ON HECKE ALGEBRA ISOMORPHISMS AND TYPES FOR DEPTH-ZERO PRINCIPAL SERIES

THOMAS J. HAINES

Abstract. These lectures describe Hecke algebra isomorphisms and types for depth-zero principal series blocks, a.k.a. Bernstein components $\mathcal{R}_{\mathfrak{s}}(G)$ for $\mathfrak{s} = \mathfrak{s}_\chi = [T, \widetilde{\chi}]_G$, where χ is a depth-zero character on $T(\mathcal{O})$. (Here T is a split maximal torus in a p-adic group G.) We follow closely the treatment of A. Roche [Ro] with input from D. Goldstein [Gol] and L. Morris [Mor]. We give an elementary proof that (I, ρ_χ) is a type for $\mathfrak{s}_\chi$, in the sense of Bushnell-Kutzko [BK]. This is a very special case of a result of Roche [Ro]. Our method is to imitate Casselman's proof of Borel's theorem on unramified principal series (the case $\chi = 1$ of the present theorem).

In contrast to the situation for general principal series blocks (see [Ro]), in the depth-zero case there is no restriction on the residual characteristic of F.

1. Notation

We let F denote an arbitrary p-adic field with ring of integers $\mathcal{O}$, and residue field k_F. Let q denote the cardinality of k_F. Write ϖ for a uniformizer.

Let G denote a connected reductive group, defined and split over $\mathcal{O}$. Fix an F-split maximal torus T and a Borel subgroup B containing T; assume T and B are defined over $\mathcal{O}$. Let ${}^\circ T = T(\mathcal{O})$ denote the maximal compact subgroup of $T(F)$. Let $\Phi \subset X^*(T)$ resp. $\Phi^\vee \subset X_*(T)$ denote the set of roots resp. coroots for G, T. Let U resp. $\overline{U}$ denote the unipotent radical of B resp. the Borel subgroup $\overline{B} \supset T$ opposite to B.

The symbol I will stand for an Iwahori subgroup of $G(F)$, which we shall assume it is in "good position" with respect to T: the alcove $\mathbf{a}$ in the building for $G(F)$ which is fixed by I is contained in the apartment corresponding to T.

Let dx denote a Haar measure on G. Denote the group of unramified characters of $G(F)$ by $X^{\mathrm{ur}}(G)$ (see [BD] or [Be92] for the definition).

Let $\mathcal{R}(G)$ denote the category of smooth representations of $G(F)$.

Let L denote an F-Levi subgroup of G (by definition, $L = C_G(A_L)$ for some F-split torus A_L in G). Let $P = LN$ denote an F-parabolic subgroup, that is, a parabolic subgroup defined over F, with unipotent radical N and with L as a Levi factor. Let σ denote any smooth representation of L, and define the normalized parabolic induction by

$$i_P^G(\sigma) = \mathrm{Ind}_P^G(\delta_P^{1/2}\, \sigma),$$

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where $\delta_P(l) := |\det(\mathrm{Ad}(l); \mathrm{Lie}(N(F)))|_F$. Here $|\cdot|_F$ denotes the normalized absolute value on F.

Throughout these notes, we will frequently write G (resp. B,T, etc.) when we really mean $G(F)$ (resp. $B(F)$, $T(F)$, etc.).

2. Bernstein decomposition (review)

A cuspidal pair (L, σ) consists of an F-Levi subgroup L of G, together with a supercuspidal representation σ of $L(F)$.

The group $G = G(F)$ acts "by conjugation" on cuspidal pairs: $g \cdot (L, \sigma) = ({}^gL, {}^g\sigma)$, where ${}^gL = gLg^{-1}$ and ${}^g\sigma(\cdot) = \sigma(g^{-1} \cdot g)$. Denote by $(L, \sigma)_G$ the G-conjugation class of (L, σ).

Let (L, σ) denote a cuspidal pair. We say (L_1, σ_1) is *inertially equivalent* to (L_2, σ_2) if there exists $g \in G(F)$ and $\chi \in X^{\mathrm{ur}}(L_2)$ such that ${}^gL_1 = L_2$ and ${}^g\sigma_1 \otimes \chi = \sigma_2$.

Let $\mathfrak{s} = [L, \sigma]_G$ denote the inertial equivalence class of (L, σ) (with respect to G). Note that $\mathfrak{s}$ depends only on $(L, \sigma)_G$. Also $\mathfrak{s}$ is a union of G-conjugacy classes of cuspidal pairs.

Fact: For $\pi \in \mathcal{R}(G)$ irreducible, there exist a (unique up to G-conjugacy) cuspidal pair (L, σ) such that π is a subquotient of $i_P^G(\sigma)$. Here $P = LN$ is an F-parabolic with unipotent radical N which has L as a Levi factor.

We call the class $(L, \sigma)_G$ as above the *supercuspidal support* of π.

Denote by $\mathcal{R}_{\mathfrak{s}}(G)$ the full subcategory of $\mathcal{R}(G)$ whose objects are the representations π each of whose irreducible subquotients has supercuspidal support belonging to the inertial class $\mathfrak{s}$. Once we fix a cuspidal pair (L, σ) in $\mathfrak{s}$, we may reformulate the condition for π to belong to $\mathcal{R}_{\mathfrak{s}}(G)$ as: every irreducible subquotient of π is a subquotient of some $i_P^G(\sigma\chi)$, $\chi \in X^{\mathrm{ur}}(L)$.

Theorem 2.0.1 (Bernstein decomposition)**.** $\mathcal{R}(G) = \prod_{\mathfrak{s}} \mathcal{R}_{\mathfrak{s}}(G)$.

Definition 2.0.2. An $\mathfrak{s}$-*type* is a pair (K, ρ) consisting of a compact open subgroup $K \subset G$ together with an irreducible smooth representation $\rho : K \to \mathrm{End}_{\mathbb{C}}(W)$ such that an irreducible $\pi \in \mathcal{R}(G)$ belongs to $\mathcal{R}_{\mathfrak{s}}(G)$ iff $\pi|_K \supset \rho$.

