ON HECKE ALGEBRA ISOMORPHISMS AND TYPES
FOR DEPTH-ZERO PRINCIPAL SERIES

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ABSTRACT. These lectures describe Hecke algebra isomorphisms and types for depth-zero principal series blocks, a.k.a. Bernstein components \( R_s(G) \) for \( s = s_\chi = [T, \chi]_G \), where \( \chi \) is a depth-zero character on \( T(O) \). (Here \( T \) is a split maximal torus in a \( p \)-adic group \( G \).) We follow closely the treatment of A. Roche [Ro] with input from D. Goldstein [Gol] and L. Morris [Mor]. We give an elementary proof that \((I, \rho_\chi)\) is a type for \( s_\chi \), in the sense of Bushnell-Kutzko [BK]. This is a very special case of a result of Roche [Ro]. Our method is to imitate Casselman’s proof of Borel’s theorem on unramified principal series (the case \( \chi = 1 \) of the present theorem).

In contrast to the situation for general principal series blocks (see [Ro]), in the depth-zero case there is no restriction on the residual characteristic of \( F \).

1. Notation

We let \( F \) denote an arbitrary \( p \)-adic field with ring of integers \( \mathcal{O} \), and residue field \( k_F \). Let \( q \) denote the cardinality of \( k_F \). Write \( \varpi \) for a uniformizer.

Let \( G \) denote a connected reductive group, defined and split over \( \mathcal{O} \). Fix an \( F \)-split maximal torus \( T \) and a Borel subgroup \( B \) containing \( T \); assume \( T \) and \( B \) are defined over \( \mathcal{O} \). Let \( T = T(O) \) denote the maximal compact subgroup of \( T(F) \). Let \( \Phi \subset X^*(T) \) resp. \( \Phi^\vee \subset X_*(T) \) denote the set of roots resp. coroots for \( G,T \). Let \( U \) resp. \( \overline{U} \) denote the unipotent radical of \( B \) resp. the Borel subgroup \( \overline{B} \supset T \) opposite to \( B \).

The symbol \( I \) will stand for an Iwahori subgroup of \( G(F) \), which we shall assume it is in “good position” with respect to \( T \): the alcove \( \mathfrak{a} \) in the building for \( G(F) \) which is fixed by \( I \) is contained in the apartment corresponding to \( T \).

Let \( dx \) denote a Haar measure on \( G \). Denote the group of unramified characters of \( G(F) \) by \( X^\text{ur}(G) \) (see [BD] or [Be92] for the definition).

Let \( \mathcal{R}(G) \) denote the category of smooth representations of \( G(F) \).

Let \( L \) denote an \( F \)-Levi subgroup of \( G \) (by definition, \( L = C_G(A_L) \) for some \( F \)-split torus \( A_L \) in \( G \)). Let \( P = LN \) denote an \( F \)-parabolic subgroup, that is, a parabolic subgroup defined over \( F \), with unipotent radical \( N \) and with \( L \) as a Levi factor. Let \( \sigma \) denote any smooth representation of \( L \), and define the normalized parabolic induction by

\[
{\text{Ind}}_P^G(\delta_P^{1/2} \sigma),
\]
where $\delta_F(l) := |\det(\text{Ad}(l); \text{Lie}(N(F)))|_F$. Here $| \cdot |_F$ denotes the normalized absolute value on $F$.

Throughout these notes, we will frequently write $G$ (resp. $B,T$, etc.) when we really mean $G(F)$ (resp. $B(F), T(F)$, etc.).

2. Bernstein decomposition (review)

A cuspidal pair $(L, \sigma)$ consists of an $F$-Levi subgroup $L$ of $G$, together with a supercuspidal representation $\sigma$ of $L(F)$.

The group $G = G(F)$ acts “by conjugation” on cuspidal pairs: $g \cdot (L, \sigma) = (gL, g^* \sigma)$, where $gL = gLg^{-1}$ and $g^* \sigma(\cdot) = \sigma(g^{-1} \cdot g)$. Denote by $(L, \sigma)_G$ the $G$-conjugation class of $(L, \sigma)$.

Let $(L, \sigma)$ denote a cuspidal pair. We say $(L_1, \sigma_1)$ is inertially equivalent to $(L_2, \sigma_2)$ if there exists $g \in G(F)$ and $\chi \in X^u(L_2)$ such that $gL_1 = L_2$ and $g^* \sigma_1 \otimes \chi = \sigma_2$.

Let $s = [L, \sigma]_G$ denote the inertial equivalence class of $(L, \sigma)$ (with respect to $G$). Note that $s$ depends only on $(L, \sigma)_G$. Also $s$ is a union of $G$-conjugacy classes of cuspidal pairs.

**Fact:** For $\pi \in \mathcal{R}(G)$ irreducible, there exist a (unique up to $G$-conjugacy) cuspidal pair $(L, \sigma)$ such that $\pi$ is a subquotient of $i^G_F(\sigma)$. Here $P = LN$ is an $F$-parabolic with unipotent radical $N$ which has $L$ as a Levi factor.

We call the class $(L, \sigma)_G$ as above the supercuspidal support of $\pi$.

Denote by $\mathcal{R}_s(G)$ the full subcategory of $\mathcal{R}(G)$ whose objects are the representations $\pi$ each of whose irreducible subquotients has supercuspidal support belonging to the inertial class $s$. Once we fix a cuspidal pair $(L, \sigma)$ in $s$, we may reformulate the condition for $\pi$ to belong to $\mathcal{R}_s(G)$ as: every irreducible subquotient of $\pi$ is a subquotient of some $i^G_F(\sigma \chi)$, $\chi \in X^u(L)$.

**Theorem 2.0.1** (Bernstein decomposition). $\mathcal{R}(G) = \prod_s \mathcal{R}_s(G)$.

**Definition 2.0.2.** An $s$-type is a pair $(K, \rho)$ consisting of a compact open subgroup $K \subset G$ together with an irreducible smooth representation $\rho : K \to \text{End}_C(W)$ such that an irreducible $\pi \in \mathcal{R}(G)$ belongs to $\mathcal{R}_s(G)$ iff $\pi|_K \supset \rho$.

