ON PARAHORIC SUBGROUPS

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Abstract We give the proofs of some simple facts on parahoric subgroups and on Iwahori Weyl groups used in [H], [PR] and in [R].

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Let G be a connected reductive group over a strictly henselian discretely valued field L. Kottwitz defines in [Ko] a functorial surjective homomorphism

$$\kappa_G : G(L) \longrightarrow X^*(\hat{Z}(G)^I). \tag{1}$$

Here $I = Gal(\bar{L}/L)$ denotes the absolute Galois group of L.

Let $\mathcal{B}$ be the Bruhat-Tits building of the adjoint group of G. Then $G(L)$ operates on $\mathcal{B}$.

Definition 1. A *parahoric subgroup* of $G(L)$ is a subgroup of the form

$$K_F = \mathrm{Fix}\,(F) \cap \mathrm{Ker}\ \kappa_G,$$

for a facet F of $\mathcal{B}$. An *Iwahori subgroup* of $G(L)$ is the parahoric subgroup associated to an alcove of $\mathcal{B}$.

Remark 2. If $F' = gF$, then

$$K_{F'} = gK_F g^{-1}.$$

In particular, since $G(L)$ acts transitively on the set of all alcoves, all Iwahori subgroups are conjugate.

We shall see presently that this definition coincides with the one of Bruhat and Tits [BTII], 5.2.6. They associate to a facet F in $\mathcal{B}$ a smooth group scheme $\mathcal{G}_F$ over Spec O_L with generic fiber G and the open subgroup $\mathcal{G}_F^\circ$ of it with the same generic fiber and with connected special fiber, and define the parahoric subgroup attached to F as $P_F^\circ = \mathcal{G}_F^\circ(O_L)$; from their definition it follows that $P_F^\circ \subseteq \mathrm{Fix}(F)$. We denote by $G(L)_1$ the kernel of κ_G.

For a facet F, let as above P_F° be the associated parahoric subgroup in the BT-sense. Our first goal is to prove the following proposition.

Proposition 3. *For any facet F in $\mathcal{B}$, we have $P_F^\circ = K_F$.*

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Proof. a) If $G = T$ is a torus, then $T(L)_1$ is the unique Iwahori subgroup, cf. Notes at the end of [R], n°1. Hence the result follows in this case.

b) If G is semisimple and simply connected, then $G(L)_1 = G(L)$. The assertion therefore follows from [BTII], 4.6.32, which also proves that $\mathcal{G}_F = \mathcal{G}_F^\circ$, where $\mathcal{G}_F$ is the group scheme defined in loc. cit. 4.6.26.

c) Let G be such that G_{der} is simply connected. Let $D = G/G_{\mathrm{der}}$. We claim that there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{G}_{\mathrm{der},F}(O_L) & \longrightarrow & \mathcal{G}_F^\circ(O_L) & \longrightarrow & \mathcal{D}^\circ(O_L) & \longrightarrow & 1 \\ & & \| & & \downarrow & & \| & & \\ 1 & \longrightarrow & K_{\mathrm{der}} & \longrightarrow & K_F & \longrightarrow & D(L)_1 & & . \end{array}$$

The bottom row involves parahoric subgroups associated in our sense to the facet F, and the top row involves those defined in [BTII] and comes by restricting the exact sequence

$$1 \longrightarrow G_{\mathrm{der}}(L) \longrightarrow G(L) \longrightarrow D(L) \longrightarrow 1$$

to O_L-points of the appropriate Bruhat-Tits group schemes (and more precisely in the case of $\mathcal{D}^\circ(O_L)$, the group $\mathcal{D}$ is the lft Neron model of D_L; cf. the notes at the end of [R]). The vertical equalities result from a) and b) above. The vertical arrow is an inclusion (but we need to justify its existence, see below) making the entire diagram commutative.

Let us first *construct* the top row. The key point is to show that the map $G(L) \longrightarrow D(L)$ restricts canonically to a surjective map $\mathcal{G}_F^\circ(O_L) \longrightarrow \mathcal{D}^\circ(O_L)$. We shall derive this from the corresponding statement involving the lft Neron models of D and of a maximal torus $T \subset G$. Let S denote a fixed maximal L-split torus in G, and define $T = \mathrm{Cent}_G(S)$, a maximal torus in G since by Steinberg's theorem G is quasisplit. Also define $T_{\mathrm{der}} := G_{\mathrm{der}} \cap T = \mathrm{Cent}_{G_{\mathrm{der}}}(S_{\mathrm{der}})$, where $S_{\mathrm{der}} = (S \cap G_{\mathrm{der}})^\circ$, a maximal L-split torus in G_{der}.