Now let ρ be any irreducible smooth representation of K, on a vector space W. We define $e_\rho \in \mathcal{H}(G) = C_c^\infty(G, dx)$ by

$$e_\rho(x) = \begin{cases} dx(K)^{-1} \dim(\rho) \, \mathrm{tr}_W(\rho(x^{-1})), & x \in K \\ 0, & x \notin K. \end{cases}$$

For any irreducible smooth representations ρ, ρ' of K, we have $e_\rho *_{dx} e_{\rho'} = \delta_{\rho,\rho'} e_\rho$, where $\delta_{\rho,\rho'} \in \{0, 1\}$ vanishes unless ρ and ρ' are equivalent. This is an exercise using the Schur orthogonality relations on the group K. In particular, e_ρ is an idempotent of the algebra $\mathcal{H}(G)$.

If $\rho = 1$ (the trivial character) we write e_K in place of e_ρ.

For any $(\pi, V) \in \mathcal{R}(G)$, denote by V^ρ the ρ-isotypical component of V. We have $V^\rho = e_\rho V$. Also, we let $V[\rho] = \mathcal{H}(G) \cdot V^\rho$, the G-submodule of V generated by V^ρ. Below we will often write π^ρ in place of V^ρ.

We define $\mathcal{R}_\rho(G)$ to be the full subcategory of $\mathcal{R}(G)$ whose objects (π, V) satisfy $V = V[\rho]$. There is a functor

$$\begin{aligned} \mathcal{R}_\rho(G) &\to e_\rho \mathcal{H}(G) e_\rho\text{-Mod} \\ (\pi, V) &\mapsto \pi^\rho. \end{aligned} \tag{2.0.1}$$

Proposition 2.0.3. *If (K,ρ) is an $\mathfrak{s}$-type, then (2.0.1) is an equivalence of categories. Moreover, in that case $\mathcal{R}_\mathfrak{s}(G) = \mathcal{R}_\rho(G)$ as subcategories of $\mathcal{R}(G)$.*

We will postpone the proof of this proposition to section 4.

3. Depth-zero principal series blocks

Example. Consider an Iwahori subgroup I in good position with respect to the torus T (this means that I fixes an alcove $\mathbf{a}$ in the apartment of the building for $G(F)$ corresponding to T). Also, for any Borel subgroup $B = TU$ containing T, with opposite Borel $\overline{B} = T\overline{U}$, we have the Iwahori decomposition

$$I = I_U \cdot {}^\circ T \cdot I_{\overline{U}}, \tag{3.0.2}$$

where $I_U := U \cap I$, $I_{\overline{U}} := \overline{U} \cap I$, and ${}^\circ T := T(\mathcal{O}) = T \cap I$.

The inertial class $\mathfrak{s} := [T, 1]_G$ indexes the *Iwahori block* $\mathcal{R}_\mathfrak{s}(G)$. A famous theorem of Borel asserts that an irreducible $\pi \in \mathcal{R}(G)$ is a constituent of an unramified principal series $i_B^G(\eta)$, $\eta \in X^{\mathrm{ur}}(T)$, if and only if $\pi^I \neq 0$. That is, $(I, 1)$ is an $\mathfrak{s}$-type. This is a special case of the theorem we will prove below (Theorem 3.0.2).

It turns out that $e_I \mathcal{H}(G) e_I = \mathcal{H}(G, I)$, the Iwahori-Hecke algebra (see below). In conjunction with the Proposition 2.0.3, we thus recover the finer result of Borel which asserts that

$$\pi \mapsto \pi^I$$

gives an equivalence of categories between the Iwahori block and the category $\mathcal{H}(G,I)$-Mod.

Fix a character $\chi : {}^\circ T \to \mathbb{C}^\times$.

Definition 3.0.1. We say χ is *depth-zero* if χ factors through the quotient ${}^\circ T \to T(k_F)$ (and we denote the factoring $T(k_F) \to \mathbb{C}^\times$ also by χ).

Choose any extension of χ to a character $\widetilde{\chi} : T(F) \to \mathbb{C}^\times$. Consider the inertial class

$$\mathfrak{s} := [T, \widetilde{\chi}]_G.$$

Since $\mathfrak{s}$ depends only on the W-orbit of χ, we may also write $\mathfrak{s}_\chi$ for $\mathfrak{s}$.

Let I be an Iwahori in good position relative to T, as above. Let I^+ denote the pro-unipotent radical of I. There is an obvious isomorphism

$${}^\circ T / {}^\circ T \cap I^+ \xrightarrow{\sim} I/I^+$$

so that χ determines a character $\rho = \rho_\chi : I \to \mathbb{C}^\times$, which is trivial on I^+. In terms of the Iwahori decomposition (3.0.2), ρ is given by

$$\rho(u \cdot t_0 \cdot \overline{u}) = \chi(t_0),$$

for $u \in I_U$, $t_0 \in {}^\circ T$, and $\overline{u} \in I_{\overline{U}}$.

Theorem 3.0.2. *If $\mathfrak{s} = \mathfrak{s}_\chi$ as above, then (I, ρ) is an $\mathfrak{s}$-type.*

We shall prove this by imitating Casselman's proof of Borel's theorem on unramified principal series. One crucial ingredient is the theory of Hecke algebra isomorphisms for depth-zero principal series types, which we will review in section 5.

4. Proof of Proposition 2.0.3

We are in the general situation, where (K, ρ) is a smooth irreducible representation on a vector space W (ie. ρ is not necessarily a character).

Lemma 4.0.1. *Fix an inertial class $\mathfrak{s}$.*

- (i) *(K, ρ) is an $\mathfrak{s}$-type $\iff$ $\mathrm{ind}\,\rho := \mathrm{c{-}Ind}_K^G\,\rho$ is a generator for $\mathcal{R}_\mathfrak{s}(G)$, i.e., $\mathrm{ind}\,\rho \in \mathcal{R}_\mathfrak{s}(G)$ and $\mathrm{Hom}_G(\mathrm{ind}\,\rho,\, \pi) \neq 0$ for all $\pi \neq 0$ in $\mathcal{R}_\mathfrak{s}(G)$.*
- (ii) *In that case $\mathcal{R}_\mathfrak{s}(G) = \mathcal{R}_\rho(G)$ as subcategories of $\mathcal{R}(G)$. In particular $\mathcal{R}_\rho(G)$ is closed under extensions and subquotients.*

Proof. First, by Frobenius reciprocity (cf. [Ro],(7.1)) we have

$$\mathrm{Hom}_G(\mathrm{ind}\,\rho, \pi) = \mathrm{Hom}_K(\rho, \pi).$$

This implies that $\mathrm{ind}\,\rho$ is a projective object in $\mathcal{R}(G)$. (It is also true that $\mathrm{ind}\,\rho$ is finitely-generated as a G-module.)

