Homework 1 - Math 603 – Fall 05
Solutions

1. Atiyah-Macdonald, Ex. 21, Chapter 1.

(i): For $\mathfrak{p} \in X, \mathfrak{q} \in Y$ such that $\mathfrak{q}^c = \mathfrak{p}$, we have $f \notin \mathfrak{p} \Leftrightarrow \phi(f) \notin \mathfrak{q}$. So $\mathfrak{p} \in X_f \Leftrightarrow \mathfrak{q} \in Y_{\phi(f)}$. Thus $(\phi^*)^{-1}X_f = Y_{\phi(f)}$.

(ii): Similarly,

$$\begin{aligned} \mathfrak{a} \subset \mathfrak{p} &\Leftrightarrow \phi(\mathfrak{a}) \subset \mathfrak{q} \\ &\Leftrightarrow \mathfrak{a}^e \subset \mathfrak{q}. \end{aligned}$$

So $\mathfrak{p} \in V(\mathfrak{a}) \Leftrightarrow \mathfrak{q} \in V(\mathfrak{a}^e)$. Thus $(\phi^*)^{-1}V(\mathfrak{a}) = V(\mathfrak{a}^e)$.

(iii): WLOG $\mathfrak{b} = \sqrt{\mathfrak{b}}$. Then $\overline{\phi^*(V(\mathfrak{b}))}$ is the intersection of all closed sets $V(\mathfrak{a})$ containing $\phi^*(V(\mathfrak{b}))$. But

$$\begin{aligned} \phi^*V(\mathfrak{b}) \subset V(\mathfrak{a}) &\Leftrightarrow V(\mathfrak{b}) \subset (\phi^*)^{-1}V(\mathfrak{a}) = V(\mathfrak{a}^e) \\ &\Leftrightarrow \sqrt{\mathfrak{a}^e} \subset \sqrt{\mathfrak{b}} = \mathfrak{b} \\ &\Rightarrow \mathfrak{a}^e \subset \mathfrak{b} \\ &\Rightarrow \mathfrak{a} \subset \mathfrak{a}^{ec} \subset \mathfrak{b}^c \\ &\Rightarrow V(\mathfrak{b}^c) \subset V(\mathfrak{a}). \end{aligned}$$

This shows that $V(\mathfrak{b}^c) \subset \overline{\phi^*V(\mathfrak{b})}$. On the other hand, it is clear that $\phi^*V(\mathfrak{b}) \subset V(\mathfrak{b}^c)$. So $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$.

(iv): (Using (vi) below.) Consider the diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\pi"'] & B \\ A/\mathrm{ker}\phi \arrow[ur, "\overline{\phi}"'] & \end{tikzcd}$$

The map $\overline{\phi}$ is a ring isomorphism, so by (i), it induces a homeomorphism on spectra. This together with $\phi^* = \pi^* \circ (\overline{\phi})^*$ reduces us to considering π in place of ϕ, where π is the canonical projection $\pi : A \to A/I$, where I is any ideal. Recall that prime ideals in A/I are precisely the images $\overline{\mathfrak{p}}$ of prime ideals $\mathfrak{p}$ containing I. From this it is clear that π^* is continuous and bijective onto $V(I) \subset \mathrm{Spec}(A)$. Since π^* clearly sends the closed set $V(\overline{\mathfrak{a}})$ to $V(\mathfrak{a})$ (where $\mathfrak{a} \supset I$), we see π^* is also closed, hence is a homeomorphism onto $V(I)$.

(v): (Using (iii) above.) Note that $\overline{\phi^*(Y)} = \overline{\phi^* V(0_B)} = V(0_B^c) = V(\ker(\phi))$. This is $X = V(0_A)$ iff $\ker(\phi) \subset \sqrt{0_A} = \mathfrak{N}$. Hence $\phi^* Y$ is dense in X iff $\ker(\phi) \subset \mathfrak{N}$.

If ϕ is injective, note that $\ker(\phi) \subset \mathfrak{N}$.

