Homework 5 – due 03/05/08

Math 601

18. Dummit-Foote, 17.1, #7.

19. For integers $n, m \geq 1$, compute $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$.

20. Let R be a domain, and $I = (a)$ a principal ideal. Compute $\mathrm{Ext}^i_R(R/I, M)$, using the projective resolution

$$0 \longrightarrow R \xrightarrow{a} R \longrightarrow R/I \longrightarrow 0.$$

21. Dummit-Foote, 17.1, #22-24. (Despite appearances, this really is just one problem.)

22. Let $(R, \mathfrak{m})$ be a local ring, with residue field $k = R/\mathfrak{m}$. Let M be an R-module. Consider the following statements:

(i) M is free;

(ii) M is projective;

(iii) M is flat;

(iv) $\mathrm{Tor}_1^R(M, k) = 0$.

(a) (5 points) Show that $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$.

(b) (5 points) Assume M is finitely-generated and R is Noetherian. Show that $(iv) \Rightarrow (i)$. HINT: Let $\overline{x_1}, \cdots, \overline{x_n}$ give a basis for the k-vector space $M/\mathfrak{m}M$. Show that $x_1, \cdots, x_n$ generate the R-module M. Define a surjective R-linear map $R^n \twoheadrightarrow M$ by $e_i \mapsto x_i$; let K denote the kernel. Show that $K \otimes_R R/\mathfrak{m} = 0$, and then that $K = 0$.

(c) (5 points) For each of the following situations, find an example of a Noetherian ring R and an R-module M which:

(a) is finitely generated and projective but not free;

(b) is flat but not projective;

(c) satisfies $\mathrm{Tor}_1^R(M, R/\mathfrak{m}) = 0$ for some maximal ideal $\mathfrak{m}$, but is not flat.