Now let us prove (i).

($\Rightarrow$): Suppose $(\pi, V) \in \mathcal{R}_\mathfrak{s}$ is non-zero. Since all irreducible subquotients of π are also in $\mathcal{R}_\mathfrak{s}$ (hence contain ρ) and representations of K are completely reducible, it follows that $\mathrm{Hom}_K(\rho, \pi) \neq 0$ and hence $\mathrm{Hom}_G(\mathrm{ind}\,\rho, \pi) \neq 0$.

Next we claim that $\mathrm{ind}\,\rho \in \mathcal{R}_\mathfrak{s}$. If not, then $\mathrm{ind}\,\rho$ possesses a non-zero quotient τ in some $\mathcal{R}_\mathfrak{t}$ with $\mathfrak{t} \neq \mathfrak{s}$. Since τ is finitely-generated (as $\mathrm{ind}\,\rho$ is), it possesses an irreducible quotient; we may assume τ is itself irreducible. But then $\mathrm{Hom}_K(\rho, \tau) \neq 0$ implies that $\tau \supset \rho$ and this means that (K, ρ) is not an $\mathfrak{s}$-type.

($\Leftarrow$): Let $(\pi, V) \in \mathcal{R}(G)$ be irreducible and non-zero. Then

$$\begin{aligned}\pi \in \mathcal{R}_\mathfrak{s}(G) &\iff \mathrm{Hom}_G(\mathrm{ind}\,\rho, \pi) \neq 0 \\ &\iff \mathrm{Hom}_K(\rho, \pi) \neq 0 \\ &\iff \pi \in \mathcal{R}_\rho(G).\end{aligned}$$

The first ($\Longleftarrow$) holds because $\mathrm{ind}\,\rho$, hence any of its quotients, lies in $\mathcal{R}_\mathfrak{s}(G)$.

This completes the proof of (i).

Now let us prove (ii). Suppose $(\pi, V) \in \mathcal{R}_{\mathfrak{s}}(G)$. We have $(V/V[\rho])^\rho = 0$. But then $V/V[\rho] = 0$, since non-zero objects in $\mathcal{R}_{\mathfrak{s}}(G)$ contain ρ. So $V = V[\rho]$, that is, $\pi \in \mathcal{R}_\rho(G)$.

Conversely, if $V = V[\rho]$, then π is a quotient of a direct sum of copies of $\operatorname{ind}\rho \in \mathcal{R}_{\mathfrak{s}}(G)$, hence $\pi \in \mathcal{R}_{\mathfrak{s}}(G)$.

□

Exercise: Since $\operatorname{ind}\rho$ is projective in $\mathcal{R}(G)$ and a generator for $\mathcal{R}_{\mathfrak{s}}(G)$ (i.e. $\operatorname{ind}\rho \in \mathcal{R}_{\mathfrak{s}}(G)$ and $\operatorname{Hom}_G(\operatorname{ind}\rho, \pi) \neq 0$ for every $\pi \neq 0$ in $\mathcal{R}_{\mathfrak{s}}(G)$), every $\pi \in \mathcal{R}_{\mathfrak{s}}(G)$ is a quotient of a direct sum of copies of $\operatorname{ind}\rho$. (Consider the maximal subobject in π which is a quotient of a direct sum of copies of $\operatorname{ind}\rho$.)

We have shown that $\operatorname{ind}\rho$ is a f.g. projective generator of $\mathcal{R}_{\mathfrak{s}}(G)$. From this, general categorical arguments ([Ba]) give (Morita) equivalences of categories

$$\begin{aligned} \mathcal{R}_{\mathfrak{s}}(G) &\approx \operatorname{End}_G(\operatorname{ind}\rho)^{\mathrm{opp}}\text{-Mod} \approx \operatorname{End}_G(\operatorname{ind}\rho)^{\mathrm{opp}} \otimes \operatorname{End}_{\mathbb{C}} W\text{-Mod} \\ \pi &\mapsto \operatorname{Hom}_G(\operatorname{ind}\rho, \pi) \mapsto \operatorname{Hom}_G(\operatorname{ind}\rho, \pi) \otimes W \\ & t.f = f \circ t. \end{aligned}$$

Therefore, we need to relate $\operatorname{End}_G(\operatorname{ind}\rho)^{\mathrm{opp}} \otimes \operatorname{End}(W)$ to $e_\rho \mathcal{H}(G) e_\rho$. First we define

$$\mathcal{H}(G, \rho^\vee) = \{\Phi : G \to \operatorname{End}(W) \mid \Phi(k_1 g k_2) = \rho(k_1)\Phi(g)\rho(k_2),\ \forall k_i \in K, g \in G\}.$$

Here the functions Φ are assumed to be smooth with compact support. Also, $(\rho^\vee, W^\vee)$ is the representation given by $\rho^\vee(k) := \rho(k^{-1})^\vee \in \operatorname{End}(W^\vee)$. We view $\mathcal{H}(G, \rho^\vee)$ as a convolution algebra using the Haar measure dx giving K volume 1.

The following lemma is left to the reader.

Lemma 4.0.2. *We have mutually inverse algebra isomorphisms*

$$\phi \mapsto t_\phi \ :\ \mathcal{H}(G, \rho^\vee) \overset{\longrightarrow}{\longleftarrow} \operatorname{End}_G(\operatorname{ind}\rho) \ :\ t \mapsto \phi_t,$$

where

$$\begin{aligned} t_\phi(f)(g) &= \int_G \phi(x)(f(x^{-1}g))\, dx && (f \in \operatorname{ind}\rho,\ g \in G) \\ \phi_t(g)(w) &= t(e_w)(g) && (g \in G, w \in W). \end{aligned}$$

Here $e_w \in \operatorname{ind}\rho$ *is defined by*

$$e_w(g) = \begin{cases} \rho(k)w, & g = k \in K \\ 0, & g \notin K. \end{cases}$$

Furthermore, there is an anti-isomorphism of algebras

$$\begin{aligned} \mathcal{H}(G, \rho^\vee) &\xrightarrow{\sim} \mathcal{H}(G, \rho) \\ \Phi &\mapsto \Phi' \end{aligned}$$

given by $\Phi'(g) := \Phi(g^{-1})^\vee \in \operatorname{End}(W^\vee)$.