Now let $\rho$ be any irreducible smooth representation of $K$, on a vector space $W$. We define $e_\rho \in \mathcal{H}(G) = C^\infty_c(G, dx)$ by

$$e_\rho(x) = \begin{cases} dx(K)^{-1} \dim(\rho) \text{tr}_W(\rho(x^{-1})), & x \in K \\ 0, & x \notin K. \end{cases}$$

For any irreducible smooth representations $\rho, \rho'$ of $K$, we have $e_\rho *_{dx} e_{\rho'} = \delta_{\rho, \rho'} e_\rho$, where $\delta_{\rho, \rho'} \in \{0,1\}$ vanishes unless $\rho$ and $\rho'$ are equivalent. This is an exercise using the Schur orthogonality relations on the group $K$. In particular, $e_\rho$ is an idempotent of the algebra $\mathcal{H}(G)$.

If $\rho = 1$ (the trivial character) we write $e_K$ in place of $e_\rho$.

For any $(\pi, V) \in \mathcal{R}(G)$, denote by $V^\rho$ the $\rho$-isotypical component of $V$. We have $V^\rho = e_\rho V$. Also, we let $V[\rho] = \mathcal{H}(G) \cdot V^\rho$, the $G$-submodule of $V$ generated by $V^\rho$. Below we will often write $\pi^\rho$ in place of $V^\rho$. 
We define $\mathcal{R}_\rho(G)$ to be the full subcategory of $\mathcal{R}(G)$ whose objects $(\pi, V)$ satisfy $V = V[\rho]$.

There is a functor $\mathcal{R}_\rho(G) \rightarrow e_\rho \mathcal{H}(G)e_\rho \text{-Mod}$

$$\pi \mapsto \pi^\rho.$$

**Proposition 2.0.3.** If $(K, \rho)$ is an $s$-type, then (2.0.1) is an equivalence of categories. Moreover, in that case $\mathcal{R}_s(G) = \mathcal{R}_\rho(G)$ as subcategories of $\mathcal{R}(G)$.

We will postpone the proof of this proposition to section 4.

3. Depth-zero principal series blocks

**Example.** Consider an Iwahori subgroup $I$ in good position with respect to the torus $T$ (this means that $I$ fixes an alcove $a$ in the apartment of the building for $G(F)$ corresponding to $T$). Also, for any Borel subgroup $B = TU$ containing $T$, with opposite Borel $\overline{B} = T\overline{U}$, we have the Iwahori decomposition

$$I = I_U \cdot ^T \cdot I_{\overline{U}},$$

where $I_U := U \cap I$, $I_{\overline{U}} := \overline{U} \cap I$, and $^T := T(\mathcal{O}) = T \cap I$.

The inertial class $s := [T, 1]_G$ indexes the Iwahori block $\mathcal{R}_s(G)$. A famous theorem of Borel asserts that an irreducible $\pi \in \mathcal{R}(G)$ is a constituent of an unramified principal series $i^G_B(\eta)$, $\eta \in X^{ur}(T)$, if and only if $\pi^I \neq 0$. That is, $(I, 1)$ is an $s$-type. This is a special case of the theorem we will prove below (Theorem 3.0.2).

It turns out that $e_I \mathcal{H}(G)e_I = \mathcal{H}(G, I)$, the Iwahori-Hecke algebra (see below). In conjunction with the Proposition 2.0.3, we thus recover the finer result of Borel which asserts that

$$\pi \mapsto \pi^I$$

gives an equivalence of categories between the Iwahori block and the category $\mathcal{H}(G, I)$-Mod.

Fix a character $\chi : ^T \rightarrow \mathbb{C}^\times$.

**Definition 3.0.1.** We say $\chi$ is **depth-zero** if $\chi$ factors through the quotient $^T \rightarrow T(k_F)$ (and we denote the factoring $T(k_F) \rightarrow \mathbb{C}^\times$ also by $\chi$).

Choose any extension of $\chi$ to a character $\widetilde{\chi} : T(F) \rightarrow \mathbb{C}^\times$. Consider the inertial class

$$s := [T, \widetilde{\chi}]_G.$$ 

Since $s$ depends only on the $W$-orbit of $\chi$, we may also write $s_\chi$ for $s$.

Let $I$ be an Iwahori in good position relative to $T$, as above. Let $I^+$ denote the pro-unipotent radical of $I$. There is an obvious isomorphism

$$^T / ^T \cap I^+ \cong I/I^+$$
so that \( \chi \) determines a character \( \rho = \rho_\chi : I \to \mathbb{C}^X \), which is trivial on \( I^+ \). In terms of the Iwahori decomposition (3.0.2), \( \rho \) is given by

\[
\rho(u \cdot t_0 \cdot \overline{\pi}) = \chi(t_0),
\]

for \( u \in I_U, t_0 \in \mathcal{O}T \), and \( \pi \in I_T \).

**Theorem 3.0.2.** If \( s = s_\chi \) as above, then \((I, \rho)\) is an \( s \)-type.

We shall prove this by imitating Casselman’s proof of Borel’s theorem on unramified principal series. One crucial ingredient is the theory of Hecke algebra isomorphisms for depth-zero principal series types, which we will review in section 5.

4. **Proof of Proposition 2.0.3**

We are in the general situation, where \((K, \rho)\) is a smooth irreducible representation on a vector space \( W \) (ie. \( \rho \) is not necessarily a character).

**Lemma 4.0.1.** Fix an inertial class \( s \).

(i) \((K, \rho)\) is an \( s \)-type \iff \( \text{ind} \rho := c - \text{Ind}_{K}^{G} \rho \) is a generator for \( \mathcal{R}_{s}(G) \), i.e., \( \text{ind} \rho \in \mathcal{R}_{s}(G) \) and \( \text{Hom}_{G}(\text{ind} \rho, \pi) \neq 0 \) for all \( \pi \neq 0 \) in \( \mathcal{R}_{s}(G) \).

(ii) In that case \( \mathcal{R}_{s}(G) = \mathcal{R}_{\rho}(G) \) as subcategories of \( \mathcal{R}(G) \). In particular \( \mathcal{R}_{\rho}(G) \) is closed under extensions and subquotients.

