Homework 2 - Math 603 – Fall 05
Solutions

1. (a): In the notation of Atiyah-Macdonald, Prop. 5.17, we have $B \subset \sum_{j=1}^n Av_j$. Since A is Noetherian, this implies that B is f.g. as an A-module.

(b): By Noether normalization we find a finite integral extension $A \subset B$, where $A = k[X_1, \ldots, X_n]$. Set $K = \mathrm{Frac}(A)$. We claim that L/K is a finite separable extension. Indeed, $K \otimes_A B$ is a localization of the domain B, hence is a domain, and $L \supset K \otimes_A B$. Since moreover $K \otimes_A B$ is f.g. hence integral over K (since B is f.g. over A), we see by Lemma 3.3.5 that $K \otimes_A B$ is a field. Since L is the smallest field containing B, we have $L = K \otimes_A B$, hence L/K is a finite extension. It is obviously separable since $\mathrm{char}(K) = 0$ by hypothesis.

Note that $\widetilde{B}$ is the integral closure of A in L. Thus, we can apply part (a) to see that $\widetilde{B}$ is f.g. as an A-module. But then it is obviously f.g. as a B-module. It follows from this that $\widetilde{B}$ is f.g. as a k-algebra, since B is.

(c): Let $\widetilde{A}$ denote the integral closure of A in K; so $\widetilde{A}$ is a normal domain. Note that B is the integral closure of $\widetilde{A}$ in L So by (a), B is a f.g. $\widetilde{A}$-module. By (b), $\widetilde{A}$ is a f.g. A-module. It follows that B is a f.g. A-module. It is then automatically f.g. as a k-algebra, since A is.

2. First, we prove $\Omega_{A'/k'} = \Omega_{A/k} \otimes_A A'$ (the latter is clearly also $\Omega_{A/k} \otimes_k k'$). If $M' \in A'$-mod, then we can regard it as an A-module via the canonical map $A \to A'$. We claim that there is an isomorphism

$$\mathrm{Der}_k(A, M') = \mathrm{Der}_{k'}(A', M').$$

Indeed, this is given by the map $D \mapsto D'$, where by definition $D'(a \otimes \alpha') := \alpha' D(a)$. This shows that there are canonical isomorphisms, functorial in M',

$$\begin{aligned}
\mathrm{Hom}_{A'}(\Omega_{A/k} \otimes_A A', M') &= \mathrm{Hom}_A(\Omega_{A/k}, M') \\
&= \mathrm{Der}_k(A, M') \\
&= \mathrm{Der}_{k'}(A', M') \\
&= \mathrm{Hom}_{A'}(\Omega_{A'/k'}, M').
\end{aligned}$$

It follows (using e.g. Yoneda's lemma), that there is a natural isomorphism of the representing objects, $\Omega_{A/k} \otimes_A A' = \Omega_{A'/k'}$.

Next, we prove $\Omega_{A_S/k} = \Omega_{A/k} \otimes_A A_S$. We proved in class that A_S/A is 0-étale, and that hence $\Omega_{A_S/A} = 0$ (since in particular A_S/A is 0-unramified). Thus the first fundamental exact sequence for $k \to A \to A_S$ is split exact and has $\Omega_{A_S/A} = 0$, and this yields

$$\Omega_{A/k} \otimes_A A_S = \Omega_{A_S/k},$$

as desired.

3. (NOTE: the problem is supposed to ask for a NON-ZERO prime ideal.) In a suitable coordinate system, we can suppose the $\mathfrak{m}_i$'s correspond to points in the plane $(x_i, y_i) \in k^2$, with all the x_i's distinct. Then by Lagrange interpolation (hint given in class), there is a polynomial $f \in k[X]$ such that $f(x_i) = y_i$ for all i. Now consider $g(X, Y) = Y - f(X) \in k[X, Y]$. Note that g is non-zero, and is

irreducible (if g factors, then considerations of Y-degrees show it must factor as $g = (a(X)Y + b(X)) \cdot c(X)$, and then we would have $a(X)c(X) = 1$, i.e. $c(X)$ is a unit). Thus $\mathfrak{p} := (g)$ is a non-zero prime ideal contained in each $\mathfrak{m}_i$. Indeed, $g(x_i, y_i) = 0$ for all i by construction.

4. Atiyah-Macdonald, Chapter 11, #2. Suppose $\dim(A) = d$, and let $x_1, \dots, x_d$ denote the system of parameters we're given. By definition, $(x_1, \dots, x_d) = I$, for some $\mathfrak{m}$-primary ideal I.