Finally, Roche checks in [Ro], p. 390, that there is an algebra isomorphism

$$\mathcal{H}(G,\rho)\otimes_{\mathbb{C}}\mathrm{End}(W)\ \widetilde{\to}\ e_\rho\mathcal{H}(G)e_\rho$$
$$\Phi\otimes(w\otimes w^\vee)\ \mapsto\ (g\mapsto \dim\rho\,\langle w,\Phi(g)w^\vee\rangle)\qquad (w\in W,\ w^\vee\in W^\vee).$$

In case ρ is a character, the last isomorphism gives $\mathcal{H}(G,\rho)\cong e_\rho\mathcal{H}(G)e_\rho$ and is immediate.

Putting these isomorphisms together, we get isomorphisms

$$\mathrm{End}_G(\mathrm{ind}\,\rho)^{\mathrm{opp}}\otimes\mathrm{End}(W)\ \widetilde{\to}\ \mathcal{H}(G,\rho)\otimes\mathrm{End}(W)\ \widetilde{\to}\ e_\rho\mathcal{H}(G)e_\rho.$$

In loc. cit. Roche checks that the induced categorical equivalence

$$\mathcal{R}_{\mathfrak{s}}(G)=\mathcal{R}_\rho(G)\ \widetilde{\to}\ e_\rho\mathcal{H}(G)e_\rho\text{-Mod}$$

is

$$(\pi,V)\mapsto\mathrm{Hom}_G(\mathrm{ind}\,\rho,\pi)=\mathrm{Hom}_K(\rho,\pi)=\pi^\rho.$$

(Again, this is quite immediate in the case where ρ is a character.) This completes the proof of Proposition 2.0.3.

5. Hecke algebra isomorphisms

To prove Theorem 3.0.2, we need to review Hecke algebra isomorphisms. We follow Roche's treatment [Ro].

5.1. **Preliminaries.** As before, fix a depth-zero character $\chi:{}^\circ T\to\mathbb{C}^\times$, and let $\mathfrak{s}=[T,\widetilde{\chi}]_G=\mathfrak{s}_\chi$, for any extension $\widetilde{\chi}:T(F)\to\mathbb{C}^\times$ of χ. Also, write $\rho=\rho_\chi$ for the associated character $\rho: I=I_U\cdot{}^\circ T\cdot I_{\overline{U}}\to\mathbb{C}^\times,\ ut\overline{u}\mapsto\chi(t)$.

Let N denote the normalizer of T in G, let $W=N/T=N(F)/T(F)$ denote the Weyl group, and write $\widetilde{W}=N(F)/{}^\circ T$ for the Iwahori-Weyl group. There is a canonical isomorphism $X_*(T)=T(F)/{}^\circ T,\ \lambda\mapsto\varpi^\lambda:=\lambda(\varpi)$ (independent of the choice of ϖ). The canonical homomorphism $N(F)/{}^\circ T=\widetilde{W}\to W=N(F)/T(F)$ has a (non-canonical) section, hence there is a (non-canonical) isomorphism $\widetilde{W}=X_*(T)\rtimes W$.

Clearly $N(F)$, $\widetilde{W}$ and W act on the set of depth-zero characters. We define

$$N_\chi=\{n\in N(F)\mid n\chi=\chi\}$$
$$\widetilde{W}_\chi=\{w\in\widetilde{W}\mid w\chi=\chi\}$$
$$W_\chi=\{w\in W\mid w\chi=\chi\}.$$

There are obvious surjective homomorphisms $N_\chi\to\widetilde{W}_\chi\to W_\chi$.

Define Φ_χ (resp. $\Phi_\chi^\vee$ resp. $\Phi_{\chi,\mathrm{aff}}$) to be the set of roots $\alpha\in\Phi$ (resp. coroots $\alpha^\vee\in\Phi^\vee$ resp. affine roots $a=\alpha+k$, where $\alpha\in\Phi$, $k\in\mathbb{Z}$) such that $\chi\circ\alpha^\vee|_{\mathcal{O}_F^\times}=1$. Note that $\widetilde{W}_\chi$ acts in an obvious way on $\Phi_{\chi,\mathrm{aff}}$. Define the following subgroups of the group of affine-linear automorphisms of $V:=X_*(T)\otimes\mathbb{R}$:

$$W_\chi^\circ=\langle s_\alpha\mid\alpha\in\Phi_\chi\rangle$$
$$W_{\chi,\mathrm{aff}}=\langle s_a\mid a\in\Phi_{\chi,\mathrm{aff}}\rangle.$$

Here s_a and s_α are the reflections on V corresponding to a and α.

Let Φ^+ denote the B-positive roots in Φ, and set $\Phi_\chi^+ = \Phi_\chi \cap \Phi^+$. Then let $\mathcal{C}_\chi$ resp. $\mathbf{a}_\chi$ denote the subsets in V defined by

$$\mathcal{C}_\chi = \{v \in V \mid 0 < \alpha(v),\ \forall \alpha \in \Phi_\chi^+\},\ \text{resp.}$$
$$\mathbf{a}_\chi = \{v \in V \mid 0 < \alpha(v) < 1,\ \forall \alpha \in \Phi_\chi^+\}.$$

For $a \in \Phi_{\chi,\mathrm{aff}}$ we write $a > 0$ if $a(v) > 0$ for all $v \in \mathbf{a}_\chi$. Similarly, we define an ordering on the set $\Phi_{\chi,\mathrm{aff}}$. Then let $\Pi_{\chi,\mathrm{aff}} = \{a \in \Phi_{\chi,\mathrm{aff}} \mid a \text{ is a minimal positive element}\}$. Define

$$S_{\chi,\mathrm{aff}} = \{s_a \mid a \in \Pi_{\chi,\mathrm{aff}}\}$$
$$\Omega_\chi = \{w \in \widetilde{W}_\chi \mid w\mathbf{a}_\chi = \mathbf{a}_\chi\}.$$

It is clear that Φ_χ is a root system with Weyl group W_χ°, and that $W_\chi^\circ \subseteq W_\chi$. In general, W_χ can be larger that W_χ° and is not even a Weyl group (see Example 8.3 in [Ro] and Remark 5.1.2 below). The following results are contained in [Ro].