**Proof.** First, by Frobenius reciprocity (cf. [Ro],(7.1)) we have

\[
\text{Hom}_{G}(\text{ind} \rho, \pi) = \text{Hom}_{K}(\rho, \pi).
\]

This implies that \( \text{ind} \rho \) is a projective object in \( \mathcal{R}(G) \). (It is also true that \( \text{ind} \rho \) is finitely-generated as a \( G \)-module.)

Now let us prove (i).

(\( \Rightarrow \)): Suppose \((\pi, V) \in \mathcal{R}_{s} \) is non-zero. Since all irreducible subquotients of \( \pi \) are also in \( \mathcal{R}_{s} \) (hence contain \( \rho \)) and representations of \( K \) are completely reducible, it follows that \( \text{Hom}_{K}(\rho, \pi) \neq 0 \) and hence \( \text{Hom}_{G}(\text{ind} \rho, \pi) \neq 0 \).

Next we claim that \( \text{ind} \rho \in \mathcal{R}_{s} \). If not, then \( \text{ind} \rho \) possesses a non-zero quotient \( \tau \) in some \( \mathcal{R}_{t} \) with \( t \neq s \). Since \( \tau \) is finitely-generated (as \( \text{ind} \rho \) is), it possesses an irreducible quotient; we may assume \( \tau \) is itself irreducible. But then \( \text{Hom}_{K}(\rho, \tau) \neq 0 \) implies that \( \tau \supset \rho \) and this means that \((K, \rho)\) is not an \( s \)-type.

(\( \Leftarrow \)): Let \((\pi, V) \in \mathcal{R}(G) \) be irreducible and non-zero. Then

\[
\pi \in \mathcal{R}_{s}(G) \iff \text{Hom}_{G}(\text{ind} \rho, \pi) \neq 0
\]

\[
\iff \text{Hom}_{K}(\rho, \pi) \neq 0
\]

\[
\iff \pi \in \mathcal{R}_{\rho}(G).
\]

The first (\( \iff \)) holds because \( \text{ind} \rho \), hence any of its quotients, lies in \( \mathcal{R}_{s}(G) \).

This completes the proof of (i).
Now let us prove (ii). Suppose \((\pi, V) \in \mathcal{R}_s(G)\). We have \((V/V[\rho])^p = 0\). But then \(V/V[\rho] = 0\), since non-zero objects in \(\mathcal{R}_s(G)\) contain \(\rho\). So \(V = V[\rho]\), that is, \(\pi \in \mathcal{R}_p(G)\).

Conversely, if \(V = V[\rho]\), then \(\pi\) is a quotient of a direct sum of copies of \(\text{ind} \rho \in \mathcal{R}_s(G)\), hence \(\pi \in \mathcal{R}_s(G)\).

\[\square\]

**Exercise:** Since \(\text{ind} \rho\) is projective in \(\mathcal{R}(G)\) and a generator for \(\mathcal{R}_s(G)\) (i.e. \(\text{ind} \rho \in \mathcal{R}_s(G)\) and \(\text{Hom}_G(\text{ind} \rho, \pi) \neq 0\) for every \(\pi \neq 0\) in \(\mathcal{R}_s(G)\)), every \(\pi \in \mathcal{R}_s(G)\) is a quotient of a direct sum of copies of \(\text{ind} \rho\). (Consider the maximal subobject in \(\pi\) which is a quotient of a direct sum of copies of \(\text{ind} \rho\).)

We have shown that \(\text{ind} \rho\) is a f.g. projective generator of \(\mathcal{R}_s(G)\). From this, general categorical arguments ([Ba]) give (Morita) equivalences of categories

\[
\mathcal{R}_s(G) \cong \text{End}_G(\text{ind} \rho)^\text{op} \otimes \text{End}_C W\text{-Mod} \\
\pi \mapsto \text{Hom}_G(\text{ind} \rho, \pi) \mapsto \text{Hom}(\text{ind} \rho, \pi) \otimes W \\
t.f = f \circ t.
\]

Therefore, we need to relate \(\text{End}_G(\text{ind} \rho)^\text{op} \otimes \text{End}(W)\) to \(e_\rho \mathcal{H}(G)e_\rho\). First we define

\[
\mathcal{H}(G, \rho^\vee) = \{ \Phi : G \to \text{End}(W) \mid \Phi(k_1 g k_2) = \rho(k_1) \Phi(g) \rho(k_2), \forall k_i \in K, g \in G \}.
\]

Here the functions \(\Phi\) are assumed to be smooth with compact support. Also, \((\rho^\vee, W^\vee)\) is the representation given by \(\rho^\vee(k) := \rho(k^{-1})^\vee \in \text{End}(W^\vee)\). We view \(\mathcal{H}(G, \rho^\vee)\) as a convolution algebra using the Haar measure \(dx\) giving \(K\) volume 1.

The following lemma is left to the reader.

**Lemma 4.0.2.** We have mutually inverse algebra isomorphisms

\[
\phi \mapsto t_\phi : \mathcal{H}(G, \rho^\vee) \leftrightarrow \text{End}_G(\text{ind} \rho) : t \mapsto \phi_t,
\]

where

\[
t_\phi(f)(g) = \int_G \phi(x)(f(x^{-1}g)) \, dx \quad (f \in \text{ind} \rho, g \in G) \\
\phi_t(g)(w) = t(e_w)(g) \quad (g \in G, w \in W).
\]

Here \(e_w \in \text{ind} \rho\) is defined by

\[
e_w(g) = \begin{cases} 
\rho(k)w, & g = k \in K \\
0, & g \notin K.
\end{cases}
\]

Furthermore, there is an anti-isomorphism of algebras

\[
\mathcal{H}(G, \rho^\vee) \cong \mathcal{H}(G, \rho) \\
\Phi \mapsto \Phi' 
\]

given by \(\Phi'(g) := \Phi(g^{-1})^\vee \in \text{End}(W^\vee)\).
Finally, Roche checks in [Ro], p. 390, that there is an algebra isomorphism
\[ \mathcal{H}(G, \rho) \otimes_{\mathbb{C}} \text{End}(W) \cong e_{\rho} \mathcal{H}(G)e_{\rho} \]
\[ \Phi \otimes (w \otimes w^\vee) \mapsto (g \mapsto \dim \rho \langle w, \Phi(g)w^\vee \rangle) \quad (w \in W, w^\vee \in W^\vee). \]
In case \( \rho \) is a character, the last isomorphism gives \( \mathcal{H}(G, \rho) \cong e_{\rho} \mathcal{H}(G)e_{\rho} \) and is immediate.

