THE TEST FUNCTION CONJECTURE FOR LOCAL MODELS OF WEIL-RESTRICTED GROUPS

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ABSTRACT. We prove the test function conjecture of Kottwitz and the first named author for local models of Shimura varieties with parahoric level structure attached to Weil-restricted groups, as defined by B. Levin. This finishes the proof of the test function conjecture for all known parahoric local models, by handling the remaining cases. In addition, we give a self-contained study of relative affine Grassmannians and loop groups formed using general relative effective Cartier divisors in a relative curve over an arbitrary Noetherian affine scheme.

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1. Introduction

Building upon the work of Pappas and Zhu [PZ13], B. Levin defines in [Lev16] candidates for parahoric local models of Shimura varieties for reductive groups of the form $\mathrm{Res}_{K/F}(G)$ where G splits over a tamely ramified extension of K, and K/F is a totally (possibly wildly) ramified extension. The present manuscript is a follow-up of [HaRi], in which we prove the test function conjecture for these local models. The method follows closely [HaRi], and we only explain new arguments in detail, but repeat as much as necessary for readability. For a detailed introduction and further references we refer the reader to the introduction of [HaRi].

Let us mention that the manuscript is supplemented in §3 by a general study of relative affine Grassmannians and loop groups formed using a general Cartier divisor as in the work of Beilinson and Drinfeld [BD]. This unifies the frameworks of [PZ13, Lev16] in mixed characteristic, of [He10, Zhu14, Zhu15, Ri16b] in equal characteristic, and of the work of Fedorov and Panin [FP15, Fe] on the Grothendieck-Serre conjecture, cf. Examples 3.1 below. As an application, we identify the torus fixed points and their attractor and repeller loci in the sense of Drinfeld [Dr] (cf. also [He80]) for these relative affine Grassmannians, cf. Theorem 3.16.

1.1. **Formulation of the main result.** Let p be a prime number. Let $F/\mathbb{Q}_p$ be a finite extension with residue field k_F of cardinality q. Let $\bar{F}/F$ be a separable closure, and denote by Γ_F the Galois group with inertia subgroup I_F and fixed geometric Frobenius lift $\Phi_F \in \Gamma_F$.

Let K/F be a totally (possibly wildly) ramified finite extension, and let G be a connected reductive K-group which splits over a tamely ramified extension. We are interested in the group of Weil restrictions $\tilde{G} = \mathrm{Res}_{K/F}(G)$ which is connected reductive F-group but now possibly wildly ramified depending on K/F. When $p \geq 5$ then any adjoint reductive F-group is of this form, cf. [Ti77, §1.12; §4] (see also [PR08, §7.a]).

Let $\tilde{\mathcal{G}}$ be a parahoric $\mathcal{O}_F$-group scheme in the sense of Bruhat-Tits [BT84] with generic fiber $\tilde{G}$. Note that $\tilde{\mathcal{G}} = \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}(\mathcal{G})$ for a unique parahoric $\mathcal{O}_K$-group scheme $\mathcal{G}$ with generic fiber G, cf. Corollary 4.8. We fix $\{\mu\}$ a (not necessarily minuscule) conjugacy class of geometric cocharacters in $\tilde{G}$ defined over a finite (separable) extension E/F.

Attached to the triple $(\tilde{G}, \{\mu\}, \tilde{\mathcal{G}})$ is the (flat) *local model*

$$M_{\{\mu\}} = M_{(\tilde{G}, \{\mu\}, \tilde{\mathcal{G}})},$$

which is a flat projective $\mathcal{O}_E$-scheme, cf. [PZ13] if $K = F$ and [Lev16] for general K/F (cf. also Definition 4.18). The generic fiber $M_{\{\mu\},E}$ is naturally the Schubert variety in the affine Grassmannian of $\tilde{G}/E$ associated with the class $\{\mu\}$. The special fiber $M_{\{\mu\},k_E}$ is equidimensional, but neither irreducible nor a divisor with normal crossings in general.

