Homework 1 – due 02/06/08

Math 601

1. (10 points) Let R be a commutative ring with unit. Let V, W be R-modules. In class we proposed a definition of multiplication on the R-module $E(V) \otimes_R E(W)$, which should make it into an R-algebra (where R belongs to the center). On generators, the multiplication we proposed was given by defining

$$[(v_1 \wedge \cdots \wedge v_k) \otimes (w_1 \wedge \cdots \wedge w_l)]\,[(x_1 \wedge \cdots \wedge x_t) \otimes (y_1 \wedge \cdots \wedge y_s)]$$

to be

$$(-1)^{lt}\,(v_1 \wedge \cdots \wedge v_k \wedge x_1 \wedge \cdots \wedge x_t) \otimes (w_1 \wedge \cdots \wedge w_t \wedge y_1 \wedge \cdots \wedge y_s).$$

Show that this does indeed give a well-defined product on $E(V) \otimes_R E(W)$, making into an R-algebra such that R belongs to the center. Warning: in this general situation, $E(V)$ and $E(W)$ are not *freely* generated by the above generators, so you have to take care in proving the product is well-defined.

2. (20 points) In Math 600, Homework 12, #61, we proved the following fact: let R denote a commutative ring with unit. Let $V = R^n$, and suppose $v_1, \ldots, v_n$ generate V over R. Then $v_1, \ldots, v_n$ form a basis for V. We proved this using a method which ultimately relied on the theory of determinants (over commutative rings). The purpose of this exercise is to provide an alternative approach to the same proposition, which avoids determinants. Instead, it relies on tensor products.

(a) Write $R^n = V$, and consider the surjective R-module homomorphism $\phi : V \to V$ given by $\phi(e_i) = v_i$ (here v_i is one of the generators above, and e_i is the standard basis vector in R^n). Let $N := \ker(\phi)$. Show that N is a f.g. R-module. HINT: show N is a direct summand of V, by constructing a splitting of ϕ.

(b) Let $\mathfrak{m}$ denote a maximal ideal of R, and set $k = R/\mathfrak{m}$. Prove the exactness of the following sequence:

$$0 \longrightarrow k \otimes_R N \longrightarrow k \otimes_R V \xrightarrow{1 \otimes \phi} k \otimes_R V \longrightarrow 0.$$

(c) Prove, in part by using Nakayama's lemma in the form we proved in Math 600, that $N_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$.

(d) Conclude that $N = 0$, and deduce that $v_1, \ldots, v_n$ are linearly independent over R, hence a basis. HINT: if $x \in N$ has $x \neq 0$, then its annihilator $\mathrm{Ann}(x)$ is a proper ideal in R, hence is contained in some maximal ideal.

3. (10 points) Dummit-Foote, 11.5, #1.

4. (10 points) Dummit-Foote, 11.5, #13.