Putting these isomorphisms together, we get isomorphisms
\[ \text{End}_G(\text{ind}\rho)^{\text{opp}} \otimes \text{End}(W) \cong \mathcal{H}(G, \rho) \otimes \text{End}(W) \cong e_{\rho} \mathcal{H}(G)e_{\rho}. \]
In loc. cit. Roche checks that the induced categorical equivalence
\[ \mathcal{R}_\rho(G) = \mathcal{R}_\rho(G) \cong e_{\rho} \mathcal{H}(G)e_{\rho} \text{-Mod} \]
is
\[ (\pi, V) \mapsto \text{Hom}_G(\text{ind}\rho, \pi) = \text{Hom}_{K}(\rho, \pi) = \pi^\rho. \]
(Again, this is quite immediate in the case where \( \rho \) is a character.) This completes the proof of Proposition 2.0.3.

5. Hecke algebra isomorphisms

To prove Theorem 3.0.2, we need to review Hecke algebra isomorphisms. We follow Roche’s treatment [Ro].

5.1. Preliminaries. As before, fix a depth-zero character \( \chi : \circ T \to \mathbb{C}^\times \), and let \( s = [T, \chi]_G = s_\chi \), for any extension \( \bar{\chi} : T(F) \to \mathbb{C}^\times \) of \( \chi \). Also, write \( \rho = \rho_\chi \) for the associated character \( \rho : I = I_U \cdot \circ T \cdot \bar{I}_U \to \mathbb{C}^\times \), \( u t \rho \mapsto \chi(t) \).

Let \( N \) denote the normalizer of \( T \) in \( G \), let \( W = N/T = N(F)/T(F) \) denote the Weyl group, and write \( \tilde{W} = N(F)/\circ T \) for the Iwahori-Weyl group. There is a canonical isomorphism \( X_s(T) = T(F)/\circ T, \lambda \mapsto \varpi^\lambda := \lambda(\varpi) \) (independent of the choice of \( \varpi \)). The canonical homomorphism \( N(F)/\circ T = \tilde{W} \to W = N(F)/T(F) \) has a (non-canonical) section, hence there is a (non-canonical) isomorphism \( \tilde{W} = X_s(T) \rtimes W \).

Clearly \( N(F), \tilde{W} \) and \( W \) act on the set of depth-zero characters. We define
\[ N_\chi = \{ n \in N(F) \mid n\chi = \chi \} \]
\[ \tilde{W}_\chi = \{ w \in \tilde{W} \mid w\chi = \chi \} \]
\[ W_\chi = \{ w \in W \mid w\chi = \chi \}. \]

There are obvious surjective homomorphisms \( N_\chi \to \tilde{W}_\chi \to W_\chi \).

Define \( \Phi_\chi \) (resp. \( \Phi_\chi^\vee \) resp. \( \Phi_{\chi, \text{aff}} \)) to be the set of roots \( \alpha \in \Phi \) (resp. coroots \( \alpha^\vee \in \Phi^\vee \) resp. affine roots \( a = \alpha + k \), where \( \alpha \in \Phi, k \in \mathbb{Z} \)) such that \( \chi \circ \alpha^\vee|_{\mathfrak{g}_\rho} = 1 \). Note that \( \tilde{W}_\chi \) acts in an obvious way on \( \Phi_{\chi, \text{aff}} \). Define the following subgroups of the group of affine-linear automorphisms of \( V := X_s(T) \otimes \mathbb{R} \):
\[ W_\chi^\circ = \langle s_\alpha \mid \alpha \in \Phi_\chi \rangle \]
\[ W_{\chi, \text{aff}} = \langle s_a \mid a \in \Phi_{\chi, \text{aff}} \rangle. \]
Here \( s_a \) and \( s_\alpha \) are the reflections on \( V \) corresponding to \( a \) and \( \alpha \).

Let \( \Phi^+ \) denote the \( B \)-positive roots in \( \Phi \), and set \( \Phi^+_\chi = \Phi \cap \Phi^+ \). Then let \( C_\chi \) resp. \( a_\chi \) denote the subsets in \( V \) defined by

\[
C_\chi = \{ v \in V \mid 0 < \alpha(v), \forall \alpha \in \Phi^+_\chi \}, \text{ resp.}
\]

\[
a_\chi = \{ v \in V \mid 0 < \alpha(v) < 1, \forall \alpha \in \Phi^+_\chi \}.
\]

For \( a \in \Phi^+_{\chi, \text{aff}} \) we write \( a > 0 \) if \( a(v) > 0 \) for all \( v \in a_\chi \). Similarly, we define an ordering on the set \( \Phi^+_{\chi, \text{aff}} \). Then let \( \Pi_{\chi, \text{aff}} = \{ a \in \Phi^+_{\chi, \text{aff}} \mid a \) is a minimal positive element \}. Define

\[
S_{\chi, \text{aff}} = \{ s_a \mid a \in \Pi_{\chi, \text{aff}} \}
\]

\[
\Omega_\chi = \{ w \in \tilde{W}_\chi \mid wa_\chi = a_\chi \}.
\]

It is clear that \( \Phi^+_{\chi} \) is a root system with Weyl group \( \tilde{W}_\chi \), and that \( \tilde{W}_\chi \subseteq W_\chi \). In general, \( W_\chi \) can be larger that \( \tilde{W}_\chi \) and is not even a Weyl group (see Example 8.3 in [Ro] and Remark 5.1.2 below). The following results are contained in [Ro].

