Let R be a Euclidean domain, and $a \in R$ be of minimum norm, then by the Euclidean algorithm, we have that a divides every $b \in R$. In particular a divides 1 and hence is a unit. Since the norm is nonnegative, any element of norm zero has minimum norm and would be a unit.