Lemma 5.1.1. (1) *The group $W_{\chi,\mathrm{aff}}$ is a Coxeter group with system of generators $S_{\chi,\mathrm{aff}}$;*

(2) *there is a canonical decomposition $\widetilde{W}_\chi = W_{\chi,\mathrm{aff}} \rtimes \Omega_\chi$, and the Bruhat order $\leq_\chi$ and length function ℓ_χ on $W_{\chi,\mathrm{aff}}$ can be extended in an obvious way to $\widetilde{W}_\chi$ such that Ω_χ consists of the length-zero elements;*

(3) *if $W_\chi^\circ = W_\chi$, then $W_{\chi,\mathrm{aff}}$ (resp. $\widetilde{W}_\chi$) is the affine (resp. extended affine) Weyl group associated to the root system $\Phi_\chi \subset V^*$, and $\mathcal{C}_\chi$ resp. $\mathbf{a}_\chi$ is the dominant Weyl chamber resp. base alcove in V corresponding to a set of simple positive affine roots, which can be identified with $\Pi_{\chi,\mathrm{aff}}$.*

In the situation of (3), let Π_χ denote the set of minimal elements of Φ_χ^+. This is then a set of simple positive roots for the root system Φ_χ.

Remark 5.1.2. In [Ro], pp. 393-6, Roche proves that $W_\chi^\circ = W_\chi$ at least when G has connected center and when p is not a torsion prime for $\Phi^\vee$ (see loc. cit. p. 396). It is easy to see that $W_\chi^\circ = W_\chi$ always holds when $G = \mathrm{GL}_d$ (with no restrictions on p).

On the other hand, $W_\chi \neq W_\chi^\circ$ in general, even for $G = \mathrm{SL}_n$. Indeed, suppose $G = \mathrm{SL}_n$ with $n \geq 3$. Suppose $n | q-1$ and that χ_1 is a character of $\mathbb{F}_q^\times$ of order n. Consider

$$\chi(a_1, \ldots, a_n) := \chi_1(a_1)\chi_1^2(a_2)\cdots\chi_1^n(a_n).$$

It is clear that $W_\chi^\circ = \{1\}$, but that, since $a_1 \cdots a_n = 1$, we have $W_\chi \ni (12\cdots n)$. In fact W_χ is the cyclic group of order n generated by $(12\cdots n)$.

5.2. **Statement.** Let $\mathcal{H}(W_{\chi,\mathrm{aff}})$ denote the affine Hecke algebra associated to the Coxeter group $(W_{\chi,\mathrm{aff}}, S_{\chi,\mathrm{aff}})$. It has the usual generators T_w, $w \in W_{\chi,\mathrm{aff}}$, and relations

$$T_{w_1w_2} = T_{w_1}T_{w_2}, \qquad \text{if } \ell_\chi(w_1w_2) = \ell_\chi(w_1) + \ell_\chi(w_2)$$
$$T_s^2 = (q-1)T_s + qT_1. \qquad \text{if } s \in S_{\chi,\mathrm{aff}}.$$

Let $\mathcal{H}_\chi := H(W_{\chi,\mathrm{aff}})\widetilde{\otimes}\mathbb{C}[\Omega_\chi]$, where the twisted tensor product is the usual tensor product on the underlying vector spaces, but where multiplication is given by

$$(T_{w_1} \otimes e_{\omega_1})(T_{w_1} \otimes e_{\omega_2}) = T_{w_1\omega_1(w_2)} \otimes e_{\omega_1\omega_2}$$

where $\omega(\cdot)$ refers the conjugation action of $\omega \in \Omega_\chi$ on $W_{\chi,\mathrm{aff}}$.

We write $T_{w\omega} := T_w \otimes e_\omega$.

The Hecke algebra isomorphism depends on a choice of extension $\breve{\chi} : N_\chi \to \mathbb{C}^\times$ of χ (this always exists: see [HL] 6.11 and [HR09]). Fix such a $\breve{\chi}$. Then for any $n \in N_\chi \mapsto w \in \widetilde{W}_\chi$, define

$$[InI]_{\breve{\chi}} \in \mathcal{H}(G, \rho)$$

to be the unique element in $\mathcal{H}(G, \rho)$ supported on InI and having value $\breve{\chi}^{-1}(n)$ at n. Note that $[InI]_{\breve{\chi}}$ depends on $w \in \widetilde{W}_\chi$ but not on the choice of $n \in N_\chi$ mapping to w.

Theorem 5.2.1 (Goldstein [Gol], Morris [Mor], Roche [Ro])**.** *Let χ be a depth-zero character as above. For any extension $\breve{\chi}$ of χ as above, there is an algebra isomorphism*

$$\mathcal{H}(G, \rho) \xrightarrow{\sim} \mathcal{H}_\chi,$$

which sends $q^{-\ell(w)/2}[InI]_{\breve{\chi}}$ to $q^{-\ell_\chi(w)/2}T_w$.

Let $\Phi_n := \breve{\chi}(n)[InI]_{\breve{\chi}}$, the unique element in $\mathcal{H}(G, \rho)$ supported on InI and having $\Phi_n(n) = 1$.

Corollary 5.2.2. *For any $n \in N_\chi$, the element $[InI]_{\breve{\chi}}$ (or equivalently, Φ_n) is invertible in $\mathcal{H}(G, \rho)$.*

6. The morphism $V^\rho \to V_U^\chi$

We assume $B = TU$ and I are in "good position": I fixes an alcove **a** contained in the apartment corresponding to T, and B is any Borel subgroup containing T. From χ we get ρ as usual.

For $(\pi, V) \in \mathcal{R}(G)$, let $V_U \in \mathcal{R}(T)$ denote the Jacquet module.

Proposition 6.0.1. *Suppose (π, V) is irreducible (hence, cf.* [Be92]*, admissible). Then the map $V \to V_U$ induces a $^\circ T$-equivariant isomorphism*

$$V^\rho \xrightarrow{\sim} V_U^\chi. \tag{6.0.1}$$

Remark 6.0.2. Since $B = TU$ may be replaced with any ${}^wB = T\,{}^wU$ $(w \in W)$, it follows that we may also hold B fixed and replace I with ${}^wI$. That is, we may replace χ with ${}^w\chi$ and ρ with ${}^w\rho$, where the latter is the character on ${}^wI$ defined by ${}^w\rho(\cdot) = \rho(w^{-1} \cdot w)$. Such a replacement causes no harm for the proof of the main theorem (cf. section 7) because $\pi(w) : V^\rho \xrightarrow{\sim} V^{{}^w\rho}$.