**Lemma 5.1.1.**

1. The group \( W^\text{aff}_{\chi, \text{aff}} \) is a Coxeter group with system of generators \( S_{\chi, \text{aff}} \);
2. there is a canonical decomposition \( \tilde{W}_\chi = W^\text{aff}_{\chi, \text{aff}} \rtimes \Omega_\chi \), and the Bruhat order \( \leq_\chi \) and length function \( \ell_\chi \) on \( W^\text{aff}_{\chi, \text{aff}} \) can be extended in an obvious way to \( \tilde{W}_\chi \) such that \( \Omega_\chi \) consists of the length-zero elements;
3. if \( W^0_\chi = W_\chi \), then \( W^\text{aff}_{\chi, \text{aff}} \) is the affine Weyl group associated to the root system \( \Phi^+_{\chi} \subseteq V^* \), and \( C_\chi \) resp. \( a_\chi \) is the dominant Weyl chamber resp. base alcove in \( V \) corresponding to a set of simple positive affine roots, which can be identified with \( \Pi_{\chi, \text{aff}} \).

In the situation of (3), let \( \Pi_\chi \) denote the set of minimal elements of \( \Phi^+_{\chi} \). This is then a set of simple positive roots for the root system \( \Phi^+_{\chi} \).

**Remark 5.1.2.** In [Ro], pp. 393-6, Roche proves that \( W_\chi^0 = W_\chi \) at least when \( G \) has connected center and when \( p \) is not a torsion prime for \( \Phi^V \) (see loc. cit. p. 396). It is easy to see that \( W_\chi^0 = W_\chi \) always holds when \( G = \text{GL}_d \) (with no restrictions on \( p \)).

On the other hand, \( W_\chi \neq W_\chi^0 \) in general, even for \( G = \text{SL}_n \). Indeed, suppose \( G = \text{SL}_n \) with \( n \geq 3 \). Suppose \( n|q-1 \) and that \( \chi_1 \) is a character of \( \mathbb{F}_q^\times \) of order \( n \). Consider

\[
\chi(a_1, \ldots, a_n) := \chi_1(a_1)\chi_1^2(a_2) \cdots \chi_1^n(a_n).
\]

It is clear that \( W_\chi^0 = \{1\} \), but that, since \( a_1 \cdots a_n = 1 \), we have \( W_\chi \supseteq (1 \cdots n) \). In fact \( W_\chi \) is the cyclic group of order \( n \) generated by \( (1 \cdots n) \).

5.2. **Statement.** Let \( \mathcal{H}(W^\text{aff}_{\chi, \text{aff}}) \) denote the affine Hecke algebra associated to the Coxeter group \( (W^\text{aff}_{\chi, \text{aff}}, S_{\chi, \text{aff}}) \). It has the usual generators \( T_w, w \in W^\text{aff}_{\chi, \text{aff}} \), and relations

\[
T_{w_1w_2} = T_{w_1}T_{w_2}, \quad \text{if } \ell_\chi(w_1w_2) = \ell_\chi(w_1) + \ell_\chi(w_2),
\]

\[
T_s^2 = (q-1)T_s + qT_1, \quad \text{if } s \in S_{\chi, \text{aff}}.
\]
Let $\mathcal{H}_\chi := H(W_{\chi, \text{aff}}) \otimes \mathbb{C}[\Omega_\chi]$, where the twisted tensor product is the usual tensor product on the underlying vector spaces, but where multiplication is given by

$$(T_{w_1} \otimes e_{\omega_1})(T_{w_1} \otimes e_{\omega_2}) = T_{w_1 \omega_1(w_2)} \otimes e_{\omega_1 \omega_2}$$

where $\omega(\cdot)$ refers the conjugation action of $\omega \in \Omega_\chi$ on $W_{\chi, \text{aff}}$.

We write $T_{w \omega} := T_w \otimes e_{\omega}$.

The Hecke algebra isomorphism depends on a choice of extension $\hat{\chi}: N_\chi \to \mathbb{C}^\times$ of $\chi$ (this always exists: see [HL] 6.11 and [HR09]). Fix such a $\hat{\chi}$. Then for any $n \in N_\chi \mapsto w \in \hat{W}_\chi$, define

$$[InI]_{\hat{\chi}} \in \mathcal{H}(G, \rho)$$

to be the unique element in $\mathcal{H}(G, \rho)$ supported on $InI$ and having value $\hat{\chi}^{-1}(n)$ at $n$. Note that $[InI]_{\hat{\chi}}$ depends on $w \in \hat{W}_\chi$ but not on the choice of $n \in N_\chi$ mapping to $w$.

**Theorem 5.2.1** (Goldstein [Gol], Morris [Mor], Roche [Ro]). Let $\chi$ be a depth-zero character as above. For any extension $\hat{\chi}$ of $\chi$ as above, there is an algebra isomorphism

$$\mathcal{H}(G, \rho) \cong \mathcal{H}_\chi,$$

which sends $q^{-\ell(w)/2}[InI]_{\hat{\chi}}$ to $q^{-\ell(w)/2} T_w$.

Let $\Phi_n := \hat{\chi}(n)[InI]_{\hat{\chi}}$, the unique element in $\mathcal{H}(G, \rho)$ supported on $InI$ and having $\Phi_n(n) = 1$.

**Corollary 5.2.2.** For any $n \in N_\chi$, the element $[InI]_{\hat{\chi}}$ (or equivalently, $\Phi_n$) is invertible in $\mathcal{H}(G, \rho)$.

6. The morphism $V^\rho \to V_U^\chi$

We assume $B = TU$ and $I$ are in “good position”: $I$ fixes an alcove $a$ contained in the apartment corresponding to $T$, and $B$ is any Borel subgroup containing $T$. From $\chi$ we get $\rho$ as usual.

For $(\pi, V) \in \mathcal{R}(G)$, let $V_U \in \mathcal{R}(T)$ denote the Jacquet module.

**Proposition 6.0.1.** Suppose $(\pi, V)$ is irreducible (hence, cf. [Be92], admissible). Then the map $V \to V_U^\chi$ induces a $^oT$-equivariant isomorphism

$$(6.0.1) \quad V^\rho \cong V_U^\chi.$$

**Remark 6.0.2.** Since $B = TU$ may be replaced with any $^wB = T^wU$ ($w \in W$), it follows that we may also hold $B$ fixed and replace $I$ with $^wI$. That is, we may replace $\chi$ with $^w\chi$ and $\rho$ with $^w\rho$, where the latter is the character on $^wI$ defined by $^w\rho(\cdot) = \rho(w^{-1} \cdot w)$. Such a replacement causes no harm for the proof of the main theorem (cf. section 7) because $\pi(w) : V^\rho \cong V^{^w\rho}$.