Fix a prime number $\ell \neq p$, and fix $q^{-1/2} \in \bar{\mathbb{Q}}_\ell$ in order to define half Tate twists. Let d_μ be the dimension of the generic fiber $M_{\{\mu\},E}$, and denote the normalized intersection complex by

$$\mathrm{IC}_{\{\mu\}} \overset{\mathrm{def}}{=} j_{!*}\bar{\mathbb{Q}}_\ell[d_\mu](\frac{d_\mu}{2}) \in D_c^b(M_{\{\mu\},E}, \bar{\mathbb{Q}}_\ell)$$

cf. §5.2.1. Under the geometric Satake equivalence [Gi, Lu81, BD, MV07, Ri14a, RZ15, Zhu], the complex $\mathrm{IC}_{\{\mu\}}$ corresponds to the ${}^L\tilde{G}_E = \tilde{G}^\vee \rtimes \Gamma_E$-representation $V_{\{\mu\}}$ of highest weight $\{\mu\}$ defined in [Hai14, 6.1], cf. [HaRi, Cor 3.12]. Note that we have $\tilde{G}^\vee = \mathrm{Ind}_{\Gamma_K}^{\Gamma_F}(G^\vee)$ and $V_{\{\mu\}} = \boxtimes_\psi V_{\mu_\psi}$ (cf. Lemma 5.6), and both are taken over $\bar{\mathbb{Q}}_\ell$.

Let E_0/F be the maximal unramified subextension of E/F, and let $\Phi_E = \Phi_{E_0} = \Phi_F^{[E_0:F]}$ and $q_E = q_{E_0} = q^{[E_0:F]}$. The semi-simple trace of Frobenius function on the sheaf of nearby cycles

$$\tau^{\mathrm{ss}}_{\{\mu\}} \colon M_{\{\mu\}}(k_E) \to \bar{\mathbb{Q}}_\ell, \quad x \mapsto (-1)^{d_\mu}\, \mathrm{tr}^{\mathrm{ss}}(\Phi_E \,|\, \Psi_{M_{\{\mu\}}}(\mathrm{IC}_{\{\mu\}})_{\bar{x}}),$$

is naturally a function in the center $\mathcal{Z}(\tilde{G}(E_0), \tilde{\mathcal{G}}(\mathcal{O}_{E_0}))$ of the parahoric Hecke algebra, cf. [PZ13, Thm 10.14], [Lev16, Thm 5.3.3] and §6.3. We remark that $\tau^{\mathrm{ss}}_{\{\mu\}}$ lives in the center of the $\bar{\mathbb{Q}}_\ell$-valued Hecke algebra attached to *function field* analogues of $(\tilde{G}_{E_0}, \tilde{\mathcal{G}}_{\mathcal{O}_{E_0}}, E_0)$; we are implicitly identifying this with $\mathcal{Z}(\tilde{G}(E_0), \tilde{\mathcal{G}}(\mathcal{O}_{E_0}))$ via Corollary 4.9 and Lemma 4.12.

Our main result, *the test function conjecture for local models for Weil restricted groups*, characterizes the function $\tau^{\mathrm{ss}}_{\{\mu\}}$, extending the main result of [HaRi] to the Weil-restricted situation. It confirms that even for these exotic local models, the local geometry of Shimura varieties at places of parahoric bad reduction can be related to automorphic-type data, as required by the Langlands-Kottwitz method.

Main Theorem. *Let $(\tilde{G}, \{\mu\}, \tilde{\mathcal{G}})$ be a triple as above. Let E/F be a finite separable extension over which $\{\mu\}$ is defined, and let E_0/F be the maximal unramified subextension. Then*

$$\tau^{\mathrm{ss}}_{\{\mu\}} = z^{\mathrm{ss}}_{\{\mu\}}$$

where $z^{\mathrm{ss}}_{\{\mu\}} = z^{\mathrm{ss}}_{\tilde{\mathcal{G}},\{\mu\}} \in \mathcal{Z}(\tilde{G}(E_0), \tilde{\mathcal{G}}(\mathcal{O}_{E_0}))$ is the unique function which acts on any $\tilde{\mathcal{G}}(\mathcal{O}_{E_0})$-spherical smooth irreducible $\bar{\mathbb{Q}}_\ell$-representation π by the scalar

$$\mathrm{tr}\Big(s(\pi) \;\Big|\; \mathrm{Ind}_{{}^L\tilde{G}_E}^{{}^L\tilde{G}_{E_0}}(V_{\{\mu\}})^{1\rtimes I_{E_0}}\Big),$$

where $s(\pi) \in [(\tilde{G}^\vee)^{I_{E_0}} \rtimes \Phi_{E_0}]_{\mathrm{ss}}/(\tilde{G}^\vee)^{I_{E_0}}$ is the Satake parameter for π [Hai15]. The function $q_{E_0}^{d_\mu/2}\tau^{\mathrm{ss}}_{\{\mu\}}$ takes values in $\mathbb{Z}$ and is independent of $\ell \neq p$ and $q^{1/2} \in \bar{\mathbb{Q}}_\ell$.