(vi): This is trivial.

(vii): Note that $\mathrm{Spec}(B)$ has two points $\{0\} \times K$ and $A/\mathfrak{p} \times \{0\}$. The map ϕ^* sends these to $\mathfrak{p}$ and (0) respectively. Hence ϕ^* is bijective.

On the other hand (0) is dense in $\mathrm{Spec}(A)$ (A is a domain), but $A/\mathfrak{p} \times \{0\}$ is not dense in $\mathrm{Spec}(B)$ (this is the disjoint union of two points, each both open and closed). Hence ϕ^* is not a homeomorphism.

2. Atiyah-Macdonald, Ex. 16, Chapter 3. The hints in Atiyah-Macdonald should be sufficient, so I am not going to write out a solution.

3. *Joining a point to an open subset with a curve.*

Step 1: Assume $B = k[X_1, \ldots, X_n]$, where $n \geq 1$. Then we can join any two distinct points x and x' with a curve $V(\mathfrak{p})$ (in fact, with a "straight line"). Indeed, by performing a translation we may assume x is the origin. By performing a linear change of variables, we may assume $x' = (1, 0, \ldots, 0)$. In terms of ideals, $\mathfrak{m}_x = (X_1, X_2, \ldots, X_n)$ and $\mathfrak{m}_{x'} = (X_1 - 1, X_2, \ldots, X_n)$. Then let $\mathfrak{p} = (X_2, \ldots, X_n)$. Then $V(\mathfrak{p})$ is a curve containing x and x'.

Step 2: Reduction step: Using Noether normalization, there is a finite integral extension $A = k[X_1, \ldots, X_n] \subset B$. By Step 1, we know we can find a curve joining any closed point in $\mathrm{Spec}(A)$ to any non-empty open subset of $\mathrm{Spec}(A)$.

In $\mathrm{Spec}(B)$, we have a closed point x and a non-empty open set U. WLOG $U = X(f)$ for some $f \in B - 0$. Let $\pi : X := \mathrm{Spec}(B) \to \mathrm{Spec}(A) =: Y$ be the map on spectra induced by the inclusion $A \subset B$. Let $y = \pi(x)$; in terms of ideals, x corresponds to a maximal ideal $\mathfrak{m} \subset B$, and y corresponds to the maximal ideal $\mathfrak{n} := \mathfrak{m}^c$ in $\mathrm{Spec}(A)$.

We need to produce a non-empty open subset in $\mathrm{Spec}(A)$ to which we can apply Step 1. The map π is actually open (see Atiyah-Macdonald, Chap. 7, Ex. 24), so we can simply take $\pi(X(f))$. We don't need to quote this exercise. Instead, we proceed as follows. The element $f \in B$ is integral over A, so we there is an equality

$$f^N + a_{N-1} f^{N-1} + \cdots + a_0 = 0,$$

for some $a_i \in A$. We choose such an equation where N is *minimal.* It's easy to see that

$$\pi(V(f)) \subset V(a_0),$$

since $f \in \mathfrak{q} \implies a_0 \in \mathfrak{q} \cap A = \mathfrak{q}^c = \pi(\mathfrak{q})$.

We want to apply Step 1 to the open subset $X(a_0)$ and the point y in $\mathrm{Spec}(A)$. First we need to verify that $X(a_0) \neq \emptyset$. Indeed, if $V(a_0) = \mathrm{Spec}(A)$, then a_0 is nilpotent, and hence a_0 is zero, and since $f \neq 0$ and B is a domain this leads to a contradiction of the minimality of N.

Now using Step 1 choose a curve $V(\mathfrak{p})$ in $\mathrm{Spec}(A)$ joining y to a point in $X(a_0)$. We have $\mathfrak{p} \subset \mathfrak{n}$, and $\mathfrak{m}$ lies over $\mathfrak{n}$. Since A is normal, the Going-Down theorem applies to the extension $A \subset B$, and so there exists a prime ideal $\mathfrak{q} \in \mathrm{Spec}(B)$ lying over $\mathfrak{p}$, which is contained in $\mathfrak{m}$.