We will prove Proposition 6.0.1 using only a consequence of the Hecke algebra isomorphism, namely Corollary 5.2.2.
Proof. We change notation slightly and write the Iwahori decomposition as
\[ I = \mathcal{U}_0 \circ T \mathcal{U}_0 \]
where \( \mathcal{U}_0 := I_U \) and \( \mathcal{U}_0 := I_U \).

For any \((\pi, V) \in \mathcal{R}(G)\), we define a projector \( P_I^\chi : V \to V^\rho \) by
\[
P_I^\chi(v) = \frac{1}{|I|} \int_I \rho(k)^{-1} \pi(k)v \, dk.
\]
It is clear that \( P_I^\chi \) really is a projector \( V \to V^\rho \).

Write \( V^\chi_{\mathcal{U}_0} \) for the set of \( v \in V \) which are fixed by \( \pi(U_0) \) and transform under \( \pi(t) \), \( t \in \circ T \), by the scalar \( \chi(t) \). Recall that we define \( P_{U_0}(v) := \frac{1}{|U_0|} \int_{U_0} \pi(k)v \, dk \).

Lemma 6.0.3 (Jacquet’s Lemma I). Let \( v \in V^\chi_{\mathcal{U}_0} \). Then \( P_I^\chi(v) = P_{U_0}(v) \) and has the same image in \( V_U \) as \( v \).

Proof. Writing the integral over \( I = U_0 \circ T \mathcal{U}_0 \) as an iterated integral proves the desired equality. The rest follows from a basic property of the operator \( P_{U_0} \). \(\Box\)

Recall we assume \((\pi, V) \in \mathcal{R}(G)\) is irreducible, hence admissible.

\( V^\rho \to V^\chi_U \) is surjective: The \( \circ T \)-morphism \( V^\chi \to V^\chi_U \) is surjective. Since \( V^\chi_U \) is finite-dimensional, there is a finite-dimensional subspace \( W \subset V^\chi \) which still surjects onto \( V^\chi_U \).

Choose a compact open subgroup \( \mathcal{U}_1 \subset \mathcal{U}_0 \) such that \( W \subset V^\chi_{\mathcal{U}_1} \).

Let \( T^+ \) denote the monoid of “positive” elements in \( T(F) \), i.e., those in a subset of the form \( \varpi^\nu \circ T \) where \( \nu \) is \( B \)-dominant. (This notion does not depend on the choice of \( \varpi \).)

Choose \( a \in T^+ \) such that \( a^{-1} \mathcal{U}_0 \mathcal{U}_1 \). Then \( \pi(a)W \subset V^{\chi_{\mathcal{U}_0}} \), and \( \pi(a)W \) has image \( \pi(a)V^\chi_U = V^\chi_U \). So, \( V^\chi_{\mathcal{U}_0} \to V^\chi_U \).

We need to prove the smaller subset \( V^\rho \subset V^{\chi_{\mathcal{U}_0}} \) still surjects onto \( V^\chi_U \). But this follows using Lemma 6.0.3: for \( v \in V^{\chi_{\mathcal{U}_0}} \), the element \( P_I^\chi(v) \) belongs to \( V^\rho \) and has the same image in \( V_U \) as \( v \). This completes the proof of the surjectivity.

\( V^\rho \to V^\chi_U \) is injective:

Lemma 6.0.4. For \( v \in V^\rho = e_\rho V \), and \( a \in T^+ \), we have
\[
\pi(\Phi_a)v = |IaI| P_I^\chi(\pi(a)v).
\]
Here the action of \( \mathcal{H}(G, \rho) \) on \( V^\rho \) is defined using the Haar measure \( dg \) which gives \( I \) measure 1, and \( |IaI| := \text{vol}_{dg}(IaI) \).

Proof. Let \( S_a \) denote any set of representatives in \( \mathcal{U}_0 \) for \( a^{-1} \mathcal{U}_0 \mathcal{U}_0 \\backslash \mathcal{U}_0 \). There is a natural bijection
\[
S_a \sim (a^{-1}Ia \cap I) \\backslash I \sim I \backslash IaI
\]
(we used $a^{-1} U_0 a \subseteq U_0$ and $U_0 \subseteq a^{-1} U_0 a$). We have

$$\pi(\Phi_a)v = \int_{I a I} \Phi_a(g) \pi(g) v \, dg$$

$$= \sum_{s \in S_a} \int_{I a s} \Phi_a(g) \pi(g) v \, dg$$

$$= |S_a| \int_I \rho^{-1}(k) \pi(k) \pi(a) v \, dk$$

$$= |S_a| \mathcal{P}_I^\chi(\pi(a)v).$$

Suppose $U_1 \subset U$ is a compact open subgroup. Let $V(U_1) = \{v \in V \mid \mathcal{P}_{U_1}(v) = 0\}$. It is easy to see that

$$\ker(V \to V_{U_1}) = \bigcup_{U_1} V(U_1).$$

**Lemma 6.0.5** (Jacquet’s Lemma II). Suppose $v \in V^\rho \cap V(U_1)$ for some compact open subgroup $U_1 \subset U$. Suppose $a \in T^+$ satisfies $U_1 \subset a^{-1} U_0 a$. Then $\mathcal{P}_{U_0}(\pi(a)v) = 0$.