The construction of $s(\pi)$ is also reviewed in [HaRi, §7.2], and the values of $z^{\mathrm{ss}}_{\{\mu\}}$ are studied in [HaRi, §7.7], cf. §6.5. The definition of the local models $M_{\{\mu\}}$ depends on certain auxiliary choices (cf. Remark 4.19), but the function $\tau^{\mathrm{ss}}_{\{\mu\}}$ depends canonically only on the data $(\tilde{G}, \{\mu\}, \tilde{\mathcal{G}})$.

1.2. **Other results.** Our methods can be used to obtain results on the fixed point (resp. attractor and repeller) locus of $\mathbb{G}_m$-actions on Fusion Grassmannians (cf. Theorem A below), and the special fiber of local models (cf. Theorem B below).

1.2.1. *Fusion Grassmannians.* Let F be any field, and let G be a connected reductive F-group. For each $n \geq 0$, there is the fusion Grassmannian $\mathrm{Gr}_{G,n} \to \mathbb{A}^n_F$ defined in [BD] which parametrizes isomorphism classes of G-bundles on the affine line together with a trivialization away from n points. Given a cocharacter $\chi\colon \mathbb{G}_{m,F} \to G$ we obtain a fiberwise $\mathbb{G}_m$-action on the family $\mathrm{Gr}_{G,n} \to \mathbb{A}^n_F$, and we are interested in determining the diagram on the fixed point ind-scheme and attractor (resp. repeller) ind-scheme

$$(\mathrm{Gr}_{G,n})^0 \leftarrow (\mathrm{Gr}_{G,n})^{\pm} \to \mathrm{Gr}_{G,n},$$

cf. (2.1). Let $M \subset G$ be the centralizer of χ, which is a Levi subgroup. The dynamic method promulgated in [CGP10] defines a pair of parabolic subgroups (P^+, P^-) in G such that $P^+ \cap P^- = M$; see the formulation of Theorem 3.16. The natural maps $M \leftarrow P^{\pm} \to G$ induce maps of fusion Grassmannians

$$\mathrm{Gr}_{M,n} \leftarrow \mathrm{Gr}_{P^{\pm},n} \to \mathrm{Gr}_{G,n}.$$

An extension of the method used in the proof of [HaRi, Prop 3.4] allows us to prove the following result.

Theorem A. *For each $n \in \mathbb{Z}_{\geq 0}$, there is a commutative diagram of $\mathbb{A}^n_F$-ind-schemes*

$$\begin{tikzcd}
\mathrm{Gr}_{M,n} \arrow[d, "\simeq"'] & \mathrm{Gr}_{P^{\pm},n} \arrow[l] \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Gr}_{G,n} \arrow[d, "\mathrm{id}"'] \\
(\mathrm{Gr}_{G,n})^0 & (\mathrm{Gr}_{G,n})^{\pm} \arrow[l] \arrow[r] & \mathrm{Gr}_{G,n},
\end{tikzcd}$$

where the vertical maps are isomorphisms.

Theorem A is a special case of Theorem 3.16 which applies to general connected reductive group schemes over $\mathbb{A}^n_F$ which are not necessarily defined over F. Let us point out that [HaRi, Prop 3.4] implies that Theorem A holds fiberwise. However, we do not know how to prove sufficiently good flatness properties of $\mathrm{Gr}_{G,n} \to \mathbb{A}^n_F$ in order to deduce the more general result from the fiberwise result.

The tensor structure on the constant term functors in geometric Langlands is constructed in [BD, MV07]. In [Ga07, Re12], it is explained how to use the nearby cycles to define the fusion structure used in the geometric Satake isomorphism. Theorem A together with [Ri, Thm 3.3] gives another way of constructing the tensor structure on the constant term functors - even without passing to the underlying reduced ind-schemes, cf. proof of [HaRi, Thm 3.16].

