Homework 2 – due 12/08/03

Math 603

Do as many as you can!

1. (a) Suppose A is an integrally closed Noetherian domain with field of fractions K. Suppose L/K is a finite separable extension, and let B denote the integral closure of A in L. Prove that B is a finite A-module. Hint: Use Atiyah-Macdonald, Prop. 5.17.

(b) Let k be a field and suppose B is a domain which is also a finitely generated k-algebra. Let $L = \mathrm{Frac}(B)$ and let $\tilde{B}$ denote the integral closure of B in L. Assume that $\mathrm{char}(k) = 0$ (for simplicity). Prove that $\tilde{B}$ is a finite B-module and a finitely generated k-algebra. Hint: use the Noether Normalization lemma applied to B together with part (a). (Remark: $\mathrm{Spec}(\tilde{B})$ is called the *normalization* of the scheme $\mathrm{Spec}(B)$. This exercise implies, for example, that the normalization of an irreducible variety is still a variety, because it is still finite-type over the coefficient field.)

(c) Suppose A is a domain which is also a finitely generated algebra over a field k of characteristic zero. Let L be a finite extension of $K = \mathrm{Frac}(A)$, and let B be the integral closure of A in L. Prove that B is a finite A-module and a finitely-generated k-algebra. (Remark: This is essentially Theorem 3.9A in Hartshorne's book *Algebraic Geometry*, and is stated there –without proof – in the more general situation where k need not have characteristic zero.)

2. Let k be a ring, A and k' two k-algebras. Let $A' = A \otimes_k k'$. Show that $\Omega_{A'/k'} = \Omega_{A/k} \otimes_k k' = \Omega_{A/k} \otimes_A A'$. If $S \subset A$ is multiplicative, show that $\Omega_{A_S/k} = \Omega_{A/k} \otimes_A A_S$. Hint: Use the fundamental exact sequences for differentials.

3. Let $k = \overline{k}$ be an algebraically closed field. Suppose $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ are maximal ideals in the ring $A = k[X, Y]$. Is there necessarily a prime ideal of A which is contained in every $\mathfrak{m}_i$?

4. Atiyah-Macdonald, Chapter 11, # 2.

5. Atiyah-Macdonald, Chapter 11, # 3.

6. Atiyah-Macdonald, Chapter 11, # 4.

7. Atiyah-Macdonald, Chapter 10, # 11.

8. Exercise 22.2.4 from the notes.

9. Exercise 23.2.3 from the notes.