Solutions to Homework 1

Math 601, Spring 2008

1 (10 points) We will show that the map from $\wedge^k(V) \otimes \wedge^l(W) \bigtimes \wedge^t(V) \otimes \wedge^s(W) \to \wedge^{k+t}(V) \otimes \wedge^{l+s}(W)$ given by $(v \otimes w, x \otimes y) \mapsto (-1)^{lt}(v \wedge x) \otimes (w \wedge y)$, (where $v = v_1 \wedge \cdots \wedge v_k$ etc.) is well defined. We do this by observing that the map from $\times^k(V) \bigtimes \times^l(W) \bigtimes \times^t(V) \bigtimes \times^s(W) \to \wedge^{k+t}(V) \otimes \wedge^{l+s}(W)$ given by $(v', w', x', y') \mapsto (-1)^{lt}(v \wedge x) \otimes (w \wedge y)$ (where $v' = (v_1, \cdots, v_k)$ and $v = v_1 \wedge \cdots \wedge v_k$ etc.) is multilinear so that each of the three $\bigtimes$'s can be replaced by $\bigotimes$, and that it has the alternating property by which $\times^k V, \times^l W, \times^t V, \times^s W$ can be replaced resp. by $\wedge^k V, \wedge^l W, \wedge^t V, \wedge^s W$. The scalars R are identified with the subalgebra $\wedge^0(V) \otimes \wedge^0(W)$ and is easily seen to be in the center of $E(V) \otimes E(W)$

2 (20 points) Let $\phi : V \to V$ be as given. By hypothesis, there exist $b_i \in V$ with $\phi(b_i) = e_i$. Let $B \subset V$ be the submodule generated by the b_i. We note that B is free of rank n, and $V = N \oplus B$, where $N = \ker(\phi)$. Since a quotient of an f.g module is f.g., we have $N = V/B$ is f.g. as required. Now we tensor the equation $V = N \oplus B$ with $k = R/\mathfrak{m}$, where $\mathfrak{m} \subset R$ is a max'l ideal. Since tensor product distributes over direct sum and both V and B are free of rank n, we obtain $B \otimes k$ is an n-dim'l vector subspace of $V \otimes k \simeq k^n$, whence $N \otimes k = 0$. From $N \otimes k = N/\mathfrak{m}N$ we get $N = \mathfrak{m}N$. Let $S = R - \mathfrak{m}$, $N_\mathfrak{m} = S^{-1}N$, and let $\mathfrak{m}$ denote the unique max'l ideal of $S^{-1}R$ as well, then $N = \mathfrak{m}N$ implies $N_\mathfrak{m} = \mathfrak{m}N_\mathfrak{m}$ (If $n = \sum m^i n_i$ with $m^i \in \mathfrak{m}$, then $n/s = \sum m^i n_i/s$). The Nakayama lemma for f.g. local rings now implies $N_\mathfrak{m} = 0$ for all max'l ideals $\mathfrak{m}$. For any $x \in N$, there exists $s \in R - \mathfrak{m}$ with $sx = 0$, this follows from $N_\mathfrak{m} = 0$. In particular $\mathrm{Ann}(x)$ is not contained in any max'l ideal whence $\mathrm{Ann}(x) = R$. Thus $N = 0$ and therefore the generators $v_1, \cdots v_n$ are linear independent and hence are a basis for V.

3 (10 points) Dummit-Foote, 11.5.1 Since M is cyclic wlog $M = R/I$ for some ideal $I \subset R$. Then $\mathcal{S}(M) = \mathcal{T}(M)/J$ where J is the 2-sided ideal generated by elements of the form $m_1 \otimes m_2 - m_2 \otimes m_1$. Writing $m_i = r_i(1 + I)$, we see $m_1 \otimes m_2 - m_2 \otimes m_1 = 0$, whence $\mathcal{S}(M) = \mathcal{T}(M)$

4 (10 points) Dummit-Foote, 11.5.13 Let $\{e_\alpha\}$ be a basis for V. Let $s_{\alpha\beta} = (e_\alpha \otimes e_\beta + e_\beta \otimes e_\alpha)/2$ and $a_{\alpha\beta} = (e_\alpha \otimes e_\beta - e_\beta \otimes e_\alpha)/2$. Let M be the subspace of $V \otimes V$ generated by $\{s_{\alpha\alpha}\}$ and $\{s_{\alpha\beta}\}$ for $\alpha \neq \beta$ (only one of $s_{\alpha\beta}$ and $s_{\beta\alpha}$ is picked). Similarly let N be the subspace generated by $\{a_{\alpha\beta}\}$ for $\alpha \neq \beta$ (again only one of $s_{\alpha\beta}$ and $s_{\beta\alpha}$ is picked). The fact that $e_\alpha \otimes e_\beta$ form a basis for $V \otimes V$ together with $2 \neq 0$ in k, imply that the generators given above for M and N are infact linearly independent. Also $e_\alpha \otimes e_\alpha = s_{\alpha\alpha}$, and $e_\alpha \otimes e_\beta = s_{\alpha\beta} \pm a_{\alpha\beta}$, so that we have $V \otimes V = M \oplus N$. Using that maps Sym and Alt, it is easy to see that M is the subspace of symmetric 2-tensors and N the subspace of alternating 2-tensors.