1.2.2. *Special fibers of local models.* As in [HaRi, §6.3.1], we use the commutation of nearby cycles with constant terms to determine the irreducible components of the geometric special fiber $M_{\{\mu\},\bar{k}}$ of the local models. Recall that by construction (cf. Definition 4.18), there is a closed embedding

$$M_{\{\mu\}} \hookrightarrow \mathcal{F}\ell_{\mathcal{G}'},$$

where $\mathcal{F}\ell_{\mathcal{G}'}$ is the (partial) affine flag variety attached to the function field analogue $\mathcal{G}'/k_F[\![u]\!]$ of $\tilde{\mathcal{G}}/\mathcal{O}_F$, cf. Theorem 4.13 and Proposition 4.15 ii). As envisioned by Kottwitz and Rapoport, the geometric special fiber $M_{\{\mu\},\bar{k}}$ should be the union of the Schubert varieties $\mathcal{F}l^{\leq w}_{\mathcal{G}',\bar{k}} \subset \mathcal{F}\ell_{\mathcal{G}',\bar{k}}$ where w ranges over the $\{\mu\}$-admissible set $\mathrm{Adm}^{\mathbf{f}}_{\{\mu\}} \subset W_{\mathbf{f}}\backslash W/W_{\mathbf{f}}$ where $\mathcal{G} = \mathcal{G}_{\mathbf{f}}$ and $W = W(\tilde{G}, F)$ denotes the Iwahori-Weyl group. Here we are identifying the Iwahori-Weyl groups attached to $\tilde{G}/F$, G/K and $G'/k_F(\!(u)\!)$ by Lemmas 4.3 and 4.11. The following result verifies their prediction (cf. Theorem 5.14).

Theorem B. *The smooth locus $(M_{\{\mu\}})^{\mathrm{sm}}$ is fiberwise dense in $M_{\{\mu\}}$, and on reduced subschemes a union of the Schubert varieties*

$$(M_{\{\mu\},\bar{k}})_{\mathrm{red}} = \bigcup_{w\in \mathrm{Adm}^{\mathbf{f}}_{\{\mu\}}} \mathcal{F}l^{\leq w}_{\mathcal{G}',\bar{k}}.$$

In particular, the geometric special fiber $M_{\{\mu\},\bar{k}}$ is generically reduced.

If $p \nmid |\pi_1(G_{\mathrm{der}})|$, then Theorem B is [PZ13, Thm 9.3] for $K = F$, and [Lev16, Thm 2.3.5] when $K \neq F$. We have removed this condition on p and thereby conclude that the Kottwitz-Rapoport strata in the special fiber are enumerated by the $\{\mu\}$-admissible set for all *known* local models.

1.3. **Overview.** In §2 we recall a few facts about $\mathbb{G}_m$-actions for convenience. The following §3 studies relative affine Grassmannians formed using a general Cartier divisor. In §4, we recall the definition of Weil-restricted local models and results from [Lev16] which are needed in the sequel. These results are applied in §5 to study $\mathbb{G}_m$-actions on Beilinson-Drinfeld affine Grassmannians for Weil-restricted groups. In §6, we formulate and prove the test function conjecture for Weil-restricted local models.

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1.5. **Conventions on Ind-Algebraic Spaces.** Let $\mathcal{O}$ be a ring, and denote by $\mathcal{O}$-Alg the category of $\mathcal{O}$-algebras equipped with the fpqc topology. An $\mathcal{O}$-space X is sheaf on the site $\mathcal{O}$-Alg, and we denote the category of $\mathcal{O}$-spaces by $\mathrm{Sp}_{\mathcal{O}}$. As each object in the site $\mathcal{O}$-Alg is quasi-compact, the pretopology on $\mathcal{O}$-Alg is generated by finite covering families, and hence filtered colimits exist in $\mathrm{Sp}_{\mathcal{O}}$ and can be computed in the category of presheaves.

The category $\mathrm{Sp}_{\mathcal{O}}$ contains the category of $\mathcal{O}$-schemes $\mathrm{Sch}_{\mathcal{O}}$ as a full subcategory. An $\mathcal{O}$-algebraic space is a $\mathcal{O}$-space X such that $X \to X \times_{\mathcal{O}} X$ is relatively representable, and such that there exists an étale surjective map from a scheme $U \to X$. By a Theorem of Gabber [StaPro, Tag 03W8] this agrees with the usual definition of algebraic spaces using étale or fppf sheaves.

