Homework 4 – due 02/27/08

Math 601

14. (15 points) Suppose we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & R & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & N' & \longrightarrow & R' & \longrightarrow & M' & \longrightarrow & 0. \end{array}$$

Prove that we can find injective resolutions $I_M^\bullet, I_R^\bullet$ etc. in a compatible way, meaning that there is a commutative diagram of complexes with split exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_N^\bullet & \longrightarrow & I_R^\bullet & \longrightarrow & I_M^\bullet & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_{N'}^\bullet & \longrightarrow & I_{R'}^\bullet & \longrightarrow & I_{M'}^\bullet & \longrightarrow & 0. \end{array}$$

15. (15 points) *Uniqueness of right derived functors*: Let $F : R-\text{Mod} \to S-\text{Mod}$ be a left-exact functor. Let $\{F^i\}_{i=0}^{\infty}$ be a family of additive functors with the following properties:

(i) $F^0 \cong F$ (a natural equivalence of functors);

(ii) If Q is an injective R-module, then $F^i(Q) = 0$ for all $i \geq 1$;

(iii) If (*) $\quad 0 \to M' \to M \to M'' \to 0$ is exact in R-Mod, then there is a long exact sequence in S-Mod

$$\cdots \to F^i(M') \to F^i(M) \to F^i(M'') \to F^{i+1}(M') \to \cdots$$

which is natural in the exact sequence (*).

Prove that for all i we have $F^i \cong R^iF$ (again, a natural equivalence). (HINT: You will need to use problems #8 and #14.)

16. (5 points) Suppose F is any additive functor, and that $f^\bullet \sim g^\bullet$, for maps of complexes $f^\bullet$ resp. $g^\bullet$. Prove that $F(f^\bullet) \sim F(g^\bullet)$. (NOTE: We used this implicitly in lecture, during our proof of the well-definedness and functoriality of R^iF.)

17. Dummit-Foote 17.1, #10.