We are assuming A is complete. What does this mean, i.e. for which topology, I-adic or $\mathfrak{m}$-adic? Answer: there is actually no difference between these two topologies, since there exists $n > 0$ with

$$\mathfrak{m}^n \subset I \subset \mathfrak{m},$$

(and hence $\mathfrak{m}^{rn} \subset I^r \subset \mathfrak{m}^r$, for all $r \geq 0$). See [AM], Cor. 7.16. However, it is convenient here to think of A as being identical to $\widehat{A} = \varprojlim A/I^n$.

Now by [AM] (11.21), the x_i are algebraically independent, and hence $t_i \mapsto x_i$ gives an *injective* map $k[t_1, \dots, t_d] \to A$. Since A is complete for the I-adic topology, this extends to give

$$\psi : k[[t_1, \dots, t_d]] \to A,$$

which remains injective (by e.g. [AM] Prop. 10.2).

We want to use [AM] Prop. 10.24 to show that ψ makes A a f.g. module over $k[[t_1, \dots, t_d]]$. Note that in 10.24, the base ring A is $k[[t_1, \dots, t_d]]$ and the module M is the ring A. Note also that the only hypothesis that is not obvious is "$G(A)$ is f.g. over $G(k[[t_1, \dots, t_d]])$". Now, the latter graded ring is just the polynomial algebra $k[t_1, \dots, t_d]$ itself. Also, $G(A) = \oplus_{n=0}^{\infty} I^n/I^{n+1}$ is clearly f.g. over the graded ring $A/I[x_1, \dots, x_d]$, so it is enough to show that A/I is f.g. as a k-module. But since $\mathfrak{m}^n \subset I \subset \mathfrak{m}$, it is enough to show that $A/\mathfrak{m}^n$ is f.g. as a k-module. But this is clear since $A/\mathfrak{m}^n$ is *finite-dimensional as a k-vector space*: $A/\mathfrak{m}^n$ is filtered by finitely many subquotients $\mathfrak{m}^k/\mathfrak{m}^{k+1}$, and each of these has finite k-dimension, since $\mathfrak{m}^k$ is f.g as A-module.

5. Atiyah-Macdonald, Chapter 11, #3. We are being asked to prove the equality $\dim(A_\mathfrak{m}) = \text{tr.deg}_k A$, where $\mathfrak{m}$ is a maximal ideal of A, a f.g domain over a field k (not necessarily algebraically closed). Note that we already did this in the notes: it's Theorem 7.3.1! There, we needed no hypothesis on k whatsoever.

6. Atiyah-Macdonald, Chapter 11, #4. Notation: let $A = k[X_1, X_2, \dots,]$ and $A_k = k[X_1, \dots, X_{m_k}]$, for $k \geq 1$. It is clear that each $\mathfrak{p}_i$ is prime and that the set $S = A - (\cup_i \mathfrak{p}_i)$ is a multiplicative set. Note that prime ideals of $S^{-1}A$ are precisely those of the form $S^{-1}\mathfrak{p}$, where $\mathfrak{p} \subset A$ is a prime ideal contained in $\cup_i \mathfrak{p}_i$.

We need to verify the various other claims, made in the hint by Atiyah-Macdonald.

The two hypotheses of Atiyah-Macdonald, Chap. 7, Ex. 9 are satisfied here. The key point is the following lemma.

Lemma 0.0.1. *If $S^{-1}\mathfrak{p}$ is a non-zero prime ideal in $S^{-1}A$ and $\mathfrak{p} \cap A_k \neq 0$, then $\mathfrak{p} \subset \mathfrak{p}_j$, for some $j \leq k$.*

Proof. Choose $0 \neq x \in \mathfrak{p} \cap A_k$. Note that $\mathfrak{p} \cap A_k$ is contained the finite union of prime ideals $\cup_{i \leq k} \mathfrak{p}_i \cap A_k$ (since if $i > k$, then $\mathfrak{p}_i \cap A_k = 0$). Thus by Atiyah-Macdonald, Prop. 1.11, $\mathfrak{p} \cap A_k \subset \mathfrak{p}_j \cap A_k$, for some $j \leq k$.

Now for every $k' \geq k$, we have $0 \neq x \in \mathfrak{p} \cap A_{k'}$, which by the same argument is contained in some $\mathfrak{p}_{j'} \cap A_{k'}$. In fact $j' \leq k$, since $0 \neq x \in \mathfrak{p}_{j'} \cap A_k$. Thus, letting k' go to infinity, we see that $\mathfrak{p} \subset \cup_{j \leq k} \mathfrak{p}_j$. Now again by Atiyah-Macdonald, Prop. 1.11, $\mathfrak{p} \subset \mathfrak{p}_j$, for some $j \leq k$. □

It follows that if $S^{-1}\mathfrak{p}$ is maximal, then $\mathfrak{p} = \mathfrak{p}_j$ for some j. Further, each $S^{-1}\mathfrak{p}_j$ is maximal. Thus,

Corollary 0.0.2. *The maximal ideals of $S^{-1}A$ are precisely the $S^{-1}\mathfrak{p}_j$.*

