Solutions to Homework 4

Math 601, Spring 2008

14) (15 points) This problem is tricky and caution is required! Given $f : N \to N'$ and $h : M \to M'$ we may construct morphisms between the injective resolutions $f^\bullet : I^\bullet_N \to I^\bullet_{N'}$ and $h^\bullet : I^\bullet_M \to I^\bullet_{M'}$. These are unique upto homotopy. Recall that the resolutions of R and R' are $I^\bullet_R = I^\bullet_N \oplus I^\bullet_M$ and $I^\bullet_{R'} = I^\bullet_{N'} \oplus I^\bullet_{M'}$ respectively. We want a morphism between the resolutions $g^\bullet : I^\bullet_R \to I^\bullet_{R'}$. The obvious candidate is $f^\bullet \oplus h^\bullet$. It does not work! First let us take $g^\bullet = f^\bullet \oplus h^\bullet$ and then fix it.

We have to show the commutativity of the following three dimensional diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^\bullet_N & \longrightarrow & I^\bullet_R & \longrightarrow & I^\bullet_M & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle f^\bullet} & & \Big\downarrow{\scriptstyle g^\bullet} & & \Big\downarrow{\scriptstyle h^\bullet} & & \\
0 & \longrightarrow & I^\bullet_{N'} & \longrightarrow & I'^\bullet_R & \longrightarrow & I^\bullet_{M'} & \longrightarrow & 0
\end{array}$$

Such a diagram is commutative if the top/bottom, the three side panels, and the panels on the plane of the paper (one for each $n \geq 0$) are individually commutative. The top and bottom panels are split exact sequence of complexes which are part of the construction of the injective resolution of R and R'. The left and right side panels commute by the very construction of the morphisms $f^\bullet$ and $h^\bullet$. The panels in the plane of the paper look as follows and clearly commute when $g^i = f^i \oplus h^i$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & I^i_N & \longrightarrow & I^i_N \oplus I^i_M & \longrightarrow & I^i_M & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle f^i} & & \Big\downarrow{\scriptstyle g^i = f^i \oplus h^i} & & \Big\downarrow{\scriptstyle h^i} & & \\
0 & \longrightarrow & I^i_{N'} & \longrightarrow & I^i_{N'} \oplus I^i_{M'} & \longrightarrow & I^i_{M'} & \longrightarrow & 0
\end{array}$$

Observe, that if we change g^i from $f^i \oplus h^i$ to $(f^i \oplus h^i) - \mu^i$, for some $\mu^i : I^i_N \oplus I^i_M \to I^i_{N'}$ then these panels still commute.

Now consider the following diagram which has no reason to commute, in general.

$$\begin{array}{ccccc}
0 & \longrightarrow & R & \xrightarrow{\gamma} & I^0_N \oplus I^0_M \\
 & & \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle f^0 \oplus h^0} \\
0 & \longrightarrow & R' & \xrightarrow[\gamma']{} & I^0_{N'} \oplus I^0_{M'}
\end{array}$$

Consider the difference $\sigma = (f^0 \oplus h^0)\circ\gamma - \gamma'\circ g$. We have $\sigma : R \to I^0_{N'} \oplus I^0_{M'}$. Check (by projecting onto $I^0_{M'}$ and chasing) that actually, the image of σ is in the first summand $I^0_{N'}$ only. By slight abuse of notation, we thus have a map $\sigma : R \to I^0_{N'}$. Now, since $I^0_{N'}$ is injective and $0 \to R \xrightarrow{\gamma} I^0_N \oplus I^0_M$ is exact, we have a lift μ^0 of σ, where $\mu^0 : I^0_N \oplus I^0_M \to I^0_{N'}$. Now define $g^0 = (f^0 \oplus h^0) - \mu^0$ and observe that $g^0 \circ \gamma = \gamma' \circ g$. As noted before this does not affect the commutativity of the panel in the plane of the paper of level zero. The construction of $g^i = (f^i \oplus h^i) - \mu^i$, where $\mu^i : I^i_N \oplus I^i_M \to I^i_{N'}$ is constructed similar to the construction of μ^0 and the usual procedure involved in constructing f^i from f^{i-1} inductively. This completes the solution.

