Solutions to Homework 5

Math 601, Spring 2008

18) [D-F], 17.1 #7 (5 points) Let $A = A_{\text{tor}} \oplus A_{\text{free}}$ be a f.g. abelian group. We know $\text{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = \text{Ext}^1_{\mathbb{Z}}(A_{\text{free}}, \mathbb{Z}) \oplus \text{Ext}^1_{\mathbb{Z}}(A_{\text{tor}}, \mathbb{Z})$, and that the first summand is zero since a free module is projective and $\text{Ext}^n(A, -)$ vanish for $n > 0$ and A projective. We also know that $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, D) = D/mD$. Therefore, again using the fact that $\text{Ext}(A, -)$ commutes with finite direct sums, we obtain $\text{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = A_{\text{tor}}$. Thus A is free $\Leftrightarrow \text{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) = 0$.

19) (5 points) Using the fact that $\text{Tor}^{\mathbb{Z}}_1(D, \mathbb{Z}/m\mathbb{Z}) = {}_mD = m$-torsion of D, we obtain that $\text{Tor}^{\mathbb{Z}}_1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ is the m-torsion of $\mathbb{Z}/n\mathbb{Z}$ which is $\mathbb{Z}/(m, n)\mathbb{Z}$.

20) (10 points) Let R be a domain, and $I = (a)$ a principal ideal. Compute $\text{Ext}^i_R(R/I, M)$ using the projective resolution $0 \to R \xrightarrow{a} R \to R/I \to 0$. (the exactness on left requires R to be a domain)

The groups $\text{Ext}^i_R(R/I, M)$ are the homology groups of the following complex:

$$0 \to \text{Hom}_R(R, M) \xrightarrow{a} \text{Hom}_R(R, M) \to 0$$

Thus $\text{Ext}^0_R(R/I, M) = \text{Hom}_R(R/I, M) = {}_aM$ (the a-torsion of M), and $\text{Ext}^i_R(R/I, M) = 0$ for $i > 1$ and $\text{Ext}^1_R(R/I, M) = M/aM$.

21 a) [D-F], 17.1 #22 (5 points) Let $P_\bullet \to A \to 0$ be a projective resolution of A. We tensor this exact sequence with the flat R-module S to get the exact (by flatness of S) sequence $S \otimes_R P_\bullet \to S \otimes_R A \to 0$ which is a resolution of $S \otimes_R A$ by projective S-modules: to see that $S \otimes_R P_i$ are projective S-modules, let F be a free R-module with $F = P_i \oplus Q_i$. Tensoring with S we get $S \otimes_R F = S \otimes_R P_i \oplus S \otimes_R Q_i$, whence $S \otimes_R P_i$ is a direct summand of the free S-module $S \otimes_R F$. Tensoring the resolution $S \otimes_R P_\bullet \to S \otimes_R A \to 0$ with any S-module B we get the complex $(S \otimes_R P_\bullet) \otimes_R B \to 0$. However, we may rewrite $(S \otimes_R P_\bullet) \otimes_R B \cong S \otimes_S (P_\bullet \otimes_R B) \cong P_\bullet \otimes_R B$, and also on the level of the differentials we have $(id_s \otimes_R d) \otimes_R id_B \cong id_S \otimes_S (d \otimes_R id_B) \cong d \otimes_R id_B$ (this associativity of tensor product is proved in Thm. 14 on Pg 371 of [D-F], as well as Problem 75, HW14 in Math600). Therefore the complexes (of S -modules) $(S \otimes_R P_\bullet) \otimes_R B \to 0$ and $P_\bullet \otimes_R B \to 0$ are isomorphic and hence their homology groups $\text{Tor}^S_n(S \otimes_R A, B)$ and $\text{Tor}^R_n(A, B)$

are also isomorphic S-modules.

21 b) [D-F], 17.1 #23 (5 points) For a multiplicative subset $D \subset R$, let $S = D^{-1}R$ with $f : R \to S$ the canonical map. Recall that $D^{-1}R \otimes_R -$ is an exact functor. We showed this in Problem 55, HW11 of Math600, and is also proved as Proposition 42, (6) on pg. 715 of [D-F]. This means precisely that S is flat over R. So we appeal to the result of the previous problem with $D^{-1}B$ playing the role of B there, to obtain

$$\mathrm{Tor}_n^{D^{-1}R}(D^{-1}A, D^{-1}B) = \mathrm{Tor}_n^R(A, D^{-1}B)$$

as $D^{-1}R$-modules. We observe that exactness of $D^{-1}R\otimes_R -$ implies that for $M \xrightarrow{f} N$ the associated map $D^{-1}M \xrightarrow{D^{-1}f} D^{-1}N$ satisfies $\ker(D^{-1}f) = D^{-1}\,\mathrm{Ker}(f)$ and $\mathrm{im}(D^{-1}f) = D^{-1}\,\mathrm{im}(f)$. We also observe that $D^{-1}M \otimes_R N = M \otimes_R D^{-1}N = D^{-1}(M \otimes_R N)$ are natural isomorphisms. So if $P_\bullet \to B \to 0$ be a projective resolution of B, then the second observation tell us that

$$\mathrm{Tor}_*^R(A, D^{-1}B) = H_*(A \otimes_R D^{-1}P_\bullet \to 0) = H_*(D^{-1}(A \otimes_R P_\bullet) \to 0)$$

and the first observation then tells us that

$$H_*(D^{-1}(A \otimes_R P_\bullet) \to 0) = D^{-1}H_*(A \otimes_R P_\bullet \to 0) = D^{-1}\mathrm{Tor}_*^R(A, B)$$