Consider the lft Neron models $\mathcal{T}$, $\mathcal{T}_{\mathrm{der}}$, and $\mathcal{D}$ associated to T, T_{der}, and D, cf. [BLR]. The map $\mathcal{T}(O_L) \longrightarrow \mathcal{D}(O_L)$ is surjective, and this implies that $\mathcal{T}^\circ(O_L) \longrightarrow \mathcal{D}^\circ(O_L)$ is also surjective, cf. [BLR], 9.6, Lemma 2. By [BTII], 4.6.3 and 4.6.7 we have decompositions $\mathcal{G}_F^\circ(O_L) = \mathcal{T}^\circ(O_L)\mathfrak{U}_F(O_L)$ and $\mathcal{G}_{\mathrm{der},F}(O_L) = \mathcal{T}_{\mathrm{der}}(O_L)\mathfrak{U}_F(O_L)$, where $\mathfrak{U}_F$ denotes the group generated by certain root-group O_L-schemes $\mathfrak{U}_{F,a}$, which depend on F; these all fix F. These remarks show that $G(L) \longrightarrow D(L)$ restricts to a map $\mathcal{G}_F^\circ(O_L) \longrightarrow \mathcal{D}^\circ(O_L)$, and also that the latter map is *surjective*.

The kernel of the latter map is contained in the subgroup of $G_{\mathrm{der}}(L)$ which fixes F, hence is precisely $K_{\mathrm{der}} = \mathcal{G}_{\mathrm{der},F}(O_L)$. This completes our discussion of the first row of the diagram above.

Since G_{der} is simply connected, $G(L)_1$ is the inverse image of $D(L)_1$ under the natural projection $G \longrightarrow D$; hence $\mathcal{G}_F^\circ(O_L)$ belongs to $G(L)_1$ hence is contained in K_F; this yields the inclusion which fits in the above diagram and makes it commutative. A diagram chase then shows that the inclusion $\mathcal{G}_F^\circ(O_L) \longrightarrow K_F$ is a bijection.

d) To treat the general case choose a z-extension $G' \longrightarrow G$, with kernel Z where the derived group of G' is simply connected. Since $X_*(Z)_I = X^*(\hat{Z}^I)$ is torsion-free, the induced sequence

$$0 \longrightarrow X^*(\hat{Z}^I) \longrightarrow X^*(\hat{Z}(G')^I) \longrightarrow X^*(\hat{Z}(G)^I) \longrightarrow 0 \tag{2}$$

is exact. We therefore obtain an exact sequence

$$1 \longrightarrow Z(L)_1 \longrightarrow G'(L)_1 \longrightarrow G(L)_1 \longrightarrow 1.$$

It follows that we also have an exact sequence

$$1 \longrightarrow Z(L)_1 \longrightarrow K'_F \longrightarrow K_F \longrightarrow 1.$$

As in c) one shows that $G'(L) \to G(L)$ maps $\mathcal{G}'^\circ_F(O_L) = P'^\circ_F$ *onto* $\mathcal{G}^\circ_F(O_L) = P^\circ_F$; in particular, since $P'^\circ_F = K'_F$ by c), we deduce $P^\circ_F \subset G(L)_1$ and thus $P^\circ_F \subseteq K_F$. The equality $P^\circ_F = K_F$ then follows from $P'^\circ_F = K'_F$. $\square$

Remark 4. Proposition 3 makes sense and still holds true if F is replaced with any bounded non-empty subset $\Omega \subset \mathcal{B}$ which is contained in an apartment. Indeed, one can follow the same proof, making only the following adjustment in the proof of b) where $G = G_{\mathrm{sc}}$: although [BTII] 4.6.32 is restricted to Ω contained in a facet, the equality $\mathcal{G}^\circ_\Omega(O_L) = \mathcal{G}_\Omega(O_L)$ (= fixer of Ω) which we need requires only the connectedness of the group scheme $\mathcal{T}$ occurring in loc.cit., which holds here by loc. cit. 4.6.1.

