Solutions to Homework 14

Math 601, Spring 2008

63) (10 points). G is canonically isomorphic to $\hat{\hat{G}}$.

We constructed in the solution to Problem 61, HW13 an isomorphism $G \to \hat{G}$: If $\chi_{k,n}$ denotes the character of $\mathbb{Z}/n\mathbb{Z}$ taking $1 \mapsto e^{2i\pi k/n}$, then any character of $G = \sum_{i=1}^{m} \mathbb{Z}/n_i\mathbb{Z}$ is $\prod_{i=1}^{m} \chi_{k_i,n_i}$. (Since this isomorphism depends on the choice of generators for the cyclic groups, it is non-canonical). If $g \in G$ is nontrivial, then it follows that there is a character $\chi \in \hat{G}$ with $\chi(g) \neq 1$. Thus the homomorphism $G \to \hat{\hat{G}}$ defined by $g \mapsto (\chi \mapsto \chi(g))$ is an injective homomorphism of finite groups of the same order, whence an isomorphism. Since this construction depends only on the natural pairing $G \times \hat{G} \to \mathbb{C}^\times$ it is canonical.

64) (10 points). Dimensions of irreps of A_5.

A_5 has 5 conjugacy classes namely $\{(1), (123), (12345), (13524), (12)(34)\}$ with the sizes of the classes being $\{1, 20, 12, 12, 15\}$ respectively. Hence, A_5 has 5 irreps ϕ_i with characters χ_i and dimensions d_i. The number of 1-dim'l irreps is the order of the abelianization of A_5 which is the trivial group. Thus we get $d_2^2 + d_3^2 + d_4^2 + d_5^2 = 59$ with $d_i > 1$. If we additionally use the fact that $d_i \,|\, 60$ then we get $d_5 = 5$, $d_4 = 4$, $d_3 = d_2 = 3$. We can also determine the d_i, by noting that the 4-dim'l and 5-dim'l irreps. of S_5 (given in section 19.1 of [D-F]) when restricted to A_5 give a 4-dim'l and a 5-dim'l irrep of A_5 (by checking that the character has unit norm). Thus $d_4 = 4$ and $d_5 = 5$ and we get $d_2^2 + d_3^2 = 18$ which has a unique solution $d_3 = d_2 = 3$. (We did not need to use $d_i \,|\, 60$ here, but we used the character table of S_5). Since 6 does not occur as any d_i it follows that the 6 dim'l irrep of S_5 when restricted to A_5 is reducible. According to the character table of A_5 worked out below, this representation is the sum of the two 3 dim'l irreps.

65) (10 points). Character table of A_5.

From the previous problem and the character table of S_5, only the characters χ_2 and χ_3 (the 3-dim'l irreps) are to be determined. Thus we get the following incomplete table:

A_5	(1)	(123)	(12345)	(13524)	(12)(34)
χ_1	1	1	1	1	1
χ_2	3				
χ_3	3				
χ_4	4	1	-1	-1	0
χ_5	5	-1	0	0	1

We note that the faithful representation $A_5 \hookrightarrow SO(3)$ as the symmetries of the icosahedron, is irreducible for it is certainly not isomorphic to 3 copies of the trivial representation. Let χ_2 denote the character of this irrep. Since the eigenvalues of an element of $SO(3)$ are $1, e^{i\theta}, e^{-i\theta}$ (where $\pm\theta$ is the rotation angle about the $\pm$ axis of rotation) we get that $\chi(g) = 1 + 2\cos(\theta)$ with $\theta = 2k\pi/|g|$ for some k relatively prime to $|g|$. Thus χ_2 is the vector $(3, 0, \alpha, \beta, -1)$ with $\alpha, \beta \in \{1 + 2\cos(2\pi/5), 1 + 2\cos(4\pi/5)\}$. The equations $(\chi_2, \chi_1) = 0$ and $(\chi_2, \chi_2) = 1$ yield $\alpha = \frac{1+\sqrt{5}}{2} = 1 + 2\cos(2\pi/5)$, and $\beta = \frac{1-\sqrt{5}}{2} = 1 + 2\cos(4\pi/5))$ or vice-versa. Let χ_3 be the other 3-dim'l irreducible character. The equations $(\chi_3, \chi_5) = (\chi_3, \chi_4) = (\chi_3, \chi_1) = 0$ together with $(\chi_3, \chi_3) = 1$ yield $\chi_3 = (3, 0, \alpha', \beta', -1)$ where $\alpha' = \frac{1+\sqrt{5}}{2}$ and $\beta' = \frac{1-\sqrt{5}}{2}$ or vice versa. It follows that $\chi_2 = (3, 0, \alpha, \beta, -1)$ and $\chi_3 = (3, 0, \beta, \alpha, -1)$ with $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Thus the complete character table of A_5 is:

A_5	(1)	(123)	(12345)	(13524)	(12)(34)
χ_1	1	1	1	1	1
χ_2	3	0	α	β	-1
χ_3	3	0	β	α	-1
χ_4	4	1	-1	-1	0
χ_5	5	-1	0	0	1

We did not make use of the column orthogonality in the solution above. The characters χ_2 and χ_3 can also be determined using column orthogonality, without using the fact that $\chi_2(g)$ is of the form $1 + 2\cos(2k\pi/|g|)$.