Solutions to Homework 3

Math 601, Spring 2008

8) (10 points) Let $d_{A_1}, d_{A_2}, d_{B_1}$ etc denote the boundary maps of the complexes $A_1^\bullet, A_2^\bullet, B_1^\bullet$ etc. Let α, β, γ denote the morphisms $A_1^\bullet \to A_2^\bullet$, $B_1^\bullet \to B_2^\bullet$, $C_1^\bullet \to C_2^\bullet$. Let $\bar{\alpha}^i : H^i(A_1^\bullet) \to H^i(A_2^\bullet)$ etc. Let δ_1, δ_2 be the connecting homomomorphisms as shown in the problem. Also let $f_1 : A_1^\bullet \to B_1^\bullet$, $g_1 : B_1^\bullet \to C_1^\bullet$ and similarly we have f_2 and g_2.

Refering to the diagram given in the problem , we are asked to prove that $\bar{\alpha}^{i+1} \circ \delta_1^i = \delta_2^i \circ \bar{\gamma}^i$. The connecting homomorphism δ_2^i is defined to be $(f_2^{i+1})^{-1} \circ d_{B_2}^i \circ (g_2^i)^{-1}$. Let $[z_1] \in H^i(C_1{}^\bullet)$ be represented by a cocycle $z_1 \in C^i(C_1{}^\bullet)$. We have to show

$$(f_2^{i+1})^{-1} \circ d_{B_2}^i \circ (g_2^i)^{-1} \circ \gamma^i z_1 = \alpha^{i+1} \circ (f_1^{i+1})^{-1} \circ d_{B_1}^i \circ (g_1^i)^{-1} z_1$$

We will show the lhs to equal rhs. Pick $y_1 \in C^i(B_1{}^\bullet)$ with $g_1^i(y_1) = z_1$. Then $\gamma^i(z_1) = \gamma^i g_1^i y_1 = g_2^i \beta^i y_1$ so that the lhs becomes $(f_2^{i+1})^{-1} \circ d_{B_2}^i \circ \beta^i y_1$. Next $d_{B_2}^i \circ \beta^i y_1 = \beta^{i+1} \circ d_{B_1}^i$ so the lhs becomes $(f_2^{i+1})^{-1} \circ \beta^{i+1} \circ d_{B_1}^i y_1$. Now $d_{B_1}^i y_1 \in \mathrm{im}(f_1^{i+1})$ so we may rewrite the lhs as $(f_2^{i+1})^{-1} \circ \beta^{i+1} \circ f_1^{i+1} \circ (f_1^{i+1})^{-1} \circ d_{B_1}^i y_1$. Using the fact that $\beta^{i+1} \circ f_1^{i+1} = f_2^{i+1} \circ \alpha^{i+1}$, the lhs now becomes $\alpha^{i+1} \circ (f_1^{i+1})^{-1} \circ d_{B_1}^i (g_1^i)^{-1} z_i$ which is precisely the rhs of the above equation. It can be shown that the above verification does not depend on the choice of representatives z_i and y_i.

9) (5 points) Given a morphism $f : A \to B$ in an Abelian category, we have to show that the canonical map $p : B \to \mathrm{coker}(B)$ is an epimorphism. This means that given two maps $g_1, g_2 : \mathrm{coker}(B) \to C$ with $g_1 \circ p = g_2 \circ fp$ then we must prove that $g_1 = g_2$. Now $g_1 \circ p \circ f = 0$, hence there is a unique maps $h_1 : \mathrm{coker}(B) \to C$ with $h_1 \circ p = g_1 \circ p$. Clearly g_2 is also a valid candidate for h_1, whence $g_1 = g_2$.

10) Dummit-Foote, 10.5.2 (5 points) Follows from diagram chasing without difficulty.

11) Dummit-Foote, 10.5.6 (5 points) We have to show that for a ring R, the property that every R-module is projective holds iff the property that every R-module is injective holds. Indeed, either of these properties is equivalent to the property that all short exact sequences are split.