**Proof.** The vanishing of $\pi(a) \int_{U_0} \pi(a^{-1}ua)v \, du$ follows from the vanishing of $\int_{U_1} \pi(u)v \, du$, since $U_1$ is a subgroup of $a^{-1} U_0 a$. □

Now we can complete the proof of the injectivity. Suppose $v \in V^\rho$ maps to zero in $V_{U_1}^\chi$. Choose $U_1$ and $a \in T^+$ satisfying the hypotheses of Lemma 6.0.5. Note that $\pi(a)v \in V_{U_0}^\chi$. Then using Jacquet’s Lemmas I and II together with Lemma 6.0.4, we see

$$0 = \mathcal{P}_{U_0}(\pi(a)v) = \mathcal{P}_I^\chi(\pi(a)v) = |IaI|^{-1} \pi(\Phi_a)v.$$

Since $\Phi_a$ is invertible (Corollary 5.2.2), this implies $v = 0$, which is what we needed to show.

This completes the proof of Proposition 6.0.1. □

7. **Proof of Theorem 3.0.2 using Proposition 6.0.1**

Let $(\pi, V) \in \mathcal{R}(G)$ be irreducible. Replacing $\chi$ with a Weyl-conjugate if necessary (cf. Remark 6.0.2), we see that $\pi \in \mathcal{R}_d(G)$ iff there exists some $\eta \in X^\text{ur}(T)$ such that $\pi \hookrightarrow i_B^G(\tilde{\chi}\eta)$. By Frobenius reciprocity $\text{Hom}_G(V, i_B^G(\tilde{\chi}\eta)) = \text{Hom}_T(V_U, \mathbb{C}^{i_B^{1/2}\chi_{\eta}})$ this is equivalent to:

- $\exists$ non-zero $V_U \to \mathbb{C}^{i_B^{1/2}\chi_{\eta}}$, for some $\eta \in X^\text{ur}(T)$

$\iff (V_U \tilde{\chi}^{-1})^* \text{ has a } \circ T\text{-invariant vector which is an eigenvector for } T/\circ T$

$\iff (V_U \tilde{\chi}^{-1})^* \text{ has a } \circ T\text{-invariant vector (since } T/\circ T \text{ is abelian)}$

$\iff (V_U \tilde{\chi}^{-1})^\circ T \neq 0$

$\iff V_U^\chi \neq 0$

$\iff V^\rho \neq 0$

$\iff \pi \in \mathcal{R}_\rho(G)$,
where of course \((\ast)\) comes from Proposition 6.0.1. This completes the proof. □

8. Remarks on constructing the Hecke algebra isomorphisms

8.1. Intertwining sets. Let \(\rho : K \to \mathbb{C}^\times\) be a smooth character.

**Definition 8.1.1.** We define the intertwining set \(I_G(\rho) \subset G\) by requiring that \(g \in I_G(\rho)\) iff
\[
\rho|_{K \cap sK} = g\rho|_{K \cap sK}.
\]
Equivalently, there exists \(\phi \neq 0\) in \(\mathcal{H}(G, \rho)\) supported on \(KgK\). [For one direction, if such a \(\phi\) exists, note that for \(k \in K \cap gK\) we have
\[
\rho(k)^{-1}\phi(g) = \phi(kg) = \phi(gg^{-1}k) = \phi(g)\rho(g^{-1}k)^{-1}.
\]

**Lemma 8.1.2** ([Ro], Prop. 4.1). Let \(K = I\) and \(\rho = \rho_\chi\). Then

(i) \(I_G(\rho) \cap N = N_\chi\);
(ii) \(I_G(\rho) = IN_\chi I\).

The lemma shows that the set \([\mathcal{H}nI]\chi, n \in N_\chi / N_\chi \cap I \cong \widetilde{W}_\chi\) forms a \(\mathbb{C}\)-basis for \(\mathcal{H}(G, \rho)\).

**Proof.** (i): If \(n \in N \cap I_G(\rho)\), then \(n\rho|_{I \cap nI} = \rho|_{I \cap nI}\), which implies that \(n\chi|_{T} = \chi|_T\), hence \(n\chi = \chi\), i.e., \(n \in N_\chi\).

Conversely, suppose \(n \in N_\chi\) maps to \(w \in N/T = W\). We want to show: for \(i \in I \cap nI\), we have \(\rho(i) = \rho(n^{-1}in)\). Write \(i = i_- i_0 i_+ \in I\mathcal{T} T I_U\). Then
\[
I \ni n^{-1}in = n^{-1}i_- n \cdot n^{-1}i_0 n \cdot n^{-1}i_+ n \in \overline{U} T U',
\]
for \(U' := w^{-1}Uw\), and \(\overline{U} := w^{-1}Uw\). Since \(n^{-1}in \in I\) can also be expressed using the Iwahori decomposition as an element in \(I\mathcal{T} T I_U\), and the expressions in \(\overline{U} T U\) are unique, we see that
\[
n^{-1}i_- n \in I_{\overline{U}}, \quad n^{-1}i_+ n \in I_U,
\]
and in particular these elements belong to \(I^+\). Using this, we see
\[
\rho(n^{-1}in) = \chi(n^{-1}i_0 n) = n\chi(i_0) = \chi(i_0) = \rho(i).
\]
This completes part (i), and (ii) is a consequence of (i). □

8.2. Presentation for \(\text{End}(\text{ind} \rho^{-1})\). Recall there is a canonical isomorphism
\[
\mathcal{H}(G, \rho) \cong \text{End}_G(\text{ind} \rho^{-1})
\]
(Lemma 4.0.2). Therefore, we just need to find generators and relations for the right hand side.

Fix an extension \(\tilde{\chi} : N_\chi \to \mathbb{C}^\times\) of \(\chi\). For \(w \in \widetilde{W}_\chi\), choose an element \(n \in N_\chi\) mapping to it. We consider the element \(\Theta_n \in \text{End}_G(\text{ind} \rho^{-1})\) defined by
\[
(8.2.1) \quad \Theta_n(f)(x) = \frac{1}{|I^+|} \int_{I^+} f(n^{-1}ux) \, du, \quad (f \in \text{ind} \rho^{-1}).
\]
Here, $|I^+| := \text{vol}_d(I^+)$. Write $I_w^+ := I^+ \cap wI^+ \backslash I^+$ and $|I_w^+| := |I^+|/|I^+ \cap wI^+|$ (the ratio of the volumes). Since $\rho$ is trivial on $I^+$, we see that

$$
(8.2.2) \quad \Theta_n(f)(x) = \frac{1}{|I_w^+|} \int_{I_w^+} f(n^{-1}ux) \, du.
$$

Note that since $I = {}^tTI^+ = (I \cap wI)I^+$ and $I^+ \cap wI = I^+ \cap wI^+$, there is a canonical isomorphism

$$
I_w^+ \cong I \cap wI \backslash I,
$$

and

$$
(8.2.3) \quad |I_w^+| = |I : I \cap wI| = q^{\ell(w)}.
$$

**Lemma 8.2.1.** Let $n \in N_G$ and let $w$ denote its image in $\widetilde{W}_G$ (and write $n = n_w$).

(i) $\Theta_n \in \text{End}_G(\text{ind} \rho^{-1})$.