Let F be a facet contained in the apartment associated to the maximal split torus S. Let T be the centralizer of S, a torus since by Steinberg's theorem G is quasisplit. Let N be the normalizer of S. Let K_F be the parahoric subgroup associated to F. Let

$$\kappa_T : T(L) \longrightarrow X^*(\hat{T}^I) = X_*(T)_I$$

be the Kottwitz homomorphism associated to T, and $T(L)_1$ its kernel.

Lemma 5.

$$T(L) \cap K_F = T(L)_1.$$

Proof. By functoriality of the Kottwitz homomorphisms, we see $T(L)_1 \subset G(L)_1$. The elements of Ker κ_T act trivially on the apartment associated to S, hence the inclusion "$\supset$" is obvious. The converse follows from the fact that $T(L)_1$ equals $\mathcal{T}^\circ(O_L)$ where $\mathcal{T}^\circ$ is the identity component of the lft Neron model of T (cf. Notes at the end of [R], n$^{\text{o}}$1.) and the fact that $\mathcal{T}^\circ(O_L)$ is the centralizer of S in K_F, comp. [BTII], 4.6.4 or [L], 6.3. $\square$

Let K_0 be the Iwahori subgroup associated to an alcove contained in the apartment associated to S.

Lemma 6.

$$N(L) \cap K_0 = T(L)_1.$$

Proof. An element of the LHS acts trivially on the apartment associated to S, hence is contained in $T(L) \cap K_0 = T(L)_1$ by the previous lemma. □

Definition 7. Let $S \subset T \subset N$ be as before (maximal split torus, contained in a maximal torus, contained in its normalizer). The *Iwahori-Weyl group* associated to S is

$$\widetilde{W} = N(L)/T(L)_1.$$

Let $W_0 = N(L)/T(L)$ (relative vector Weyl group). We obtain an obvious exact sequence

$$0 \longrightarrow X_*(T)_I \longrightarrow \widetilde{W} \longrightarrow W_0 \longrightarrow 0. \tag{3}$$

Proposition 8. *Let K_0 be the Iwahori subgroup associated to an alcove contained in the apartment associated to the maximal split torus S. Then*

$$G(L) = K_0.N(L).K_0$$

and the map $K_0 n K_0 \mapsto n \in \widetilde{W}$ induces a bijection

$$K_0 \backslash G(L)/K_0 \simeq \widetilde{W}.$$

More generally, let K resp. K' be parahoric subgroups associated to facets F resp. F' contained in the apartment associated to S. Let

$$\begin{array}{rcl} \widetilde{W}^K & = & (N(L) \cap K)/T(L)_1 \ , \\ \widetilde{W}^{K'} & = & (N(L) \cap K')/T(L)_1 \ . \end{array}$$

Then

$$K \backslash G(L)/K' \simeq \widetilde{W}^K \backslash \widetilde{W} / \widetilde{W}^{K'} \ .$$

Proof. To F there are associated the subgroups

$$U_F \subset P_F \subset G(L),$$

cf. [BTI], 7.1 (where P_F is denoted $\hat{P}_F$) or [L], 8.8. By [BTI], 7.1.8 or [L], 8.10 we have

$$P_F \subset U_F.N(L).$$

Since U_F is contained in the parahoric subgroup $P_F^\circ \subset P_F$, the equality $G(L) = P_F.N(L).P_{F'}$ ([BTI], 7.4.15 resp. [L], 8.17) implies the first assertion

$$G(L) = K_F.N(L).K_{F'}.$$

To see the second assertion, we follow closely the proof of [BTI], 7.3.4. Assume that $n, n' \in N(L)$ with $n' \in P_F^\circ \, n \, P_{F'}^\circ$, i.e.

$$n'.n^{-1} \in P_F^\circ \cdot P_{nF'}^\circ.$$

We choose points $f \in F$ and $f' \in n \cdot F'$ with

$$P_F^\circ = P_f^\circ, \quad P_{n.F'}^\circ = P_{f'}^\circ.$$

We choose an order on the root system of S, with associated vector chamber D such that

$$f \in f' + D.$$

Then $U_f^- \cdot U_{f'}^- \subset U_{f'}^-$, in the notation of [L], 8.8. Hence, by [BTII], 4.6.7 (comp. [L], 8.10),

$$\begin{aligned} P_f^\circ . P_{f'}^\circ &= \Big[(N(L) \cap P_f^\circ) \cdot U_f^+ . U_f^-\Big] \cdot \Big[U_{f'}^- U_{f'}^+ \cdot (N(L) \cap P_{f'}^\circ)\Big] \\ &= (N(L) \cap P_f^\circ) \cdot U_f^+ . P_{f'}^\circ \\ &= (N(L) \cap P_f^\circ) \cdot U_f^+ . U_{f'}^+ . U_{f'}^- (N(L) \cap P_{f'}^\circ). \end{aligned}$$