12) Dummit-Foote, 10.5.15 (15 points) The problem needs to be reworded to make it free of errors and confusion. The changed problem is as follows: (it can also be found at http://www.emba.uvm.edu/˜foote/errata_3rd_edition.pdf

Let M be a left $\mathbb{Z}$-module and let R be a ring with 1.

(a) Show that $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ is a left R-module under the action $(r\phi)(r') = \phi(r'r)$ (no change here).

(b) Suppose that $0 \to A \xrightarrow{\psi} B$ is an exact sequence of R-modules. Prove that if every $\mathbb{Z}$-module homomorphism f from A to M lifts to a $\mathbb{Z}$-module homomorphism F from B to M with $f = F \circ \psi$, then every R-module homomorphism f' from A to $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ lifts to an R-module homomorphism F' from B to $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ with $f' = F' \circ \psi$. (note changes to f, F) [Given f', show that $f(a) = f'(a)(1_R)$ defines a $\mathbb{Z}$-module homomorphism of A to M. If F is the associated lift of f to B, show that $F'(b)(r) = F(rb)$ defines an R-module homomorphism from B to $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ that lifts f'.]

(c) Prove that if Q is an injective $\mathbb{Z}$-module then $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$ is an injective R-module. (note change in Q)

Solution: a) One checks $rs \cdot \phi(t) = \phi(trs) = s \cdot \phi(tr) = r \cdot (s \cdot \phi(r))$.

b) Given $f' \in \mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbb{Z}}(R, M))$ define $f \in \mathrm{Hom}_{\mathbb{Z}}(A, M)$ by $f(a) = f'(a)(1_R)$. Note that f is not R-linear, whence the change in the problem. By hypothesis, we have $F \in \mathrm{Hom}_{\mathbb{Z}}(B, M)$ with $f = F \circ \psi$. Define F' by the formula $F'(b)(r) = F(rb)$. A quick calculation shows that $F' \in \mathrm{Hom}_R(B, \mathrm{Hom}_{\mathbb{Z}}(R, M))$.

c) Given an injective $\mathbb{Z}$-module Q, letting Q play the role of M in part b), shows that to every $f : A \to Q$ there is a lift $F : B \to Q$.Now given $f' \in \mathrm{Hom}_R(A, \mathrm{Hom}_{\mathbb{Z}}(R, Q))$, we construct the f, F, F' as in part b) and obtain $F' \in \mathrm{Hom}_R(B, \mathrm{Hom}_{\mathbb{Z}}(R, Q))$. Let us check that F' is a lift of f:

$$F'(\psi(a))(r) = F(r\psi(a)) = F(\psi(ra)) = f(ra) = f'(ra)(1_R) = (rf'(a))(1_R) = f'(a)(r)$$

as required. This shows that for an injective abelian group Q, we get an injective R-module $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$

13) Dummit-Foote, 10.5.16 (10 points) a) By corollary 37 of Section 10.5 of the text, every abelian group is a subgroup of an injective abelian group. Given a left R-module, M we get that M (also being an abelian group) is contained in an injective abelian group Q.

b) Both inclusions $\mathrm{Hom}_R(R, M) \subset \mathrm{Hom}_{\mathbb{Z}}(R, M) \subset \mathrm{Hom}_{\mathbb{Z}}(R, Q)$ are easily seen to be homomorphisms of abelian groups. The left R-module structure on $\mathrm{Hom}_R(R, M)$ as well as its isomorphism with M is defined in Problem 10.5.10 of the text. The left R-module structure on $\mathrm{Hom}_{\mathbb{Z}}(R, A)$ for any abelian group A is defined in Problem 10.5.15a) of the text. Both inclusions $\mathrm{Hom}_R(R, M) \subset \mathrm{Hom}_{\mathbb{Z}}(R, M) \subset \mathrm{Hom}_{\mathbb{Z}}(R, Q)$ are easily seen to be left R-module monomorphisms.

c) Since we have proved in part b) that $M \cong \mathrm{Hom}_R(R, M)$ is a R-submodule of $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$, and in part c) of the previous problem that for an injective abelian group Q, the R-module $\mathrm{Hom}_{\mathbb{Z}}(R, Q)$ is an injective R-module, we have proved that every R-module M is a submodule of an injective R-module.