

## Homework 2 – due 02/13/08

### Math 601

5. (a) (5 points) Let  $R$  be a commutative ring with identity, and for an  $R$ -module  $M$  define the  $R$ -module  $M^* = \text{Hom}_R(M, R)$ . Define a natural  $R$ -linear homomorphism  $M \otimes_R M^* \rightarrow R$ . Then use adjointness to construct a natural morphism  $M \rightarrow M^{**}$ .

(b) (5 points) Discuss whether  $M \rightarrow M^{**}$  is an isomorphism, injective, or surjective. Can you find examples of rings  $R$  and certain conditions on  $M$  such that some or all of these properties hold, and other examples showing that one or more can fail?

6. Prove the Cauchy-Binet formula. Recall that this says the following. Suppose  $R$  is as above, and  $A$  is an  $n \times m$  matrix over  $R$ , and  $B$  is an  $m \times n$  matrix over  $R$ .

(a) (5 points) If  $n > m$ , then  $\det(AB) = 0$ .

(b) (15 points) If  $n \leq m$ , then  $\det(AB) = \sum_{\mathbf{k}} \det(A_{\mathbf{k}}) \det(B_{\mathbf{k}})$ . Here,  $\mathbf{k}$  ranges over increasing  $n$ -tuples  $1 \leq i_1 < \dots < i_n \leq m$ , and  $A_{\mathbf{k}}$  (resp.  $B_{\mathbf{k}}$ ) denotes the  $n \times n$  submatrix of  $A$  (resp.  $B$ ) formed by selecting from  $A$  (resp.  $B$ ) the columns (resp. rows) corresponding to the  $n$ -tuple  $\mathbf{k}$ . HINT: Use exterior algebras.

7. Let  $\mathcal{A}$  be an abelian category. Recall that this means the following:

- $\mathcal{A}$  is an additive category: for objects  $A, B, C$ , the set  $\text{Hom}(A, B)$  is an abelian group and the composition map  $\text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$  is a group homomorphism;
- in  $\mathcal{A}$  we can form direct sums, kernels, and cokernels (each of which satisfies a certain universal property, discussed in class). Also, if we define  $\text{im}(f : A \rightarrow B) := \ker(B \rightarrow \text{cok}(f))$  and  $\text{coim}(f : A \rightarrow B) := \text{cok}(\ker(f) \rightarrow A)$ , then the canonical map

$$\text{coim}(f) \rightarrow \text{im}(f)$$

is an isomorphism.

(a) (5 points) If  $\mathcal{A}$  is an abelian category, define the category  $\mathbf{C}(\mathcal{A})$  of complexes over  $\mathcal{A}$ .

(b) (15 points) Show that  $\mathbf{C}(\mathcal{A})$  is an abelian category.