It follows that there exist $m_1 \in N(L) \cap P_f^\circ$ and $m_2 \in N(L) \cap P_{f'}^\circ$ such that

$$m_1 \cdot n' n^{-1} m_2 \in U^+(L).U^-(L).$$

¿From the usual Bruhat decomposition it follows that

$$\begin{aligned} m_1 \cdot n' n^{-1} m_2 &= 1, \qquad \text{i.e.,} \\ m_1 n' (n^{-1} m_2 n) &= n\ . \end{aligned}$$

Since $m_1 \in N(L) \cap K_F$ and $n^{-1} m_2 n \in N(L) \cap K_{F'}$, the last equality means that

$$n \equiv n' \quad \text{in} \quad \widetilde{W}^K \backslash \widetilde{W} / \widetilde{W}^{K'}.$$

$\square$

Remark 9 (Descent)**.** Let σ denote an automorphism of L having fixed field $L^\natural$ such that L is the strict henselization of $L^\natural$. Let us assume G is defined over $L^\natural$; we may assume S, and hence T and N, are likewise defined over $L^\natural$ ([BTII], 5.1.12). Assume F and F' are σ-invariant facets in $\mathcal{B}$. Write $K(L^\natural) = K^\sigma$ and $K'(L^\natural) = K'^\sigma$. We have a canonical bijection

$$K(L^\natural) \backslash G(L^\natural) / K'(L^\natural) \xrightarrow{\sim} [K \backslash G(L) / K']^\sigma. \tag{4}$$

To prove that the map is surjective, we use the vanishing of $H^1(\langle\sigma\rangle, K)$ and $H^1(\langle\sigma\rangle, K')$. To prove it is injective, we use the vanishing of $H^1(\langle\sigma\rangle, K \cap gK'g^{-1})$ for all $g \in G(L^\natural)$. The vanishing statements hold because K, K', and $K \cap gK'g^{-1}$ are the O_L-points of group schemes over $O_{L^\natural}$ with connected fibers, by virtue of Proposition 3 and Remark 4.

Next, note that, using a similar cohomology vanishing argument,

$$\widetilde{W}^\sigma = N(L^\natural)/T(L^\natural) \cap T(L)_1 =: \widetilde{W}(L^\natural)$$
$$(\widetilde{W}^K)^\sigma = N(L^\natural) \cap K / T(L^\natural) \cap T(L)_1 =: \widetilde{W}^K(L^\natural).$$

Now suppose that F and F' are σ-invariant facets contained in the closure of a σ-invariant alcove in the apartment of $\mathcal{B}$ associated to S. Then the canonical map

$$\widetilde{W}^K(L^\natural) \backslash \widetilde{W}(L^\natural) / \widetilde{W}^{K'}(L^\natural) \ \rightarrow\ [\widetilde{W}^K \backslash \widetilde{W} / \widetilde{W}^{K'}]^\sigma \tag{5}$$

is bijective. Indeed, note first that $\widetilde{W}^K$ and $\widetilde{W}^{K'}$ are parabolic subgroups of the quasi-Coxeter group $\widetilde{W}$ (see Lemma 14 below), and that any element $x \in \widetilde{W}$ has a *unique* expression in the form wx_0w' where $w \in \widetilde{W}^K$, $w' \in \widetilde{W}^{K'}$, such that x_0 is the unique minimal-length element in $\widetilde{W}^K x_0 \widetilde{W}^{K'}$, and wx_0 is the unique minimal-length element in

$wx_0\widetilde{W}^{K'}$. Secondly, note that σ preserves these parabolic subgroups as well as the quasi-Coxeter structure and therefore the Bruhat-order on $\widetilde{W}$. These remarks imply the bijectivity just claimed. Putting (4) and (5) together, we obtain a bijection

$$K(L^\natural)\backslash G(L^\natural)/K'(L^\natural) \xrightarrow{\sim} \widetilde{W}^K(L^\natural)\backslash\widetilde{W}(L^\natural)/\widetilde{W}^{K'}(L^\natural).$$

Remark 10. We now compare the Iwahori Weyl group with a variant of it in [T].

