8) (10 points) Let $d_{A_1}, d_{A_2}, d_{B_1}$ etc denote the boundary maps of the complexes $A_1^\bullet, A_2^\bullet, B_1^\bullet$ etc. Let $\alpha, \beta, \gamma$ denote the morphisms $A_1^\bullet \rightarrow A_2^\bullet, B_1^\bullet \rightarrow B_2^\bullet, C_1^\bullet \rightarrow C_2^\bullet$. Let $\tilde{\alpha} : H^i(A_1^\bullet) \rightarrow H^i(A_2^\bullet)$ etc. Let $\delta_1, \delta_2$ be the connecting homomorphisms as shown in the problem. Also let $f_1 : A_1^\bullet \rightarrow B_1^\bullet, g_1 : B_1^\bullet \rightarrow C_1^\bullet$ and similarly we have $f_2$ and $g_2$.

Refering to the diagram given in the problem, we are asked to prove that $\tilde{\alpha} + 1 \circ \delta_1^i = \delta_2^i \circ \gamma^i$. 

The connecting homomorphism $\delta_1^i$ is defined to be $(f_2^{i+1})^{-1} \circ d_{B_2}^i \circ (g_2^i)^{-1}$. Let $[z_1] \in H^i(C_1^\bullet)$ be represented by a cocycle $z_1 \in C^i(C_1^\bullet)$. We have to show

$$(f_2^{i+1})^{-1} \circ d_{B_2}^i \circ (g_2^i)^{-1} \circ \gamma^i z_1 = \alpha^i+1 \circ (f_1^{i+1})^{-1} \circ d_{B_1}^i \circ (g_1^i)^{-1} z_1$$

We will show the lhs to equal rhs. Pick $y_1 \in C^i(B_1^\bullet)$ with $g_1^i(y_1) = z_1$. Then $\gamma^i(z_1) = \gamma^i g_1^i y_1 = g_2^i \beta^i y_1$ so that the lhs becomes $(f_2^{i+1})^{-1} \circ d_{B_2}^i \circ \beta^i y_1$. Next $d_{B_2}^i \circ \beta^i y_1 = \beta^{i+1} \circ d_{B_1}^i$, so the lhs becomes $(f_2^{i+1})^{-1} \circ \beta^{i+1} \circ d_{B_1}^i y_1$. Now $d_{B_1}^i y_1 \in \text{im}(f_1^{i+1})$ so we may rewrite the lhs as $(f_2^{i+1})^{-1} \circ \beta^{i+1} \circ f_1^{i+1} \circ (f_1^{i+1})^{-1} \circ d_{B_1}^i y_1$. Using the fact that $\beta^{i+1} \circ f_1^{i+1} = f_2^{i+1} \circ \alpha + 1$, the lhs now becomes $\alpha^{i+1} \circ (f_1^{i+1})^{-1} \circ d_{B_1}^i (g_1^i)^{-1} z_1$ which is precisely the rhs of the above equation. It can be shown that the above verification does not depend on the choice of representatives $z_i$ and $y_i$.

9) (5 points) Given a morphism $f : A \rightarrow B$ in an Abelian category, we have to show that the canonical map $p : B \rightarrow \text{coker}(B)$ is an epimorphism. This means that given two maps $g_1, g_2 : \text{coker}(B) \rightarrow C$ with $g_1 \circ p = g_2 \circ f p$ then we must prove that $g_1 = g_2$. Now $g_1 \circ p \circ f = 0$, hence there is a unique maps $h_1 : \text{coker}(B) \rightarrow C$ with $h_1 \circ p = g_1 \circ p$. Clearly $g_2$ is also a valid candidate for $h_1$, whence $g_1 = g_2$.

10) Dummit-Foote, 10.5.2 (5 points) Follows from diagram chasing without difficulty.

11) Dummit-Foote, 10.5.6 (5 points) We have to show that for a ring $R$, the property that every $R$-module is projective holds if the property that every $R$-module is injective holds. Indeed, either of these properties is equivalent to the property that all short exact sequences are split.
Let $M$ be a left $\mathbb{Z}$-module and let $R$ be a ring with 1.

(a) Show that $\text{Hom}_\mathbb{Z}(R, M)$ is a left $R$-module under the action $(r \phi)(r') = \phi(r'r)$ (no change here).

