

Solutions to Homework 13

Math 601, Spring 2008

59) (5 points). Let $\phi : V \rightarrow V$ and $\psi : W \rightarrow W$ be F -linear endomorphisms of finite-dimensional F -vector spaces V and W . Show that $\text{Tr}(\phi \otimes \psi) = \text{Tr}(\phi) \cdot \text{Tr}(\psi)$.

Let A and B be $m \times m$ and $n \times n$ matrices representing the endomorphisms ϕ and ψ respectively, in some choice of bases for $V \simeq F^m$ and $W \simeq F^n$. The matrix of $\phi \otimes \psi$ is the Kronecker product $A \otimes B$ of size $mn \times mn$ consisting of $n \times n$ blocks $a_{ij}B$. Thus $\text{Tr}(\phi \otimes \psi) = \text{Tr}(A) \cdot \text{Tr}(B) = \text{Tr}(\phi) \cdot \text{Tr}(\psi)$.

60) (10 points). Let F be a field. We say $u \in GL_n(F)$ is unipotent if $u - 1$ is nilpotent, that is, $(u - 1)^N = 0$ for some $N \geq 1$.

a) Show that if $\text{char}(F) = 0$, then the only finite-order unipotent element is the identity matrix.

The minimal polynomial of a nilpotent matrix η is X^k , where η is nilpotent of order k . The characteristic polynomial is X^n . Thus the $F[X]$ -module F^n is of the form $F[X]/(X^{k_1}) \oplus \cdots \oplus F[X]/(X^{k_l})$ with $\max\{k_i \mid 1 \leq i \leq l\} = k$. Thus there is a basis for F^n such that the matrix η can be chosen to be block diagonal with l blocks η_i , which are standard nilpotent matrices of size $k_i \times k_i$ (with 1's on the sub-diagonal and 0's elsewhere). Let $\eta := u - 1$. Let $u^n = 1$ where n is a multiple of the order of u such that $n \geq k$. Then $1 = u^n = \sum_{i=0}^{k-1} \binom{n}{i} \eta^i$. If $k = 1$, then $\eta = 0$ and $u = 1$, hence we are done. If $k > 1$, then the linear independence of the matrices η^i for $1 \leq i \leq k - 1$ implies $\binom{n}{i} = 0$, which is possible only in characteristic p . Thus, if $\text{char}(F) = 0$, then $k = 1$ and $u = 1$.

b): Use (a) to prove that if ρ is a representation of a finite group G over \mathbb{C} , then $\rho(g)$ is similar to a diagonal matrix. HINT: use the Jordan-normal form of $\rho(g)$.

We choose a basis of \mathbb{C}^n such that the matrix $\rho(g)$ (for a particular $g \in G$, that is) is in Jordan canonical form. Thus, we have $\mathbb{C}^n = \bigoplus_{i=1}^l V_i$ with $\rho(g)_i : V_i \rightarrow V_i$ given by $\rho(g)_i = \lambda_i I + \eta_i$, where I is the identity matrix and η_i is a (standard) nilpotent matrix (of size $\dim(V_i) \times \dim(V_i)$). Since $\rho(g)$ is nonsingular, we have $\lambda_i \neq 0$ (in fact λ_i is a root of unity), therefore $\rho(g)_i / \lambda_i : V_i \rightarrow V_i$ is unipotent and by part a) $\rho(g)_i = \lambda_i I$. Thus $\rho(g)$ is diagonal.

(c): Show that the conclusion of (a) is false if $\text{char}(F) = p$.

We take $u \in GL_2(\mathbb{F}_p)$ to be the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Clearly $u^p = 1$ and $(u - 1)^2 = 0$, so that u is a finite order unipotent matrix with $u \neq 1$.

61) [D-F] 18.1 #16 (5 points). One-dim'l \mathbb{C} -reps. of a finite abelian group G .