In [T], p. 32, the following group is introduced. Let $T(L)_b$ be the maximal bounded subgroup of $T(L)$. The *affine Weyl group* in the sense of [T], p.32, associated to S is the quotient $\widetilde{W}' = N(L)/T(L)_b$. We obtain a morphism of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_*(T)_I & \longrightarrow & \widetilde{W} & \longrightarrow & W_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \Lambda & \longrightarrow & \widetilde{W}' & \longrightarrow & W_0 & \longrightarrow & 0\ . \end{array}$$

Here $\Lambda = X_*(T)_I/torsion$. Of course,

$$\Lambda = \mathrm{Hom}(X^*(T)^I, \mathbf{Z}) = T(L)/T(L)_b\ ,$$

(see [Ko], 7.2.) It follows that the natural homomorphism from $\widetilde{W}$ to $\widetilde{W}'$ is surjective, with finite kernel isomorphic to $T(L)_b/T(L)_1$.

The affine Weyl group in the sense of [T] also appears in a kind of Bruhat decomposition, as follows. Let

$$v_G : G(L) \longrightarrow X^*(\hat{Z}(G)^I)/torsion$$

be derived from κ_G in the obvious way. Let C be an alcove in the apartment corresponding to S. We consider the subgroup

$$\widetilde{K}_0 = \mathrm{Fix}(C) \cap \mathrm{Ker}\ v_G. \tag{6}$$

Then K_0, the Iwahori subgroup corresponding to C, is a normal subgroup of finite index in $\widetilde{K}_0$. In fact Ker v_G is the group denoted G^1 in [BTII] 4.2.16, and so by loc. cit. 4.6.28, $\widetilde{K}_0$ is the group denoted there $\widehat{P}^1_C$, in other words the fixer of C in G^1. Using loc. cit. 4.6.3, 4.6.7, we have $K_0 = T(L)_1\,\mathfrak{U}_C(O_L)$ and $\widetilde{K}_0 = T(L)_b\,\mathfrak{U}_C(O_L)$, where $\mathfrak{U}_C$ is the group generated by certain root-group O_L-schemes $\mathfrak{U}_{C,a}$ which fix C. It follows that $\widetilde{K}_0 = T(L)_b K_0$, and

$$T(L)_b/T(L)_1 \simeq \widetilde{K}_0/K_0.$$

In [T], 3.3.1, the affine Weyl group $\widetilde{W}'$ appears in the Bruhat decomposition with respect to $\widetilde{K}_0$

$$\widetilde{K}_0\backslash G(L)/\widetilde{K}_0 \simeq \widetilde{W}'. \tag{7}$$

The group $T(L)_b/T(L_1)$ acts freely on both sides of the Bruhat decomposition

$$K_0\backslash G(L)/K_0 = \widetilde{W}$$

of Proposition 8, and we obtain (7) by taking the quotients.

Remark 11. In [BTII] the building of $G(L)$ (sometimes called the enlarged building $\mathcal{B}^1$) is also considered; it carries an action of $G(L)$. There is an isomorphism $\mathcal{B}^1 = \mathcal{B} \times V_G$, where $V_G := X_*(Z(G))_I \otimes \mathbb{R}$. Given a bounded subset Ω in the apartment of $\mathcal{B}$ associated to S, there is a smooth group scheme $\widehat{\mathcal{G}}_\Omega$ whose generic fiber is G and whose O_L-points $\widehat{\mathcal{G}}(O_L)$ is the subgroup of $G(L)$ fixing $\Omega \times V_G$, in other words the subgroup $\widehat{P}^1_\Omega$ in G^1 fixing Ω. We have $(\widehat{\mathcal{G}}_\Omega)^\circ = \mathcal{G}^\circ_\Omega$. Thus, the above discussion and Proposition 3 show that

$$(\widehat{\mathcal{G}}_\Omega)^\circ(O_L) = \widehat{\mathcal{G}}_\Omega(O_L) \cap \ker \kappa_G.$$

Proposition 12. *Let K be associated to F as above. Let $\mathcal{G}^\circ_F$ be the O_L-form with connected fibres of G associated to F, and let $\bar{\mathcal{G}}^\circ_F$ be its special fiber. Then $\widetilde{W}^K$ is isomorphic to the Weyl group of $\bar{\mathcal{G}}^\circ_F$.*