The category of $\mathcal{O}$-algebraic spaces is denoted $\mathrm{AlgSp}_{\mathcal{O}}$. There are full embeddings $\mathrm{Sch}_{\mathcal{O}} \subset \mathrm{AlgSp}_{\mathcal{O}} \subset \mathrm{Sp}_{\mathcal{O}}$. A map of $\mathcal{O}$-spaces $X \to Y$ is called representable (resp. schematic) if for every scheme $T \to Y$ the fiber product $X \times_Y T$ is representable by an algebraic space (resp. scheme).

An $\mathcal{O}$-ind-algebraic space (resp. $\mathcal{O}$-ind-scheme) is a contravariant functor

$$X\colon \mathcal{O}\text{-Alg} \to \mathrm{Sets}$$

such that there exists a presentation as presheaves $X = \mathrm{colim}_i X_i$ where $\{X_i\}_{i\in I}$ is a filtered system of quasi-compact $\mathcal{O}$-algebraic spaces (resp. quasi-compact $\mathcal{O}$-schemes) X_i with transition maps being (schematic) closed immersions. Since filtered colimits in $\mathrm{Sp}_{\mathcal{O}}$ can be computed in presheaves, every $\mathcal{O}$-ind-algebraic space (resp. $\mathcal{O}$-ind-scheme) is an $\mathcal{O}$-space. The category of $\mathcal{O}$-ind-algebraic spaces (resp. $\mathcal{O}$-ind-schemes) $\mathrm{IndAlgSp}_{\mathcal{O}}$ (resp. $\mathrm{IndSch}_{\mathcal{O}}$) is the full subcategory of $\mathrm{Sp}_{\mathcal{O}}$ whose objects are $\mathcal{O}$-ind-algebraic spaces (resp. $\mathcal{O}$-ind-schemes). If $X = \mathrm{colim}_i X_i$ and $Y = \mathrm{colim}_j Y_j$ are presentations of ind-algebraic spaces (resp. ind-schemes), then as sets

$$\mathrm{Hom}_{\mathrm{Sp}_{\mathcal{O}}}(X,Y) = \lim_i \mathrm{colim}_j \mathrm{Hom}_{\mathrm{Sp}_{\mathcal{O}}}(X_i, Y_j),$$

because every map $X_i \to Y$ factors over some Y_j by quasi-compactness of X_i. The category $\mathrm{IndAlgSp}_{\mathcal{O}}$ (resp. $\mathrm{IndSch}_{\mathcal{O}}$) is closed under fiber products, i.e., $\mathrm{colim}_{(\mathrm{i},\mathrm{j})}(X_i \times_{\mathcal{O}} Y_j)$ is a presentation of $X \times_{\mathcal{O}} Y$. If $\mathcal{P}$ is a property of algebraic spaces (resp. schemes), then an $\mathcal{O}$-ind-algebraic space (resp. $\mathcal{O}$-ind-scheme) X is said to have ind-$\mathcal{P}$ if there exists a presentation $X = \mathrm{colim}_i X_i$ where each X_i has property $\mathcal{P}$. A map $f\colon X \to Y$ of $\mathcal{O}$-ind-algebraic spaces (resp. $\mathcal{O}$-ind-schemes) is said to have property $\mathcal{P}$ if f is representable and for all schemes $T \to Y$, the pullback $f \times_Y T$ has property $\mathcal{P}$. Note that every representable quasi-compact map of $\mathcal{O}$-ind-schemes is schematic.

2. Recollection on $\mathbb{G}_m$-actions on Ind-Algebraic Spaces

We recall some set-up and notation from [Dr] and [Ri]. Let $\mathcal{O}$ be a ring, and let X be an $\mathcal{O}$-algebraic space (or $\mathcal{O}$-ind-algebraic space) equipped with an action of $\mathbb{G}_m$ which is trivial on $\mathcal{O}$. There are the following three functors on the category of $\mathcal{O}$-algebras

$$\begin{aligned} X^0\colon\ R &\longmapsto \mathrm{Hom}_R^{\mathbb{G}_m}(R, X) \\ X^+\colon\ R &\longmapsto \mathrm{Hom}_R^{\mathbb{G}_m}((\mathbb{A}^1_R)^+, X) \\ X^-\colon\ R &\longmapsto \mathrm{Hom}_R^{\mathbb{G}_m}((\mathbb{A}^1_R)^-, X), \end{aligned} \tag{2.1}$$