We will prove Proposition 6.0.1 using only a consequence of the Hecke algebra isomorphism, namely Corollary 5.2.2.

Proof. We change notation slightly and write the Iwahori decomposition as

$$I = \overline{U}_0\, {}^\circ T\, U_0$$

where $U_0 := I_U$ and $\overline{U}_0 := I_{\overline{U}}$.

For any $(\pi, V) \in \mathcal{R}(G)$, we define a projector $\mathcal{P}_I^\chi : V \to V^\rho$ by

$$\mathcal{P}_I^\chi(v) = \frac{1}{|I|} \int_I \rho(k)^{-1} \pi(k) v \, dk.$$

It is clear that $\mathcal{P}_I^\chi$ really is a projector $V \to V^\rho$.

Write $V^{\chi \overline{U}_0}$ for the set of $v \in V$ which are fixed by $\pi(\overline{U}_0)$ and transform under $\pi(t)$, $t \in {}^\circ T$, by the scalar $\chi(t)$. Recall that we define $\mathcal{P}_{U_0}(v) := \frac{1}{|U_0|} \int_{U_0} \pi(k) v \, dk$.

Lemma 6.0.3 (Jacquet's Lemma I)**.** *Let $v \in V^{\chi \overline{U}_0}$. Then $\mathcal{P}_I^\chi(v) = \mathcal{P}_{U_0}(v)$ and has the same image in V_U as v.*

Proof. Writing the integral over $I = U_0\, {}^\circ T\, \overline{U}_0$ as an iterated integral proves the desired equality. The rest follows from a basic property of the operator $\mathcal{P}_{U_0}$. □

Recall we assume $(\pi, V) \in \mathcal{R}(G)$ is irreducible, hence admissible.

$V^\rho \to V_U^\chi$ is **surjective**: The ${}^\circ T$-morphism $V^\chi \to V_U^\chi$ is surjective. Since V_U^χ is finite-dimensional, there is a finite-dimensional subspace $W \subset V^\chi$ which still surjects onto V_U^χ. Choose a compact open subgroup $\overline{U}_1 \subset \overline{U}_0$ such that $W \subset V^{\chi \overline{U}_1}$.

Let T^+ denote the monoid of "positive" elements in $T(F)$, i.e., those in a subset of the form $\varpi^\nu\, {}^\circ T$ where ν is B-dominant. (This notion does not depend on the choice of ϖ.)

Choose $a \in T^+$ such that $a^{-1} \overline{U}_0 a \subset \overline{U}_1$. Then $\pi(a) W \subset V^{\chi \overline{U}_0}$, and $\pi(a) W$ has image $\pi(a) V_U^\chi = V_U^\chi$. So, $V^{\chi \overline{U}_0} \twoheadrightarrow V_U^\chi$.

We need to prove the smaller subset $V^\rho \subset V^{\chi \overline{U}_0}$ still surjects onto V_U^χ. But this follows using Lemma 6.0.3: for $v \in V^{\chi \overline{U}_0}$, the element $\mathcal{P}_I^\chi(v)$ belongs to V^ρ and has the same image in V_U as v. This completes the proof of the surjectivity.

$V^\rho \to V_U^\chi$ is **injective**:

Lemma 6.0.4. *For $v \in V^\rho = e_\rho V$, and $a \in T^+$, we have*

$$\pi(\Phi_a) v = |IaI| \, \mathcal{P}_I^\chi(\pi(a) v).$$

Here the action of $\mathcal{H}(G, \rho)$ on V^ρ is defined using the Haar measure dg which gives I measure 1, and $|IaI| := \mathrm{vol}_{dg}(IaI)$.

Proof. Let S_a denote any set of representatives in $\overline{U}_0$ for $a^{-1} \overline{U}_0 a \backslash \overline{U}_0$. There is a natural bijection

$$S_a \xrightarrow{\sim} (a^{-1} I a \cap I) \backslash I \xrightarrow{\sim} I \backslash IaI$$

(we used $a^{-1}\overline{U}_0 a \subseteq \overline{U}_0$ and $U_0 \subseteq a^{-1}U_0 a$). We have

$$\begin{aligned}\pi(\Phi_a)v &= \int_{IaI} \Phi_a(g)\pi(g)v\,dg \\ &= \sum_{s\in S_a}\int_{Ias} \Phi_a(g)\pi(g)v\,dg \\ &= |S_a| \int_I \rho^{-1}(k)\pi(k)\pi(a)v\,dk \\ &= |S_a|\,\mathcal{P}_I^{\chi}(\pi(a)v).\end{aligned}$$

□

Suppose $U_1 \subset U$ is a compact open subgroup. Let $V(U_1) = \{v \in V \mid \mathcal{P}_{U_1}(v) = 0\}$. It is easy to see that

$$\ker(V \to V_U) = \bigcup_{U_1} V(U_1).$$

Lemma 6.0.5 (Jacquet's Lemma II). *Suppose $v \in V^\rho \cap V(U_1)$ for some compact open subgroup $U_1 \subset U$. Suppose $a \in T^+$ satisfies $U_1 \subset a^{-1}U_0 a$. Then $\mathcal{P}_{U_0}(\pi(a)v) = 0$.*

Proof. The vanishing of $\pi(a)\int_{U_0}\pi(a^{-1}ua)v\,du$ follows from the vanishing of $\int_{U_1}\pi(u)v\,du$, since U_1 is a subgroup of $a^{-1}U_0 a$. □

Now we can complete the proof of the injectivity. Suppose $v \in V^\rho$ maps to zero in V_U^χ. Choose U_1 and $a \in T^+$ satisfying the hypotheses of Lemma 6.0.5. Note that $\pi(a)v \in V^{\chi \overline{U}_0}$. Then using Jacquet's Lemmas I and II together with Lemma 6.0.4, we see

$$0 = \mathcal{P}_{U_0}(\pi(a)v) = \mathcal{P}_I^\chi(\pi(a)v) = |IaI|^{-1}\,\pi(\Phi_a)v.$$

Since Φ_a is invertible (Corollary 5.2.2), this implies $v = 0$, which is what we needed to show.