Proof. Let $\bar{\mathcal{S}}^\circ \subset \bar{\mathcal{T}}^\circ \subset \bar{\mathcal{G}}^\circ_F$ be the tori associated to S resp. T, cf. [BTII], 4.6.4 or [L], 6.3. The natural projection $\bar{\mathcal{G}}^\circ_F \longrightarrow \bar{\mathcal{G}}^\circ_{F,\mathrm{red}} = \bar{\mathcal{G}}^\circ_F / R_u(\bar{\mathcal{G}}^\circ_F)$ induces an isomorphism

$$\bar{\mathcal{S}}^\circ \xrightarrow{\sim} \bar{\mathcal{S}}^\circ_{\mathrm{red}},$$

where the index „red" indicates the image group in $\bar{\mathcal{G}}^\circ_{F,\mathrm{red}}$. Let $\bar{W}$ denote the Weyl group of $\bar{\mathcal{S}}^\circ_{\mathrm{red}}$ and consider the natural homomorphism

$$N(L) \cap K_F \longrightarrow \bar{W}.$$

The surjectivity of this homomorphism follows from [BTII], 4.6.13 or [L], 6.10. An element n of the kernel centralizes $\bar{\mathcal{S}}^\circ_{\mathrm{red}}$ and hence also $\bar{\mathcal{S}}^\circ$. But then n centralizes $\mathcal{S}$ because this can be checked via the action of n on $X_*(\mathcal{S})$. But the centralizer of S in K_F is $T(L) \cap K_F$, cf. [BTII], 4.6.4 or [L], 6.3. Hence $n \in T(L) \cap K_F$, which proves the claim. □

We will give the Iwahori-Weyl group $\widetilde{W}$ the structure of a quasi-Coxeter group, that is, a semi-direct product of an abelian group with a Coxeter group. Consider the real vector spaces $V = X_*(T)_I \otimes \mathbb{R} = X_*(S) \otimes \mathbb{R}$ and $V' := X_*(T_{\mathrm{ad}})_I \otimes \mathbb{R}$, where T_{ad} denotes the image of T in the adjoint group G_{ad}. The relative roots $\Phi(G, S)$ for S determine hyperplanes in V (or V'), and the relative Weyl group W_0 can be identified with the group generated by the reflections through these hyperplanes. The homomorphism

$$T(L) \to X_*(T)_I \to V$$

derived from κ_T can be extended canonically to a group homomorphism

$$\nu : N(L) \to V \rtimes W_0$$

where $V \rtimes W_0$ is viewed as a group of affine-linear transformations of V; see [T], §1. Using ν, Tits defines in [T] 1.4 the set of affine roots Φ_{af}, which can be viewed as a set of affine-linear functions on V or on V'. There is a unique reduced root system Σ such that the affine roots Φ_{af} consist of the functions on V (or V') of the form $y \mapsto \alpha(y) + k$ for $\alpha \in \Sigma$, $k \in \mathbb{Z}$. The group generated by the reflections through the walls in V (or in V') through the hyperplanes coming from $\Phi_{\rm af}$ is the affine Weyl group $W_{\rm af}(\Sigma)$, which we also sometimes denote W_a. The group $W_{\rm af}(\Sigma)$ can be given the structure of a Coxeter group, as follows.

The apartment $\mathcal{A}$ in $\mathcal{B}$ associated to S is a torsor for V'; we identify it with V' by fixing a special vertex x in $\mathcal{A}$. Assume x belongs to the closure of the alcove $C \subset \mathcal{A}$. Let $\mathbf{S}$ denote the reflections in V' through the walls of C. Then $(W_{\rm af}(\Sigma), \mathbf{S})$ is a Coxeter group. Writing a superscript x to designate subgroups of the affine linear transformations $\mathrm{Aff}(V')$ of $\mathcal{A}$ with x as "origin", we have a semi-direct product

$$^xW_{\rm af}(\Sigma) = Q^\vee \rtimes {}^xW(\,\Sigma\,)\ ,$$

where $Q^\vee = Q^\vee(\Sigma)$ is the coroot lattice for Σ viewed as translations on V', and where ${}^xW(\Sigma)$ is the finite Weyl group, isomorphic to W_0 and consisting of the elements in ${}^xW_{\rm af}(\Sigma)$ which fix x.

The group $N(L)$ acts on the apartment $\mathcal{A}$ via the homomorphism ν when the latter is viewed as taking values in the group ${}^xW_{\rm af}(\Sigma)$ of affine-linear transformations on $\mathcal{A}$ endowed with x as the "origin".