15) (15 points) The isomorphisms $F^0(M) \cong F(M)$ and $R^0F(M) \cong F(M)$ are natural in M, thus we obtain that $F^0 \cong R^0F$ is a natural equivalence (isomorphism) of functors. Let $0 \to M \to Q \xrightarrow{p} Q/M \to 0$ be a short exact sequence where Q is injective. Similarly let $0 \to N \to Q' \xrightarrow{p} Q'/N \to 0$ be a short exact sequence where Q' is injective. Given $f : M \to N$ we can extend f to the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\gamma} & Q & \xrightarrow{p} & Q/M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle \bar{f}^0} & & \\
0 & \longrightarrow & N & \xrightarrow{\gamma'} & Q' & \xrightarrow{p} & Q'/N & \longrightarrow & 0
\end{array}$$

We now obtain a compatible injective resolutions as shown in the second commutative diagram on the problem sheet (Problem 14). The rows of this diagram of resolutions are split exact, and applying the functor F to this diagram, gives us the following commutative diagram with the rows still being exact.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F(I^\bullet_M) & \longrightarrow & F(I^\bullet_Q) & \longrightarrow & F(I^\bullet_{Q/M}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F(I^\bullet_N) & \longrightarrow & F(I^\bullet_{Q'}) & \longrightarrow & F(I^\bullet_{Q'/N}) & \longrightarrow & 0
\end{array}$$

We now apply the result of Problem 8 (long exact sequence of homology, naturality) to this diagram, to get the following diagram where $i \geq 1$, where as for $i = 1$ we get the first row (exact) of the diagram in equation (2) etc.

$$\begin{array}{ccccccc}
0 & \longrightarrow & R^iF(Q/M) & \longrightarrow & R^{i+1}F(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & R^iF(Q'/N) & \longrightarrow & R^{i+1}F(N) & \longrightarrow & 0
\end{array} \tag{1}$$

We get a similar diagram with R^iF etc replaced by F^i etc. We will prove below that $R^1F \cong F^1$ naturally. Assuming $R^iF \cong F^i$ naturally, and the above mentioned diagrams, we get

$$\begin{array}{ccc} R^{i+1}F(M) & \xrightarrow{\sim} & F^{i+1}(M) \\ \downarrow & & \downarrow \\ R^{i+1}F(N) & \xrightarrow{\sim} & F^{i+1}(N) \end{array}$$

Thus showing that $R^{i+1}F \cong F^{i+1}$ naturally. To prove $R^1F \cong F^1$ naturally, consider the following diagram

$$\begin{array}{ccccccc} F^0(Q) & \longrightarrow & F^0(Q/M) & \longrightarrow & F^1(M) & \longrightarrow & 0 \\ \downarrow\wr & & \downarrow\wr & & \downarrow & & \| \\ R^0F(Q) & \longrightarrow & R^0F(Q/M) & \longrightarrow & R^1F(M) & \longrightarrow & 0 \end{array} \tag{2}$$

The first two vertical arrows are isomorphisms $F^0 \cong R^0F$. The rows are exact as observed before. The exactness of the rows induces the third vertical arrow. We can put another layer below this diagram where the second layer is the exact analog of the top layer for N instead of M. We also get maps from the top layer to the bottom layer (provided by the naturality of the isomorphisms $F^0 \cong R^0F$) making the two layered diagram commutative. It is easy to show that the third vertical arrow (in each layer) is an isomorphism (say using Problem 10), and this implies that the isomorphism $R^1F \cong F^1$ is natural.

16) (5 points) Given $f^i - g^i = h^{i+1} \circ d^i + d'^{i-1} \circ h^i$, we can apply the additive functor F to this to get $(Ff)^i - (Fg)^i = (Fh)^{i+1} \circ (Fd)^i + (Fd)'^{i-1} \circ (Fh)^i$ showing that Ff is homotopic to Fg

17) Dummit-Foote, 17.1.10 (15 points) a) If P_i are projectives, choose $F_i = P_i \oplus Q_i$ with F_i free, then $\oplus_i F_i$ is free and we have $\oplus_i F_i = (\oplus_i P_i) \oplus (\oplus_i Q_i)$, thus showing that $\oplus_i P_i$ is projective. If I_i are injectives, given $0 \to M \to N$ exact and $M \to \prod_i I_i$, we get individual maps $M \to I_i$ and their lifts $N \to I_i$ which can be pieced together to get a lift $N \to \prod_i I_i$ of $M \to \prod_i I_i$, so that $\prod_i I_i$ is injective.

b) Given family of modules A_i with projective resolutions $P_i^\bullet \to A_i \to 0$ we get that $\oplus_i P_i^\bullet \to \oplus_i A_i \to 0$ is a projective resolution of $\oplus_i A_i$. Applying the functor $\mathrm{Hom}(-, B)$ to this resolution, we get a complex $\prod_i \mathrm{Hom}(P_i^\bullet, B) \to \mathrm{Hom}(\oplus_i A_i, B) \to 0$. Taking the homology of the complex $\prod_i \mathrm{Hom}(P_i^\bullet, B) \to 0$ we get $\mathrm{Ext}^n(\oplus_i A_i, B) = \prod_i \mathrm{Ext}^n(A_i, B)$ as required.

parts c) and d) are analogous to part b)