(b) Suppose that $0 \to A \xrightarrow{\psi} B$ is an exact sequence of $R$-modules. Prove that if every $\mathbb{Z}$-module homomorphism $f$ from $A$ to $M$ lifts to a $\mathbb{Z}$-module homomorphism $F$ from $B$ to $M$ with $f = F \circ \psi$, then every $R$-module homomorphism $f'$ from $A$ to $\text{Hom}_\mathbb{Z}(R, M)$ lifts to an $R$-module homomorphism $F'$ from $B$ to $\text{Hom}_\mathbb{Z}(R, M)$ with $f' = F' \circ \psi$. (Note changes to $f, F$) [Given $f'$, show that $f(a) = f'(a)(1_R)$ defines a $\mathbb{Z}$-module homomorphism of $A$ to $M$. If $F$ is the associated lift of $f$ to $B$, show that $F'(b)(r) = F(rb)$ defines an $R$-module homomorphism from $B$ to $\text{Hom}_\mathbb{Z}(R, M)$ that lifts $f'$.]

(c) Prove that if $Q$ is an injective $\mathbb{Z}$-module then $\text{Hom}_\mathbb{Z}(R, Q)$ is an injective $R$-module. (Note change in $Q$)

Solution: a) One checks $rs \cdot \phi(t) = \phi(trs) = s \cdot \phi(tr) = r \cdot (s \cdot \phi(r))$.

b) Given $f' \in \text{Hom}_R(A, \text{Hom}_\mathbb{Z}(R, M))$ define $f \in \text{Hom}_\mathbb{Z}(A, M)$ by $f(a) = f'(a)(1_R)$. Note that $f$ is not $R$-linear, whence the change in the problem. By hypothesis, we have $F \in \text{Hom}_\mathbb{Z}(B, M)$ with $f = F \circ \psi$. Define $F'$ by the formula $F'(b)(r) = F(rb)$. A quick calculation shows that $F' \in \text{Hom}_R(B, \text{Hom}_\mathbb{Z}(R, M))$.

c) Given an injective $\mathbb{Z}$-module $Q$, letting $Q$ play the role of $M$ in part b), shows that to every $f : A \to Q$ there is a lift $F : B \to Q$. Now given $f' \in \text{Hom}_R(A, \text{Hom}_\mathbb{Z}(R, Q))$, we construct the $f, F, F'$ as in part b) and obtain $F' \in \text{Hom}_R(B, \text{Hom}_\mathbb{Z}(R, Q))$. Let us check that $F'$ is a lift of $f$:

$$F'(\psi(a))(r) = F(r\psi(a)) = F(\psi(ra)) = f(ra) = f'(ra)(1_R) = (rf'(a))(1_R) = f'(a)(r)$$

as required. This shows that for an injective abelian group $Q$, we get an injective $R$-module $\text{Hom}_\mathbb{Z}(R, Q)$.
13) Dummit-Foote, 10.5.16 (10 points) a) By corollary 37 of Section 10.5 of the text, every abelian group is a subgroup of an injective abelian group. Given a left $R$-module, $M$ we get that $M$ (also being an abelian group) is contained in an injective abelian group $Q$.

b) Both inclusions $\text{Hom}_R(R, M) \subset \text{Hom}_Z(R, M) \subset \text{Hom}_Z(R, Q)$ are easily seen to be homomorphisms of abelian groups. The left $R$-module structure on $\text{Hom}_R(R, M)$ as well as its isomorphism with $M$ is defined in Problem 10.5.10 of the text. The left $R$-module structure on $\text{Hom}_Z(R, A)$ for any abelian group $A$ is defined in Problem 10.5.15a) of the text. Both inclusions $\text{Hom}_R(R, M) \subset \text{Hom}_Z(R, M) \subset \text{Hom}_Z(R, Q)$ are easily seen to be left $R$-module monomorphisms.

c) Since we have proved in part b) that $M \cong \text{Hom}_R(R, M)$ is a $R$-submodule of $\text{Hom}_Z(R, Q)$, and in part c) of the previous problem that for an injective abelian group $Q$, the $R$-module $\text{Hom}_Z(R, Q)$ is an injective $R$-module, we have proved that every $R$-module $M$ is a submodule of an injective $R$-module.