Let $T_{\rm sc}$ resp. $N_{\rm sc}$ be the inverse images of $T \cap G_{\rm der}$ resp. $N \cap G_{\rm der}$ in $G_{\rm sc}$. Let $S_{\rm sc}$ denote the split component of $T_{\rm sc}$. Then $S_{\rm sc}$ is a maximal split torus in $G_{\rm sc}$, and $T_{\rm sc}$ resp. $N_{\rm sc}$ is its centralizer resp. normalizer. Let $\widetilde{W}_{\rm sc} = N_{\rm sc}(L)/T_{\rm sc}(L)_1$ be the Iwahori Weyl group of $G_{\rm sc}$. By [BTII], 5.2.10, this can be identified with $W_{\rm af}(\Sigma)$. More precisely, note that the Kottwitz homomorphism for $G_{\rm sc}$ is trivial; also parahoric subgroups in $G_{\rm sc}(L)$ are simply the stabilizers or also the pointwise fixers of facets in $\mathcal{B}$. Let $K_{\rm sc,0}$ be the Iwahori subgroup of $G_{\rm sc}(L)$ associated to a chamber $C \subset \mathcal{A}$ and let $\mathbf{S}$ be the set of reflections about the walls of C. Then ([BTII], 5.2.10) $(G_{\rm sc}(L), K_{\rm sc,0}, N_{\rm sc}(L), \mathbf{S})$ is a double Tits system and $\nu : N_{\rm sc}(L) \to V' \rtimes {}^xW(\Sigma)$ induces an isomorphism $\widetilde{W}_{\rm sc} = {}^xW_{\rm af}(\Sigma)$. In particular, $\widetilde{W}_{\rm sc}$ is a Coxeter group.

Proposition 13. *Let $x \in \mathcal{B}$ be a special vertex in the apartment corresponding to the maximal split torus S, and let K be the associated maximal parahoric subgroup of $G(L)$. The subgroup $\widetilde{W}^K$ projects isomorphically to the factor group W_0, and the exact sequence (3) presents $\widetilde{W}$ as a semi-direct product*

$$\widetilde{W} = X_*(T)_I \rtimes W_0\ .$$

Proof. In view of Lemma 5 we need only to see that $\widetilde{W}^K$ maps surjectively to W_0. In the case of $G_{\rm sc}$, we have an isomorphism $\widetilde{W}_{\rm sc} = {}^xW_{\rm af}(\Sigma)$ which sends $\widetilde{W}^{K_{\rm sc}}$ onto ${}^xW(\Sigma)$, which is identified with $N_{\rm sc}(L)/T_{\rm sc}(L) = N(L)/T(L) = W_0$. The desired surjectivity follows. $\square$

By a result of Borovoi [Bo], there is an exact sequence

$$0 \to X_*(T_{\rm sc}) \to X_*(T) \to X^*(\hat{Z}(G)) \to 0.$$

Since $X_*(T_{\rm sc})$ is an induced Galois module ([BTII], 4.4.16), $X_*(T_{\rm sc})_I$ is torsion-free and so the map $X_*(T_{\rm sc})_I \to X_*(T)_I$ is injective, and just as with (2),

$$X_*(T)_I/X_*(T_{\rm sc})_I = X^*(\hat{Z}(G)^I).$$

Using Proposition 13 for both $\widetilde{W}$ and $\widetilde{W}_{\rm sc} \cong W_{\rm af}(\Sigma) =: W_a$, we deduce the following lemma.

Lemma 14. *There is an exact sequence*

$$1 \longrightarrow W_a \longrightarrow \widetilde{W} \longrightarrow X^*(\hat{Z}(G)^I) \longrightarrow 1.$$

The subgroup $\Omega \subset \widetilde{W}$ consisting of the elements which preserve the alcove C maps isomorphically onto $X^*(\hat{Z}(G)^I)$, defining a quasi-Coxeter structure on $\widetilde{W}$,

$$\widetilde{W} = W_a \rtimes \Omega.$$

Let ν' denote the composition of ν with the map $V \to V'$. By [T], 1.7, $\Phi_{\rm af}$ is stable under $\nu'(N(L)) \subset {\rm Aff}(V')$, hence in particular under the group of translations $\nu'(T(L))$, which may be identified with $X_*(T_{\rm ad})_I$. (Note that $X_*(T_{\rm ad})_I$ is torsion-free, by [BTII], 4.4.16.)