where $(\mathbb{A}^1_R)^+$ (resp. $(\mathbb{A}^1_R)^-$) is $\mathbb{A}^1_R$ with the usual (resp. opposite) $\mathbb{G}_m$-action. The functor X^0 is the functor of $\mathbb{G}_m$-fixed points in X, and X^+ (resp. X^-) is called the attractor (resp. repeller). Informally speaking, X^+ (resp. X^-) is the space of points x such that the limit $\lim_{\lambda\to 0} \lambda \cdot x$ (resp. $\lim_{\lambda\to\infty} \lambda \cdot x$) exists. The functors (2.1) come equipped with natural maps

$$X^0 \leftarrow X^{\pm} \to X, \tag{2.2}$$

where $X^{\pm} \to X^0$ (resp. $X^{\pm} \to X$) is given by evaluating a morphism at the zero section (resp. at the unit section). We say that the $\mathbb{G}_m$-action on an algebraic space X is étale (resp. Zariski) locally linearizable, i.e. the $\mathbb{G}_m$-action lifts - necessarily uniquely - to an étale cover which is affine and on which $\mathbb{G}_m$ acts linearly. We say that an $\mathbb{G}_m$-action on an S-ind-algebraic space X is étale (resp. Zariski) locally linearizable if there is an $\mathbb{G}_m$-stable presentation with equivariant transition maps $X = \mathrm{colim}_i X_i$ where the $\mathbb{G}_m$-action on each X_i is étale (resp. Zariski) locally linearizable. We use the following representability properties of the functors (2.1), cf. [HaRi, Thm 2.1].

Theorem 2.1. *Let $X = \mathrm{colim}_i X_i$ be an $\mathcal{O}$-ind-algebraic space equipped with an étale locally linearizable $\mathbb{G}_m$-action.*

i) The subfunctor $X^0 = \mathrm{colim}_i X_i^0$ is representable by a closed sub-ind-algebraic space of X.

ii) The functor $X^{\pm} = \mathrm{colim}_i X_i^{\pm}$ is representable, and the map $X^{\pm} \to X$ is representable and quasi-compact. The map $X^{\pm} \to X^0$ is ind-affine with geometrically connected fibers and induces a bijection on connected components $\pi_0(X^{\pm}) \simeq \pi_0(X^0)$ of the underlying topological spaces.

iii) If $X = \mathrm{colim}_i X_i$ is of ind-finite presentation (resp. an ind-scheme; resp. separated), so are X^0 and $X^{\pm}$.

The proof is like that of [HaRi, Thm. 2.1], using the representability results of [Ri, Thm. 1.8]. We record the following lemma for later use.

Lemma 2.2. *For $n \in \mathbb{Z}_{>0}$, let $X_1, \dots, X_n$ be $\mathcal{O}$-algebraic spaces (or $\mathcal{O}$-ind-algebraic spaces) equipped with an étale locally linearizable $\mathbb{G}_m$-action. Then the diagonal $\mathbb{G}_m$-action on the product $\prod_{i=1}^n X_i$ is étale locally linearizable, and the canonical maps*

$$(\prod_{i=1}^n X_i)^0 \xrightarrow{\simeq} \prod_{i=1}^n X_i^0 \quad \textit{and} \quad (\prod_{i=1}^n X_i)^{\pm} \xrightarrow{\simeq} \prod_{i=1}^n X_i^{\pm}$$

are isomorphisms.

Proof. If, for each i, the map $U_i \to X_i$ is an étale local linearization, then the product $\prod_{i=1}^n U_i \to \prod_{i=1}^n X_i$ is an étale local linearization. It is easy to check on the level of functors that the maps are isomorphisms. $\square$

3. Loop groups and affine Grassmannians for Cartier divisors

In this section, we give a self-contained treatment of affine Grassmannians for non-constant group schemes over relative curves which are formed using a formal neighborhood of a general Cartier divisor. This extends the work of Beilinson-Drinfeld [BD], and is inspired by the work of Fedorov-Panin [FP15, Fe] and Levin [Lev16].

