Solutions to Homework 2

Math 601, Spring 2008

5 a) (5 points) Let us denote typical elements of R-modules M, M^* and N, N^* by m, μ and n, η respectively. Consider the map $M \times M^* \to R$ given by $(m, \mu) \mapsto \mu(m)$, which is R-bilinear and hence gives an element of $\mathrm{Hom}_R(M \otimes M^*, R)$ which in turn by adjointness gives an element $F \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(M^*, R)) = \mathrm{Hom}_R(M, M^{**})$. Clearly $F : M \to M^{**}$ is given by $m \mapsto (\mu \mapsto \mu(m))$. We have to show F is natural. In basic category terminology: Let Id and Dd be covariant functors in the category R-Mod with $Id(M) = M$, $Dd(M) = M^{**}$ and for $f : M \to N$, let $Id(f) = f$, $Dd(f) = f^{**}$. We need to show that $F : Id \to Dd$ is a natural transformation of functors. This only means that the diagram below commutes:

$$\begin{array}{ccc} M & \xrightarrow{F} & M^{**} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{**}} \\ N & \xrightarrow{F} & N^{**} \end{array}$$

We need to know what $f^{**} \circ F : M \to N^{**}$ is. It sends m to $(\eta \mapsto (\eta \circ f)(m))$. But this is precisely $F \circ f : M \to N^{**}$. So the diagram does commute.

5 b) (5 points) If R is a domain and M is a torsion module then $M^* = 0$, hence $F : M \to M^{**}$ is surjective but not injective. Let R be a field and $M = \oplus_{i \in \mathbb{N}} Re_i$. Then $M^* = \prod_{i \in \mathbb{N}} Re_i^*$. Since every vector space has a basis, we can write $M^* \simeq \oplus_{\alpha \in \mathcal{A}} R$ where $\mathcal{A}$ is an uncountable index set for the basis of M^*. But now, $M^{**} \simeq \prod_{\alpha \in \mathcal{A}} R$. Thus M which has a countable basis cannot map surjectively into M^{**}. It is easy to see that F is injective in this case. An example when F is an isomorphism is when $M \simeq R^n$.

6 (20 points) $\det(AB) = \wedge^n A \cdot \wedge^n B$ where $\wedge^n A$ is a $1 \times l$ matrix and $\wedge^n B$ is an $l \times 1$ matrix and $l = \binom{m}{n}$. Thus, if $m < n$ then $l = 0$ and hence $\det(AB) = 0$. Let $\{e_i\}$ denote the standard bases of R^n and R^m. Let $n \geq m$ and let $\mathbf{k}(i)$ for $1 \leq i \leq l$ be the multi-indices defined in the problem, then the l multivectors $e_{\mathbf{k}} = e_{k_1} \wedge \cdots \wedge e_{k_n}$ are a basis for $\wedge^n R^m$. We have:

$$\wedge^n A \cdot \wedge^n B(e_1 \wedge \cdots e_n) = \wedge^n A(\sum_{i=1}^{l} \det(B_{\mathbf{k}(i)}) e_{\mathbf{k}(i)}) = \sum_{i=1}^{l} \det(B_{\mathbf{k}(i)}) \det(A_{\mathbf{k}(i)})$$

where $A_{\mathbf{k}}$ is the minor of A on all its rows and columns $\mathbf{k}$ and $B_{\mathbf{k}}$ is the minor of B on all its columns and rows $\mathbf{k}$.

7 (20 points) A complex (A_*, d) over $\mathcal{A}$ is a sequence $d_n : A_n \to A_{n+1}$ with $d^2 = 0$ i.e. $d_{n+1} \circ d_n = 0 \in \text{Hom}(A_n, A_{n+2})$. These complexes are the objects of the category $\mathbf{C}(\mathcal{A})$. The morphisms $f_* \in \text{Hom}((A_*, d), (B_*, d'))$ are morphims $f_n : A_n \to B_n$ such that $f \circ d = d' \circ f$. Given $f_* \in \text{Hom}((A_*, d), (B_*, d'))$ and $g_* \in \text{Hom}((B*, d'), (C_*, d''))$ we have $(g \circ f) \circ d = g \circ d' \circ f = d'' \circ (g \circ f)$, therefore we have composition $\text{Hom}((A_*, d), (B_*, d')) \times \text{Hom}((B*, d'), (C_*, d'')) \to \text{Hom}((A_*, d), (C_*, d''))$ (and identity morphism etc. so that $\mathbf{C}(\mathcal{A})$ is a Category.