This completes the proof of Proposition 6.0.1. □

7. Proof of Theorem 3.0.2 using Proposition 6.0.1

Let $(\pi, V) \in \mathcal{R}(G)$ be irreducible. Replacing χ with a Weyl-conjugate if necessary (cf. Remark 6.0.2), we see that $\pi \in \mathcal{R}_{\mathfrak{s}}(G)$ iff there exists some $\eta \in X^{\rm ur}(T)$ such that $\pi \hookrightarrow i_B^G(\widetilde{\chi}\,\eta)$. By Frobenius reciprocity $\mathrm{Hom}_G(V, i_B^G(\widetilde{\chi}\,\eta)) = \mathrm{Hom}_T(V_U, \mathbb{C}_{\delta_B^{1/2}\widetilde{\chi}\,\eta})$ this is equivalent to:

$$\begin{aligned}&\exists\,\text{non-zero}\, V_U \to \mathbb{C}_{\delta_B^{1/2}\widetilde{\chi}\eta}, \ \text{for some } \eta \in X^{\rm ur}(T) \\ \iff &(V_U\,\widetilde{\chi}^{-1})^* \text{ has a } {}^\circ T\text{-invariant vector which is an eigenvector for } T/{}^\circ T \\ \iff &(V_U\,\widetilde{\chi}^{-1})^* \text{ has a } {}^\circ T\text{-invariant vector (since } T/{}^\circ T \text{ is abelian)} \\ \iff &(V_U\,\widetilde{\chi}^{-1})^{{}^\circ T} \neq 0 \\ \iff &V_U^\chi \neq 0 \\ \overset{(\star)}{\iff} &V^\rho \neq 0 \\ \iff &\pi \in \mathcal{R}_\rho(G),\end{aligned}$$

where of course ($\star$) comes from Proposition 6.0.1. This completes the proof. $\square$

8. Remarks on constructing the Hecke algebra isomorphisms

8.1. **Intertwining sets.** Let $\rho : K \to \mathbb{C}^\times$ be a smooth character.

Definition 8.1.1. We define the *intertwining set* $I_G(\rho) \subset G$ by requiring that $g \in I_G(\rho)$ iff

$$\rho|_{K\cap {}^gK} = {}^g\rho|_{K\cap {}^gK}.$$

Equivalently, there exists $\phi \neq 0$ in $\mathcal{H}(G,\rho)$ supported on KgK. [For one direction, if such a ϕ exists, note that for $k \in K \cap {}^gK$ we have

$$\rho(k)^{-1}\phi(g) = \phi(kg) = \phi(g\,{}^{g^{-1}}k) = \phi(g)\rho({}^{g^{-1}}k)^{-1}.]$$

Lemma 8.1.2 ([Ro], Prop. 4.1)**.** *Let $K = I$ and $\rho = \rho_\chi$. Then*

(i) $I_G(\rho) \cap N = N_\chi$;
(ii) $I_G(\rho) = IN_\chi I$.

The lemma shows that the set $\{[InI]_{\breve\chi},\, n \in N_\chi/N_\chi \cap I \cong \widetilde{W}_\chi\}$ forms a $\mathbb{C}$-basis for $\mathcal{H}(G,\rho)$.

Proof. (i): If $n \in N \cap I_G(\rho)$, then ${}^n\rho|_{I\cap {}^nI} = \rho|_{I\cap {}^nI}$, which implies that ${}^n\chi|_{{}^\circ T} = \chi|_{{}^\circ T}$, hence ${}^n\chi = \chi$, i.e., $n \in N_\chi$.

Conversely, suppose $n \in N_\chi$ maps to $w \in N/T = W$. We want to show: for $i \in I \cap {}^nI$, we have $\rho(i) = \rho(n^{-1}in)$. Write $i = i_-\, i_0\, i_+ \in I_{\overline{U}}\, {}^\circ T\, I_U$. Then

$$I \ni n^{-1}in = n^{-1}i_-n \cdot n^{-1}i_0n \cdot n^{-1}i_+n \in \overline{U}'\, T\, U',$$

for $U' := w^{-1}Uw$, and $\overline{U}' := w^{-1}\overline{U}w$. Since $n^{-1}in \in I$ can also be expressed using the Iwahori decomposition as an element in $I_{\overline{U}'}\, {}^\circ T\, I_{U'}$, and the expressions in $\overline{U}'TU$ are unique, we see that

$$n^{-1}i_-n \in I_{\overline{U}'} \quad , \quad n^{-1}i_+n \in I_{U'},$$

and in particular these elements belong to I^+. Using this, we see

$$\rho(n^{-1}in) = \chi(n^{-1}i_0n) = {}^n\chi(i_0) = \chi(i_0) = \rho(i).$$

This completes part (i), and (ii) is a consequence of (i). $\square$

8.2. **Presentation for** $\mathrm{End}(\mathrm{ind}\,\rho^{-1})$**.** Recall there is a canonical isomorphism

$$\mathcal{H}(G,\rho) \cong \mathrm{End}_G(\mathrm{ind}\,\rho^{-1})$$

(Lemma 4.0.2). Therefore, we just need to find generators and relations for the right hand side.

Fix an extension $\breve\chi : N_\chi \to \mathbb{C}^\times$ of χ. For $w \in \widetilde{W}_\chi$, choose an element $n \in N_\chi$ mapping to it. We consider the element $\Theta_n \in \mathrm{End}_G(\mathrm{ind}\,\rho^{-1})$ defined by

$$\Theta_n(f)(x) = \frac{1}{|I^+|}\int_{I^+} f(n^{-1}ux)\, du, \qquad (f \in \mathrm{ind}\,\rho^{-1}). \tag{8.2.1}$$

Here, $|I^+| := \mathrm{vol}_{du}(I^+)$. Write $I_w^- := I^+ \cap {}^wI^+\backslash I^+$ and $|I_w^-| := |I^+|/|I^+ \cap {}^wI^+|$ (the ratio of the volumes). Since ρ is trivial on I^+, we see that

$$\Theta_n(f)(x) = \frac{1}{|I_w^-|}\int_{I_w^-} f(n^{-1}ux)\,du. \tag{8.2.2}$$

Note that since $I = {}^\circ TI^+ = (I \cap {}^wI)I^+$ and $I^+ \cap {}^wI = I^+ \cap {}^wI^+$, there is a canonical isomorphism

$$I_w^- \xrightarrow{\sim} I \cap {}^wI\backslash I,$$

and

$$|I_w^-| = [I : I \cap {}^wI] = q^{\ell(w)}. \tag{8.2.3}$$

Lemma 8.2.1. *Let $n \in N_\chi$ and let w denote its image in $\widetilde{W}_\chi$ (and write $n = n_w$).*

(i) $\Theta_n \in \mathrm{End}_G(\mathrm{ind}\,\rho^{-1})$.

