MACDONALD'S FORMULA IMPLIES THE MIRKOVIC-VILONEN DIMENSION FORMULA

Mirkovic-Vilonen: The dimension of $N\pi^\lambda K/K \cap K\pi^\mu K/K$ is $\langle \rho, \mu+\lambda\rangle$, and the number of irreducible components of top dimension is $m_\mu(\lambda)$, the multiplicity of the weight λ in the character E_μ.

We will deduce this using Macdonald's formula. It is enough (by, e.g., the Weil conjectures) to show that

$$\lim_{q\to\infty} \frac{\#(N\pi^\lambda K/K \cap K\pi^\mu K/K)(\mathbb{F}_q)}{q^{\langle\rho,\mu+\lambda\rangle}} = m_\mu(\lambda).$$

The numerator in the left hand side is

$$\begin{aligned}\int_G 1_{A_{\mathcal{O}}NK}(\pi^{-\lambda}y)\ 1_{K\pi^{-\mu}K}(y^{-1})\ dy &= (1_{A_{\mathcal{O}}NK} * 1_{K\pi^{-w_0\mu}K})(\pi^{-\lambda}) \\ &= (1^\vee_{K\pi^{-w_0\mu}K}\cdot 1_{A_{\mathcal{O}}NK})(\pi^{-\lambda}) \\ &= 1^\vee_{K\pi^{-w_0\mu}K}(\pi^{-\lambda})\ \delta_B^{1/2}(\pi^{-\lambda}) \\ &= 1^\vee_{K\pi^{-w_0\mu}K}(\pi^{-w_0\lambda})\ q^{\langle\rho,\lambda\rangle}.\end{aligned}$$

By Macdonald's formula, this is the coefficient of $\pi^{-w_0\lambda}$ in

$$\frac{q^{\langle\rho\,,\,\mu+\lambda\rangle}}{W_{-w_0\mu}(q^{-1})} \sum_{w\in W} w\Big(\prod_{\alpha>0}\frac{1-q^{-1}\pi^{-\alpha^\vee}}{1-\pi^{-\alpha^\vee}}\Big)\cdot \pi^{-w\,w_0\mu}.$$

Divide this by $q^{\langle\rho\,,\,\mu+\lambda\rangle}$ and take the limit as $q\to\infty$. The Weyl character formula implies that we get $m_{-w_0\mu}(-w_0\lambda) = m_\mu(\lambda)$. This completes the proof.