Given two morphisms f_*, g_*, define $(f + g)_n = f_n + g_n$, then we have $(f + g) \circ d = d' \circ (f + g)$. In this way $\text{Hom}(A_*, B_*)$ is an abelian group and the composition of morphisms mentioned above is a group homomorphism, so that $\mathbf{C}(\mathcal{A})$ is additive.

We will now use direct sums, kernels and cokernels in $\mathcal{A}$ to manufacture the same constructs in $\mathbf{C}(\mathcal{A})$. The fact that the constructs in $\mathcal{A}$ satisfy a universal property is the only fact needed prove that the manufactured constructs in $\mathbf{C}(\mathcal{A})$ satisfy the required universal property. We skip this verification. For complexes (A_*, d) and (B_*, d'), define $(A \oplus B)_n = A_n \oplus B_n$. We have inclusions i_1, i_2 from A_n, B_n into $A_n \oplus B_n$ and projections p_1, p_2 from $A_n \oplus B_n$ into A_n, B_n. Define the boundary map d'' for the complex $A_* \oplus B_*$ by $d'' = i_1 \circ d \circ p_1 + i_2 \circ d' \circ p_2$ where the addition takes place in $\text{Hom}(A_n \oplus B_n, A_{n+1} \oplus B_{n+1})$. It can be checked that $d''^2 = 0$.

Given morphism $f_* : A_* \to B_*$, we have $i_n : \ker(f_n) \to A_n$. Define $\ker(f_*)_n = \ker(f_n)$. By the universal property of ker, the boundary map d of A_* induces a unique boundary map $\tilde{d} : \ker(f_n) \to \ker(f_{n+1})$ such that $i : (\ker(f_*), \tilde{d}) \to (A_*, d)$ is a morphism of complexes. The construction of cokernel is analogous. (Note, to be notationally correct the object I am calling $\ker(f_n)$ should be called something else like K_n and reserve the name $\ker(f_n)$ for the morphism $i_n : K_n \to A_n$)

Since $\text{coim}(f_*)$ and $\text{im}(f_*)$ are cokernels and kernels of complexes (resp.) they are themselves complexes. We also have a map $f' : \text{coim}(f_*) \to \text{im}(f_*)$, which we will show to be a morphism. This is equivalent to showing the commutativity of the middle square in the diagram below: (there should be a less complicated way than what follows.)

$$
\begin{array}{ccc}
A_n & \xrightarrow{d_n} & A_{n+1} \\
\downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle \pi_{n+1}} \\
\operatorname{coim}(f_n) & \xrightarrow{d_n} & \operatorname{coim}(f_{n+1}) \\
{\scriptstyle f'_n}\downarrow{\scriptstyle \cong} & & {\scriptstyle f'_{n+1}}\downarrow{\scriptstyle \cong} \\
\operatorname{im}(f_n) & \xrightarrow{d'_n} & \operatorname{im}(f_{n+1}) \\
\downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle i_{n+1}} \\
B_n & \xrightarrow{d'_n} & B_{n+1}
\end{array}
$$

In the diagram above, consider $\alpha_1 = f'_{n+1} \circ d_n$ and $\alpha_2 = d'_n \circ f'_n$. We need to show $\alpha_1 = \alpha_2$. Observe that $i_{n+1} \circ \alpha_1 \circ \pi_n = f_{n+1} \circ d_n$ using the commutativity of the top square. Similarly $i_{n+1} \circ \alpha_2 \circ \pi_n = d'_n \circ f_n$ using the commutativity of the bottom square. Therefore, $f_{n+1} \circ d_n = d'_n \circ f_n$ implies $i_{n+1} \circ (\alpha_1 \circ \pi_n) = i_{n+1} \circ (\alpha_2 \circ \pi_n)$. The universal property of $\operatorname{im}(f_{n+1})$ (that derives from that of ker) implies $\alpha_1 \circ \pi_n = \alpha_2 \circ \pi_n$. Call this β. Let $j_n : \ker(f_n) \to A_n$, then $\pi_n \circ j_n = 0$ implies $\beta \circ j_n$. The universal property of $\operatorname{coim}(f_n)$ (derived from that of coker) now implies that $\beta = \alpha_i \circ \pi_n$ must factor uniquely through $\operatorname{coim}(f_n)$, and this in turn means $\alpha_1 = \alpha_2$. Thus we have shown $f' : \operatorname{coim}(f_*) \to \operatorname{im}(f_*)$ to be a morphism, and since each f'_n is an isomorphism we have shown that $\operatorname{coim}(f_*) \to \operatorname{im}(f_*)$ is an isomorphism.