We claim $V(\mathfrak{q})$ is a curve: indeed, $A/\mathfrak{p} \hookrightarrow B/\mathfrak{q}$ is an integral extension, so that $\dim(B/\mathfrak{q}) = \dim(A/\mathfrak{p}) = 1$. Also, we claim that $V(\mathfrak{q})$ meets $X(f)$: if not then $V(\mathfrak{q}) \subset V(f)$. By the Going-Up theorem, we have $\pi(V(\mathfrak{q}) = V(\mathfrak{p})$, so applying π we have

$$V(\mathfrak{p}) = \pi(V(\mathfrak{q})) \subset \pi(V(f)) \subset V(a_0).$$

But by construction the curve $V(\mathfrak{p})$ meets the complement $X(a_0)$ of $V(a_0)$, giving us a contradiction.

4. Exercise 4.3.3. from the notes. *Show that polynomials determine continuous maps $k^n \to k$, for the Zariski topologies.*

Let $\phi(X_1, \ldots, X_n) \in k[X_1, \ldots, X_n]$ determine the map $k^n \to k$ by evaluation. Then for any $f \in k[X]$, the composition $f \circ \phi \in k[X_1, \ldots, X_n]$. We have

$$\phi^{-1}(X_f) = X_{f \circ \phi}$$

where $X_?$ denotes the Zariski open subset of points at which the polynomial ? does not vanish. Since f was arbitrary, this shows $\phi : k^n \to k$ is continuous.

5. Exercise 6.3.3. from the notes.

Lemma 6.3.1: *Proof.* If $b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0$, then also $(\frac{b}{s})^n + \frac{a_{n-1}}{s}(\frac{b}{s})^{n-1} + \cdots + \frac{a_0}{s^n} = 0$. This shows that $S^{-1}C$ is integral over $S^{-1}A$. Conversely, if $\frac{b}{s} \in S^{-1}B$ is integral over $S^{-1}A$, then so is $\frac{b}{1}$, and multiplying an equation of form

$$\left(\frac{b}{1}\right)^n + \frac{a_{n-1}}{s_{n-1}}\left(\frac{b}{1}\right)^{n-1} + \cdots + \frac{a_0}{s_0} = 0$$

through by $(s_0 \cdots s_{n-1})^n$ shows that there is a $t \in S$ with tb integral over A, hence belongs to C. But then $\frac{b}{s} = \frac{bt}{st}$ belongs to $S^{-1}C$, and we are done.

Lemma 6.3.2: *Proof.* We have an inclusion $A/\mathfrak{p} \hookrightarrow B/\mathfrak{q}$, an integral extension of domains. By Lemma 3.3.5, one is a field iff the other is.

6. Exercise 7.6.2 from the notes.

For each $m \geq 2$, we want to find a polynomial $\phi_m \in k[X_1, \ldots, X_m]$ whose only zero is at $(0, \ldots, 0)$. It is enough to construct ϕ_2, for then we get ϕ_m recursively by setting

$$\phi_m(X_1, \ldots, X_m) = \phi_2(\phi_{m-1}(X_1, \ldots, X_{m-1})\,,\, X_m).$$

To construct ϕ_2, choose any element $\alpha \in \overline{k} - k$. Let $f(X) = X^r + a_{r-1}X^{r-1} + \cdots + a_0 \in k[X]$ be the minimal polynomial for α over k (so $r \geq 2$). Set

$$\phi_2(X, Y) = X^r + a_{r-1}X^{r-1}Y + \cdots + a_0 Y^r,$$

the homogeneous variant of f.

Suppose $x, y \in k$ and $\phi_2(x, y) = 0$. If $y = 0$, then also $x = 0$. If $y \neq 0$, then dividing by y^r shows that $f(x/y) = 0$, which shows that the irreducible non-linear polynomial f has a root in k, a contradiction. Thus $(0, 0)$ is the only zero of ϕ_2.