Lemma 15. *There are natural inclusions*

$$Q^\vee \subset X_*(T_{\rm ad})_I \subset P^\vee \ ,$$

where $P^\vee \subset V'$ is the set of coweights for Σ. Moreover, ν' gives an identification

$$X_*(T_{sc})_I = Q^\vee.$$

Proof. The translations $\nu'(T(L))$ permute the special vertices in $\mathcal{A}$, and $P^\vee$ acts simply transitively on them; hence $\nu'(T(L)) \subset P^\vee$. The equality $X_*(T_{\rm sc})_I = Q^\vee$ follows from the isomorphism $\widetilde{W}_{\rm sc} = {}^xW_{\rm af}(\Sigma)$ deduced above from [BTII], 5.2.10, and this implies $Q^\vee \subset X_*(T_{\rm ad})_I$. □

Remark 16. The affine Weyl group associated to S also appears in [BTII] in a slightly different way. In [BTII], 5.2.11 the following subgroup of $G(L)$ is introduced,

(8) $$G(L)' = \textit{ subgroup generated by all parahoric subgroups of } G(L).$$

In loc.cit. it is shown that this coincides with

(9) $$G(L)' = T(L)_1.\varphi(G_{\rm sc}(L)),$$

where $\varphi : G_{\rm sc} \longrightarrow G$ is the natural homomorphism from the simply connected cover of $G_{\rm der}$.

Lemma 17. *We have $G(L)' = G(L)_1$.*

Proof. The inclusion "$\subset$" is obvious. If $G = T$ is a torus the equality is obvious, and so is the case when G is semisimple and simply connected. Next assume that $G_{\rm der}$ is simply connected and let $D = G/G_{\rm der}$. We obtain a commutative diagram

$$\begin{array}{ccc} G(L) & \longrightarrow & X^*(\hat{Z}(G)^I) \\ \downarrow & & \| \\ D(L) & \longrightarrow & X^*(\hat{D}^I) \ . \end{array}$$

The exact sequence $1 \longrightarrow T_{\mathrm{der}} \longrightarrow T \longrightarrow D \longrightarrow 1$ induces a morphism of exact sequences

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_{\mathrm{der}}(L) & \longrightarrow & T(L) & \longrightarrow & D(L) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \kappa_T & & \downarrow \kappa_D & & \\ & & X_*(T_{\mathrm{der}})_I & \longrightarrow & X_*(T)_I & \longrightarrow & X_*(D)_I & \longrightarrow & 0\ . \end{array}$$

All vertical maps are surjective. Now let $g \in G(L)_1$. Then by a simple diagram chase there exists $t \in T(L)_1$ with the same image in D as g, and hence $g = tg' \in T(L)_1 \cdot \varphi(G_{\mathrm{der}}(L)) = G(L)'$.

To treat the general case, choose a z-extension

$$1 \longrightarrow Z \longrightarrow \widetilde{G} \xrightarrow{\pi} G \longrightarrow 1,$$

with $\widetilde{G}_{\mathrm{der}}$ simply connected. We obtain a morphism of exact sequences with surjective vertical maps

$$\begin{array}{ccccccccc} 1 & \longrightarrow & Z(L) & \longrightarrow & \widetilde{G}(L) & \longrightarrow & G(L) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & X^*(\hat{Z}^I) & \longrightarrow & X^*(\hat{Z}(\widetilde{G})^I) & \longrightarrow & X^*(\hat{Z}(G)^I) & \longrightarrow & 0\ . \end{array}$$

The diagram shows that $G(L)_1 = \pi(\widetilde{G}(L)_1)$. On the other hand, the equality $G(L)' = \pi(\widetilde{G}(L)')$ is easy to see. We conclude by the equality $\widetilde{G}(L)' = \widetilde{G}(L)_1$ which we already know. $\square$

The other way of introducing the affine Weyl group is now as follows. Let $N(L)' = N(L) \cap G(L)'$ and $W' = N(L)'/T(L)_1$. Let K_0 be the Iwahori subgroup associated to an alcove C in the apartment corresponding to S and let $\mathbf{S}$ be the set of reflections about the walls of C. Then ([BTII], 5.2.12) $(G(L)', K_0, N(L)', \mathbf{S})$ is a double Tits system and W' is the affine Weyl group of the affine root system of S. In particular, the natural homomorphism $W' \longrightarrow W_a$ is an isomorphism.

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