Since conjugation in $GL_1(\mathbb{C}) = \mathbb{C}^\times$ is trivial, distinct 1-dim'l representations of any group G are inequivalent. We can multiply two 1-dim'l reps. ρ, ρ' by the rule $(\rho\rho')(g) = \rho(g)\rho'(g)$ because \mathbb{C}^\times is abelian. This gives the set of inequivalent 1-dim'l reps of G , the structure of an abelian group, which we call \hat{G} here. (Note: the inverse of ρ in \hat{G} is clearly given by $(\rho)^{-1}(g) = \rho(g^{-1})$). If $G = \mathbb{Z}/n\mathbb{Z}$ then any $\rho : G \rightarrow \mathbb{C}^\times$ takes $1 \mapsto e^{2\pi ik/n}$, we denote this rep. by $\rho_{k,n}$. Thus $\hat{G} = G$ when $G = \mathbb{Z}/n\mathbb{Z}$. Next if G is a finite abelian group, we can write $G = \bigoplus_{i=1}^l \mathbb{Z}/n_i\mathbb{Z}$ with $n_i | n_{i+1}$. Any representation of $\rho : G \rightarrow \mathbb{C}^\times$ takes the form $\rho(g_1, \dots, g_l) = \prod_{i=1}^l \rho_{k_i, n_i}(g_i)$. Thus $\hat{G} = G$, again.

62) [D-F] 18.3 #6-8 (30 points).

6): Let χ_1, \dots, χ_r be the irreps. of G , where r is the number of conjugacy classes of G . Let ψ be the character of the given representation $\phi : G \rightarrow GL(V)$, then we know $\psi = \sum_{i=1}^r a_i \chi_i$, for some non-negative integers a_i , where a_i is the number of copies of the irrep. corresponding to χ_i in ϕ . Since the χ_i are orthonormal, we have $(\psi, \chi_1) = a_1 =$ number of copies of the trivial rep. in ϕ . The latter quantity is the dimension of the subspace $W = \{v \in V \mid \phi(g)v = v \text{ for all } g \in G\}$.

7 a): If $V = \bigoplus_{i=1}^n \mathbb{C}e_i$, with G permuting the basis vectors $\mathcal{B} = \{e_i \mid i = 1 \dots n\}$. Let \mathcal{B}_i for $i = 1 \dots t$ be the orbits of the G -action on \mathcal{B} . We have $V = \bigoplus_{i=1}^t V_i$ as $\mathbb{C}G$ -modules, where V_i is the \mathbb{C} -span of \mathcal{B}_i .

7 b): If $W = \mathbb{C}(\sum_{e_j \in \mathcal{B}_i} a_j e_j)$ is a 1-dim'l subspace of V_i which is G -invariant then clearly the a_i have to be identical, so that $W = \mathbb{C}(\sum_{e_j \in \mathcal{B}_i} e_j)$. Thus the number of copies of the trivial rep. in the representation of G on V_i is exactly one.

7 c): Since, the number of copies of the trivial rep. in the representation of G on V_i is exactly one, it follows that $t = \#G$ -orbits of $\mathcal{B} = \#$ copies of the trivial rep. in the representation of G on V .

8): Since $G < S_n$, the \mathbb{C} -span, V , of $\mathcal{B} = \{1, 2, \dots, n\}$ is a $\mathbb{C}G$ -module (permuting the basis \mathcal{B}). For $g \in G$, the number of basis elements fixed by g , is $\text{Fix}(g) = \psi(g)$, where ψ is the character of the representation V . Thus $\sum_{g \in G} \text{Fix}(g) = |G|(\psi, \chi_1)$ where χ_1 is the character of the trivial rep. By Problem 6) above, $(\psi, \chi_1) = \#$ copies of the trivial rep. in V , and by Problems 7 a) and b) above, this quantity is $t = \#$ of G -orbits of \mathcal{B} . Thus $\sum_{g \in G} \text{Fix}(g) = |G|t$ as required.