(ii) *For $n \in N_\chi$, let Φ_n denote the unique element in $\mathcal{H}(G,\rho)$ which is supported on InI and takes value 1 at n. Then $t_{\Phi_n} = q^{\ell(w)}\Theta_n$.*

(iii) $\{\Theta_{n_w}\}_{w\in\widetilde{W}_\chi}$ *is a $\mathbb{C}$-basis for* $\mathrm{End}_G(\mathrm{ind}\,\rho^{-1})$.

(iv) *Let $n_i = n_{w_i}$ for $i = 1,2$. If $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$, then $\Theta_{n_1n_2} = \Theta_{n_1} \circ \Theta_{n_2}$.*

Proof. (i): We need to check that $\Theta_n(f) \in \mathrm{ind}\,\rho^{-1}$. Write $i \in I$ as $i = ti_+$ for $t \in {}^\circ T$ and $i_+ \in I^+$ (not a unique expression). Then since $\mathrm{Ad}(t)$ is a measure-preserving automorphism of I^+, we have

$$\begin{aligned}|I^+|\,\theta_n(f)(ix) = \int_{I^+} f(n^{-1}uti^+x)\,du &= \int_{I^+} f(n^{-1}tn\cdot n^{-1}ui^+x)\,du\\ &= {}^n\chi(t)^{-1}\int_{I^+} f(n^{-1}ux)\,du\\ &= \rho(i)^{-1}\int_{I^+} f(n^{-1}ux)\,du,\end{aligned}$$

since ${}^n\chi(t) = \chi(t) = \rho(i)$.

(ii): By Lemma 4.0.2, it is enough to prove $\phi_{\Theta_n} = q^{-\ell(w)}\Phi_n$. Recalling $W = \mathbb{C}$ and letting $w = 1 \in \mathbb{C}$, we have

$$\begin{aligned}\phi_{\Theta_n}(g)(w) &= \Theta_n(e_w)(g)\\ &= \frac{1}{|I^+|}\int_{I^+} e_w(n^{-1}ug)\,du.\end{aligned}$$

This is non-zero only if $n^{-1}ug \in I$ for some $u \in I^+$, i.e., only if $g \in InI$. Therefore $\phi_{\Theta_n} \in \mathcal{H}(G,\rho)$ is supported on InI. It remains to check its value at $g = n$. We find it is

$$\frac{1}{|I^+|}\int_{I^+\cap {}^wI} e_w(n^{-1}un)\,du = \frac{|I^+ \cap {}^wI^+|}{|I^+|} = q^{-\ell(w)}$$

(cf. (8.2.3)).

(iii): This is proved in greater generality in [Mor], 5.4, 5.5. Alternatively, we can use the fact that we have proved $\{\Phi_{n_w}\}_{w\in\widetilde{W}_\chi}$ is a basis for $\mathcal{H}(G,\rho)$ (Lemma 8.1.2), together with part (ii).

(iv): This is proved in [Mor], Prop. 5.10. Alternatively, it is an easy consequence of (8.2.2) and standard calculations. □

From now on we want to choose the family $\{n_w\}$ in a compatible way: we require that $n_{w_1w_2} = n_{w_1}n_{w_2}$ whenever $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$. It is always possible to do this (see [Mor], 5.2).

Note that Θ_n depends on $n \mapsto w$ and not just on w. So, we define a new basis element in $\mathcal{H}(G,\rho)$ by

$$B_w := \check{\chi}(n)^{-1}\Theta_n.$$

This indeed depends just on w (and $\check{\chi}$, of course). We also define

$$\begin{aligned} T_w &:= q^{(\ell_\chi(w)+\ell(w))/2}B_w \\ &= q^{(\ell_\chi(w)-\ell(w))/2}\,\check{\chi}(n_w)^{-1}\,t_{\Phi_{n_w}} \end{aligned}$$

for $w \in \widetilde{W}_\chi$. The main computation in this subject shows that these elements T_w generate the algebra $\mathcal{H}_\chi$:

Theorem 8.2.2 (Goldstein [Gol], Morris [Mor])**.** *The elements T_w, $w \in \widetilde{W}_\chi$ satisfy the following relations:*

(i) $T_{w_1w_2} = T_{w_1}T_{w_2}$, *if* $\ell_\chi(w_1w_2) = \ell_\chi(w_1) + \ell_\chi(w_2)$

(ii) $T_s^2 = (q-1)T_s + qT_1$, *if* $s \in S_{\chi,\mathrm{aff}}$.

Thus, the algebra $\mathrm{End}_G(\mathrm{ind}\,\rho^{-1})$ *is isomorphic to* $\mathcal{H}_{\chi^{-1}} = \mathcal{H}_\chi$*, by an isomorphism which depends only on the choice of* $\check{\chi}$*.*

8.3. **Proof of Theorem 5.2.1.** We can now see that the isomorphism $t_\bullet$ of Lemma 4.0.2 gives the desired algebra isomorphism

$$\mathcal{H}(G,\rho) \xrightarrow{\sim} \mathcal{H}_\chi.$$

Indeed, by Lemma 8.2.1 and our definitions, $t_\bullet$ takes

$$q^{-\ell(w)/2}[InI]_{\check{\chi}} = q^{-\ell(w)/2}\,\check{\chi}^{-1}(n)\,\Phi_n$$

to

$$q^{-\ell(w)/2}\check{\chi}^{-1}(n)\,\Theta_n\,q^{\ell(w)} = q^{\ell(w)/2}B_w = q^{-\ell_\chi(w)/2}T_w.$$

This completes the (sketch of the) proof of Theorem 5.2.1. □

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University of Maryland
Department of Mathematics
College Park, MD 20742-4015 U.S.A.
email: tjh@math.umd.edu