Now we can verify the hypotheses of Atiyah-Macdonald, Ex. 9 of Chap. 7. The ring $S^{-1}A_{S^{-1}\mathfrak{p}_i}$ is a localization of the ring

$$k(X_j)_{j \notin [m_i+1,\ldots,m_{i+1}]}[X_{m_i+1}, \ldots, X_{m_{i+1}}],$$

hence is Noetherian. (In fact, one can prove that the above ring is isomorphic to $S^{-1}A_{S^{-1}\mathfrak{p}_i}$, which is also $A_{\mathfrak{p}_i}$. This also shows, shortening the argument given below, that $\mathrm{ht}(S^{-1}\mathfrak{p}_i) = m_{i+1} - m_i$.)

Secondly, suppose $0 \neq x/s$, and let $S^{-1}\mathfrak{p}_i$ be a maximal ideal containing x/s. This is the same as saying $0 \neq x \in \mathfrak{p}_i$. But x involves only finitely many variables, and remains the same if all sufficiently high variables are specialized to zero. Hence a non-zero x can only be contained in finitely many ideals $\mathfrak{p}_i$ Thus, x/s is contained in only finitely many maximal ideals of $S^{-1}A$.

We conclude that $S^{-1}A$ is Noetherian, by the quoted exercise in Atiyah-Macdonald.

Next, verify that $\mathrm{ht}(S^{-1}\mathfrak{p}_i) = m_{i+1} - m_i$. This will show that $\dim(S^{-1}A) = \infty$.

There is something to show here, because A is not a polynomial algebra in finitely many variables. It is clear that $\mathrm{ht}(S^{-1}\mathfrak{p}_i) = \mathrm{ht}(\mathfrak{p}_i)$, since every prime ideal contained in $\mathfrak{p}_i$ is contained in $\cup_i \mathfrak{p}_i$. It is also clear that $\mathrm{ht}(\mathfrak{p}_i) \geq m_{i+1} - m_i$ and further that $\mathrm{ht}(\mathfrak{p}_i \cap A_k) = m_{i+1} - m_i$ for all large k (since A_k is just a polynomial algebra in finitely many variables). Suppose $\mathfrak{q}_0 \subsetneq \cdots \subsetneq \mathfrak{q}_d = \mathfrak{p}_i$ is a chain of prime ideals in A. Then intersecting it with A_k for large k (and noting that the inclusions remain strict) shows that the chain has length $\leq \mathrm{ht}(\mathfrak{p}_i \cap A_k)$, which is $m_{i+1} - m_i$. We are now done.

7. Atiyah-Macdonald, Chapter 10, #11. Show that there is a non-Noetherian local ring $(A, \mathfrak{m})$, and an ideal $\mathfrak{a} \subset A$ such that the $\mathfrak{a}$-adic completion $\widehat{A}$ is Noetherian. Show we can even arrange things such that $\widehat{A}$ is f.g. as an A-module.

Solution: Let A be the ring of germs of C^∞ functions at $x = 0$, and let $\mathfrak{a}$ denote the ideal generated by the germ of the function x. Let $\widehat{A}$ denote the $\mathfrak{a}$-adic completion of A.

Claim 1: $\widehat{A}$ can be identified with the ring of formal power series $\mathbb{R}[[x]]$, hence is Noetherian.

Consider the map $\phi : A \to \mathbb{R}[[x]]$ given by sending a germ f to its Taylor expansion $\widehat{f} := \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$. Leibniz' rule

$$(fg)^{(n)}(0) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(0)\, g^{(n-k)}(0) \tag{1}$$

easily implies that $\widehat{fg} = \widehat{f}\,\widehat{g}$, and so the map $f \mapsto \widehat{f}$ is an $\mathbb{R}$-algebra homomorphism. The theorem of Borel quoted in the hint given by Atiyah-Macdonald shows that $\phi : f \mapsto \widehat{f}$ is *surjective*.

Let $I = \ker(\phi)$. Clearly I consists of the germs f such that $f^{(n)}(0) = 0$ for all $n \geq 0$. We claim that

$$I = \cap_{k=0}^{\infty} x^k A. \tag{2}$$

Indeed, by (1) it is clear that $f \in x^k A$ implies that $f^{(n)}(0) = 0$ for all $n < k$. Hence $I \supset \cap_k x^k A$. Conversely, if $f \in I$, it is not too hard using Taylor's remainder theorem to show that the germ f_k defined by

$$f_k(x) = \begin{cases} f(x)x^{-k}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

is C^∞. But then $f = x^k f_k$. Since this holds for all k, we have $I \subset \cap_k x^k A$. Note that I is therefore also the kernel of the natural map $A \to \widehat{A}$.