(ii) For $n \in N_G$, let $\Phi_n$ denote the unique element in $\mathcal{H}(G, \rho)$ which is supported on $\text{InI}$ and takes value 1 at $n$. Then $t_{\Phi_n} = q^{\ell(w)}\Theta_n$.

(iii) $\{\Theta_{nw}\}_{w \in \widetilde{W}_G}$ is a $\mathbb{C}$-basis for $\text{End}_G(\text{ind} \rho^{-1})$.

(iv) Let $n_i = n_{w_i}$ for $i = 1, 2$. If $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$, then $\Theta_{n_1n_2} = \Theta_{n_1} \circ \Theta_{n_2}$.

**Proof.** (i): We need to check that $\Theta_n(f) \in \text{ind} \rho^{-1}$. Write $i \in I$ as $i = ti_+$ for $t \in {}^tTI$ and $i_+ \in I^+$ (not a unique expression). Then since $\text{Ad}(t)$ is a measure-preserving automorphism of $I^+$, we have

$$
|I^+| \theta_n(f)(ix) = \int_{I^+} f(n^{-1}uti^+x) \, du = \int_{I^+} f(n^{-1}tn \cdot n^{-1}ui^+x) \, du
$$

$$
= n \chi(t)^{-1} \int_{I^+} f(n^{-1}ux) \, du
$$

$$
= \rho(i)^{-1} \int_{I^+} f(n^{-1}ux) \, du,
$$

since $n \chi(t) = \chi(t) = \rho(i)$.

(ii): By Lemma 4.0.2, it is enough to prove $\phi_{\Theta_n} = q^{-\ell(w)}\Phi_n$. Recalling $W = \mathbb{C}$ and letting $w = 1 \in \mathbb{C}$, we have

$$
\phi_{\Theta_n}(g)(w) = \Theta_n(e_w)(g)
$$

$$
= \frac{1}{|I^+|} \int_{I^+} e_w(n^{-1}ug) \, du.
$$

This is non-zero only if $n^{-1}ug \in I$ for some $u \in I^+$, i.e., only if $g \in \text{InI}$. Therefore $\phi_{\Theta_n} \in \mathcal{H}(G, \rho)$ is supported on $\text{InI}$. It remains to check its value at $g = n$. We find it is

$$
\frac{1}{|I^+|} \int_{I^+ \cap wI} e_w(n^{-1}un) \, du = \frac{|I^+ \cap wI^+|}{|I^+|} = q^{\ell(w)}
$$

(cf. (8.2.3)).
(iii): This is proved in greater generality in [Mor], 5.4, 5.5. Alternatively, we can use the fact that we have proved \( \{ \Phi_{n_w} \}_{w \in \widetilde{W}_\chi} \) is a basis for \( \mathcal{H}(G, \rho) \) (Lemma 8.1.2), together with part (ii).

(iv): This is proved in [Mor], Prop. 5.10. Alternatively, it is an easy consequence of (8.2.2) and standard calculations.

\[ \text{□} \]

From now on we want to choose the family \( \{ n_w \} \) in a compatible way: we require that \( n_{w_1 w_2} = n_{w_1} n_{w_2} \) whenever \( \ell(w_1 w_2) = \ell(w_1) + \ell(w_2) \). It is always possible to do this (see [Mor], 5.2).

Note that \( \Theta_n \) depends on \( n \mapsto w \) and not just on \( w \). So, we define a new basis element in \( \mathcal{H}(G, \rho) \) by

\[ B_w := \check{\chi}(n)^{-1} \Theta_n. \]

This indeed depends just on \( w \) (and \( \check{\chi} \), of course). We also define

\[ T_w := q^{(\ell(\chi(w)) + \ell(w))/2} B_w = q^{(\ell(\chi(w)) - \ell(w))/2} \check{\chi}(n_w)^{-1} t_{\Phi_n w} \]

for \( w \in \widetilde{W}_\chi \). The main computation in this subject shows that these elements \( T_w \) generate the algebra \( \mathcal{H}_\chi \):

**Theorem 8.2.2** (Goldstein [Gol], Morris [Mor]). The elements \( T_w, w \in \widetilde{W}_\chi \) satisfy the following relations:

(i) \( T_{w_1 w_2} = T_{w_1} T_{w_2} \), if \( \ell(\chi(w_1 w_2)) = \ell(\chi(w_1)) + \ell(\chi(w_2)) \)

(ii) \( T_s^2 = (q - 1)T_s + qT_1 \), if \( s \in S_{\chi, \text{aff}} \).

Thus, the algebra \( \text{End}_G(\text{ind} \rho^{-1}) \) is isomorphic to \( \mathcal{H}_\chi^{-1} = \mathcal{H}_\chi \), by an isomorphism which depends only on the choice of \( \check{\chi} \).

8.3. **Proof of Theorem 5.2.1.** We can now see that the isomorphism \( t_* \) of Lemma 4.0.2 gives the desired algebra isomorphism

\[ \mathcal{H}(G, \rho) \xrightarrow{\sim} \mathcal{H}_\chi. \]

Indeed, by Lemma 8.2.1 and our definitions, \( t_* \) takes

\[ q^{-\ell(w)/2} \check{\chi}^{-1}(n) \Phi_n \]

to

\[ q^{-\ell(w)/2} \check{\chi}^{-1}(n) \Theta_n q^{\ell(w)/2} B_w = q^{-\ell(w)/2} T_w. \]

This completes the (sketch of the) proof of Theorem 5.2.1. \( \text{□} \)
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