Solutions to Homework 6
Math 601, Spring 2008

23) (10 points). Let R be a domain and M a flat R-module. Show that M is torsion-free. Is there a natural condition on R and M so that the converse holds?

Since R is an integral domain, for every $r \in R - \{0\}$, the sequence $0 \to R \xrightarrow{\times r} R$ is exact. Since M is flat the sequence $0 \to M \otimes R \xrightarrow{id \otimes r} M \otimes R$ is also exact. But this sequence is the same as $0 \to M \xrightarrow{\times r} M$, and hence M has no r-torsion for any r, or M is torsion free.

Conversely let R be a PID, and M a torsion free f.g. R-module, then by the structure theorem of f.g. modules over a PID, we know that M is free, hence flat.

24) (10 points). Let F, G be adjoint additive functors R-Mod $\to$ R-Mod. Suppose that F is exact. Show by example that G need not be exact (although it is automatically left-exact).

Let $R = \mathbb{Z}$ and F be the functor $- \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, we know F is exact. Let G be the functor $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, -)$. Since $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module, G is not right exact. We also know that F is left adjoint to G as required.

26) [D-F], 13.1 #5 (5 points). Suppose α is a rational root of a monic polynomial in $\mathbb{Z}[X]$. Prove that α is an integer.
Solution: If $p(x) = a_l X^l + a_{l-1} X^{l-1} + \cdots + a_0$ is a polynomial with coefficients in a UFD and $\alpha = m/n$ is a root of $p[X]$ (with m, n being relatively prime elements of the UFD) then substituting α for X and clearing denominators, we see that $m | a_0$ and $n | a_l$. In the present problem, we have $a_l = 1$, therefore α must be an integer (and a divisor of a_0).

25) [D-F], 13.1 #1 (10 points). Show that $p(X) = X^3 + 9X + 6$ is irreducible in $\mathbb{Q}[X]$. Let θ be a root of $p[X]$. Find the inverse of $1 + \theta$ in $\mathbb{Q}(\theta)$.
Solution: $p[X]$ being a cubic is reducible iff it has a root α. As noted in the solution to the previous problem the only candidates for α are divisors of 6, moreover for integers x, we have $p(x) \equiv x \mod (3)$. Thus $\alpha \in \{0, \pm 3, \pm 6\}$ which are easily seen to be non-roots of $p[X]$. Alternatively by Eisenstein's criterion we have $p = 3$ divides 9 and p^2 does not divide the constant term 6, hence $p[X]$ is irreducible over $\mathbb{Q}$. By divison we see $(1 + \theta)(\theta^2 - \theta + 10) = 4$ in $\mathbb{Q}(\theta)$, whence $\alpha^{-1} = (\theta^2 - \theta + 10)/4 \in \mathbb{Q}(\theta)$. (Note: The problem requires an answer of the form $a + b\theta + c\theta^2$ in which any element of $\mathbb{Q}(\theta)$ may be written. Otherwise $1/(1 + \theta) \in \mathbb{Q}(\theta)$ itself is an answer.)