Homework 1 – due 10/24/03

Math 603

Do at least 3 problems to earn a grade of $\surd$. Do more to earn a $\surd+$.

1. Atiyah-Macdonald, Exercise 2, Chapter 1.

2. Atiyah-Macdonald, Exercise 21, Chapter 1.

3. Atiyah-Macdonald, Exercise 16, Chapter 3.

4. (a) Suppose A is an integrally closed Noetherian domain with field of fractions K. Suppose L/K is a finite separable extension, and let B denote the integral closure of A in L. Prove that B is a finite A-module. Hint: Use Atiyah-Macdonald, Prop. 5.17.

(b) Let k be a field and suppose B is a domain which is also a finitely generated k-algebra. Let $L = \mathrm{Frac}(B)$ and let $\tilde{B}$ denote the integral closure of B in L. Assume that $\mathrm{char}(k) = 0$ (for simplicity). Prove that $\tilde{B}$ is a finite B-module and a finitely generated k-algebra. Hint: use the Noether Normalization lemma applied to B together with part (a). (Remark: $\mathrm{Spec}(\tilde{B})$ is called the *normalization* of the scheme $\mathrm{Spec}(B)$. This exercise implies, for example, that the normalization of an irreducible variety is still a variety, because it is still finite-type over the coefficient field.)

(c) Suppose A is a domain which is also a finitely generated algebra over a field k of characteristic zero. Let L be a finite extension of $K = \mathrm{Frac}(A)$, and let B be the integral closure of A in L. Prove that B is a finite A-module and a finitely-generated k-algebra. (Remark: This is essentially Theorem 3.9A in Hartshorne's book *Algebraic Geometry*, and is stated there –without proof – in the more general situation where k need not have characteristic zero.)

5. Let k be any algebraically closed field, and let B be a domain which is a finitely generated k-algebra. Let x be a closed point of $\mathrm{Spec}(B)$, and let $U \subset \mathrm{Spec}(B)$ be a non-empty open set. Show that there is a curve in $\mathrm{Spec}(B)$ joining x to some point in U. (Note: for the purposes of this exercise, a curve in $\mathrm{Spec}(B)$ is a closed set of form $V(\mathfrak{p})$, $\mathfrak{p}$ a prime ideal, where $\dim(B/\mathfrak{p}) = 1$). Hint: do it first in the case $B = k[X_1, \ldots, X_n]$, then reduce to this case by using the Noether Normalization and the Going-Down theorem.