Next we want to show that the homomorphism $\phi : A \twoheadrightarrow \mathbb{R}[[x]]$ factors through the canonical map $A \to \widehat{A}$, giving us a surjective map $\overline{\phi} : \widehat{A} \twoheadrightarrow \mathbb{R}[[x]]$. It is enough to show that $f \mapsto \widehat{f}$ induces a compatible family of homomorphisms

$$\phi_k : A/(x^k) \to \mathbb{R}[[x]]/(x^k). \tag{3}$$

We need to show that $\widehat{x^k f} \in x^k \mathbb{R}[[x]]$. This follows easily from (1).

We claim that every ϕ_k is an isomorphism. Borel's theorem implies that every ϕ_k is surjective. To prove injectivity, suppose $f \in A$ maps into $x^k\mathbb{R}[[x]]$. Then by Borel's theorem there is a $g \in A$ such that $\widehat{f} = x^k \widehat{g}$. But then since $\widehat{x^k g} = \widehat{x^k}\ \widehat{g} = x^k\ \widehat{g}$, we have $f - x^k g \in I \subset x^k A$ (as explained in (2)). Thus $f \in x^k A$, proving the injectivity of every ϕ_k.

Now taking inverse limits, we see that $\varprojlim \phi_k$ induces the map $\overline{\phi} : \widehat{A} \to \mathbb{R}[[x]]$, which is therefore an isomorphism. Thus $\widehat{A}$ is Noetherian.

Claim 2: The ring $\widehat{A}$ is a quotient of A, so is f.g. as an A-module.
We have established the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d] & \mathbb{R}[[x]] \\ \widehat{A}. \arrow[ur, "\cong"'] & \end{tikzcd}$$

Since ϕ is surjective (Borel's theorem), we see that $A \to \widehat{A}$ is surjective too.

Claim 3: The ring A is not Noetherian.
Note that A is a domain. If it were Notherian, then since $\mathfrak{a} = (x) \neq A$, Krull's theorem would imply that $I = \cap_k x^k A = (0)$. But it is well-known that $I \neq 0$: there exist non-zero germs f such that $f^{(n)}(0) = 0$ for all $n \geq 0$. The simplest example is probably the germ

$$f(x) = \begin{cases} e^{-1/x^2}, & \text{if } x \neq 0 \\ 0 \text{ if } x = 0. \end{cases}$$

8. Exercise 22.2.4 from the notes. From the first example in the notes right before Exercise 22.2.3 it is clear that $\mathrm{Sing}(V(f)) = V(f, \frac{\partial f}{\partial X_1}, \ldots, \frac{\partial f}{\partial X_n})$. Thus, by definition (of the LHS), we have

$$\dim(\mathrm{Sing}(V(f)) = \dim(k[X_1, \ldots, X_n]/(f, \frac{\partial f}{\partial X_i})_{1 \leq i \leq n}).$$

It remains to prove that the RHS is zero iff the ring $A = k[X]/I$ (where $I = (f, \frac{\partial f}{\partial X_i})$) is finite-dimensional as a k-vector space. But this follows immediately from the notes Exercise 4.2.2. Indeed, if $\dim_k A < \infty$, then A is clearly Artin, and so $\dim(A) = 0$. Conversely, if $\dim(A) = 0$, then A is Artin and hence is a finite product of local Artin rings A_i ([AM], Thm. 8.7). Each A_i is f.g. as a k-algebra (since A is), and then by Exercise 4.2.2, each A_i is finite-dimensional. Thus A is finite-dimensional too.

9. Exercise 23.2.3 from the notes. Suppose $P = (x_1, x_2, \ldots, x_n, z)$ is a point on the variety $V(Z^2 - f)$, where $f = (X_1 - a_1)(X_2 - a_2)$. It is easy to see that the Jacobian $J(P)$ is the row vector

$$J(P) = (-x_2 + a_2, -x_1 + a_1, 0, \ldots, 0, 2z).$$

Clearly this vector is zero iff $x_1 = a_1$, $x_2 = a_2$, and $z = 0$ (recall we are assuming $\mathrm{char}(k) \neq 2$ here). Thus, $\mathrm{Sing}(V(Z^2 - f))$ is precisely the set of points of the form $(a_1, a_2, x_3, \ldots, x_n, 0)$, where $x_i \in k$ range freely. Thus the singular locus is identified with affine space $\mathbb{A}^{n-2}$, hence $\dim(\mathrm{Sing}) = n - 2$.

On the other hand, since $V(Z^2 - f)$ is a hypersurface in $n+1$-space, its dimension is $n+1-1 = n$. Thus, $\mathrm{codim}(Y, \mathrm{Sing}(Y)) = 2$. (NOTE: for a general normal variety, we have $\mathrm{codim}(Y, \mathrm{Sing}(Y)) \geq 2$; this example shows that that inequality cannot be improved.)