Thus we have shown that

$$\mathrm{Tor}_*^{D^{-1}R}(D^{-1}A, D^{-1}B) = D^{-1}\mathrm{Tor}_*^R(A, B)$$

21 c) [D-F], 17.1 #24 (5 points) Recall that for an R-module N, we have $N = 0 \Leftrightarrow N_\mathfrak{p} = 0$ for every prime (or maximal) ideal $\mathfrak{p}$. (We proved this in going from part c) to d) of Problem 2 of HW1). Using this and the previous exercise, we have

$$M \text{is flat} \Leftrightarrow \mathrm{Tor}_1^R(M, -) = 0 \Leftrightarrow (\mathrm{Tor}_1^R(M, -))_\mathfrak{p} = 0\ \forall \mathfrak{p} \Leftrightarrow \mathrm{Tor}_1^{R_\mathfrak{p}}(M_\mathfrak{p}, -) = 0\ \forall \mathfrak{p} \Leftrightarrow M_\mathfrak{p} \text{ is flat } \forall \mathfrak{p}$$

For the third implication, we implicitly use the fact that every $R_\mathfrak{p}$-module is necessarily the localization at $\mathfrak{p}$ of an R-module (using the canonical map $R \to R_\mathfrak{p}$). Let us also prove the first implication. If $\mathrm{Tor}_1^R(M, -) = 0$ then the long exact sequence of Tor groups associated with a short exact sequence $0 \to L \to M \to N \to 0$ shows that $M \otimes -$ is exact whence M is flat. Conversely if M is flat then the exactness of the projective resolution of N: $P_\bullet \to N \to 0$ implies the exactness of $M \otimes P_\bullet \to M \otimes N \to 0$ whence all groups $\mathrm{Tor}_i^R(M, -)$ are zero for $i > 0$.

22 a) (5 points) First we assume only that R is an arbitrary commutative ring. The inclusions [free $\subset$ projective $\subset$ flat] are true. We also know (as proved above) that M being flat is equivalent to $\mathrm{Tor}_1^R(M,-) = 0$, which proves $iii) \Rightarrow iv)$. (To see that a projective module M is flat, let $F = M \oplus N$ with F free, since free modules are flat we get $0 = \mathrm{Tor}_1(F,-) = \mathrm{Tor}_1(M,-) \oplus \mathrm{Tor}_1(N,-)$, whence $\mathrm{Tor}_1(M,-) = 0$ or M is flat)

22 b) (5 points) Now assume $(R, \mathfrak{m})$ is local with $k = R/\mathfrak{m}$ and that K is an f.g. R-module. Under these conditions it is not difficult to prove the equivalence (see Problem 80a) HW15, Math600) $K = 0 \Leftrightarrow K \otimes_R k = 0$. Let $0 \to K \xrightarrow{i} R^n \to M \to 0$ be a presentation of the f.g. module M, with n the minimum number of generators of M and i being inclusion. Then, it is easy to see (using $R - \mathfrak{m} \subset R^\times$ and minimality of n) that $K \subset \mathfrak{m}^n$. Now we tensor this presentation with k to get the exact sequence:

$$\mathrm{Tor}_1^R(M,k) \to K \otimes k \xrightarrow{i \otimes 1} R^n/\mathfrak{m}^n \to M/\mathfrak{m}M \to 0$$

By our observation that $i(K) \subset \mathfrak{m}^n$, we get $\ker(i \otimes 1) = K \otimes k$, whence $K \otimes k$ is a factor group of $\mathrm{Tor}_1^R(M,k)$. Therefore, if $\mathrm{Tor}_1^R(M,k) = 0$, then $K \otimes k = 0$. Assuming now that R is noetherian also, we get that K is f.g. and hence $K \otimes k = 0$ implies $K = 0$ (i.e M is free) as observed above. Thus $iv) \Rightarrow i)$

22 c) (5 points) Let $R = \mathbb{Z}/6\mathbb{Z} = M \oplus \mathbb{Z}/3\mathbb{Z}$ where $M = \mathbb{Z}/2\mathbb{Z}$. Thus R is noetherian, M is projective and f.g. but cannot be free because free R-modules have number of elements divisible by 6.

Let $R = \mathbb{Z}$ (noetherian) and $M = \mathbb{Q}$ is a flat (being $S^{-1}\mathbb{Z}$ for $S = \mathbb{Z} - \{0\}$) but can be shown to be not projective.

Let $R = \mathbb{Z}$ (noetherian), $M = \mathbb{Z}/n\mathbb{Z}$, let (p) be a maximal ideal with p not dividing n. Then $\mathrm{Tor}_1^{\mathbb{Z}}(M, \mathbb{Z}/p\mathbb{Z}) = 0$ (by Problem 19) but M is not flat because clearly $\mathrm{Tor}_1^{\mathbb{Z}}(M,-) \neq 0$