3.1. **Definitions and Examples.** Let $\mathcal{O}$ be a Noetherian ring. Let X be a smooth quasi-projective $\mathcal{O}$-curve, i.e., the map $X \to \operatorname{Spec}(\mathcal{O})$ is smooth of pure dimension 1, and factors as $X \to \mathbb{P}(\mathcal{E}) \to \operatorname{Spec}(\mathcal{O})$ where $\mathcal{E}$ is a locally free of finite rank $\mathcal{O}$-module and $X \to \mathbb{P}(\mathcal{E})$ a quasi-compact immersion. Let $D \subset X$ be a relative effective Cartier divisor which is finite and flat over $\mathcal{O}$. Let $\mathcal{G}$ be a smooth affine X-group scheme of finite presentation.

To the triple $(X, \mathcal{G}, D)$, we associate the functor $\mathrm{Gr}_{\mathcal{G}} = \mathrm{Gr}_{(X,\mathcal{G},D)}$ on the category of $\mathcal{O}$-algebras which assigns to every R the set of isomorphism classes of tuples $(\mathcal{F}, \alpha)$ with

$$\begin{cases} \mathcal{F} \text{ a } \mathcal{G}\text{-torsor on } X_R; \\ \alpha : \mathcal{F}|_{(X\backslash D)_R} \xrightarrow{\simeq} \mathcal{F}_0|_{(X\backslash D)_R} \text{ a trivialization,} \end{cases} \tag{3.1}$$

where $\mathcal{F}_0$ denotes the trivial $\mathcal{G}$-torsor. Fpqc-descent for schemes affine over X_R implies that $\mathrm{Gr}_{\mathcal{G}}$ is an $\mathcal{O}$-space. As $\mathcal{G}$ is of finite presentation, the functor $\mathrm{Gr}_{\mathcal{G}}$ commutes with filtered colimits of $\mathcal{O}$-algebras. Further, if R is a $\mathcal{O}$-algebra, then as functors on R-Alg,

$$\mathrm{Gr}_{\mathcal{G}} \times_{\operatorname{Spec}(\mathcal{O})} \operatorname{Spec}(R) = \mathrm{Gr}_{\mathcal{G}}|_{R\text{-Alg}} = \mathrm{Gr}_{(X_R, \mathcal{G}_{X_R}, D_R)}. \tag{3.2}$$

If we replace D by a positive multiple nD for some $n \geq 1$, then $X\backslash D = X\backslash nD$, and hence as $\mathcal{O}$-functors

$$\mathrm{Gr}_{(X,\mathcal{G},D)} = \mathrm{Gr}_{(X,\mathcal{G},nD)}. \tag{3.3}$$

The following examples are special cases of the general set-up.

Example 3.1. i) *Affine Grassmannians/Flag Varieties.* Let $\mathcal{O} = F$ be a field, and let $D = \{x\}$ for some point $x \in X(F)$. Then on completed local rings $\mathcal{O}_x \simeq F[\![t_x]\!]$ where t_x denotes a local parameter at $x \in X$. If $\mathcal{G} = G \otimes_F X$ for some smooth affine F-group G, then $\mathrm{Gr}_G := \mathrm{Gr}_{\mathcal{G}}$ is the "affine Grassmannian" formed using the local parameter t_x, i.e., the ind-scheme given by the fpqc sheafification of the functor $R \mapsto G(R(\!(t_x)\!))/G(R[\![t_x]\!])$. In general, the functor $\mathrm{Gr}_{\mathcal{G}}$ is the "twisted affine flag variety" for the group scheme $\mathcal{G} \otimes_X F[\![t_x]\!]$ in the sense of [PR08].

ii) *Mixed characteristic.* Let $\mathcal{O} = \mathcal{O}_F$ be the valuation ring of a finite extension $F/\mathbb{Q}_p$. Let K/F be a finite totally ramified extension with uniformizer $\varpi \in K$. Let $X = \mathbb{A}^1_{\mathcal{O}_F}$ with global coordinate denoted z, and let $D = \{Q = 0\}$, where $Q \in \mathcal{O}_F[z]$ is the minimal polynomial of ϖ over F (an Eisenstein polynomial). Let $\mathcal{G}$ be the X-group scheme constructed in [PZ13, Thm 4.1] if $K = F$, and in [Lev16, Thm 3.3.3] otherwise; here it is denoted $\underline{\mathcal{G}}$, see Theorem 4.13. Then $\mathrm{Gr}_{\mathcal{G}}$ is the $\mathcal{O}_F$-ind-scheme defined in [PZ13, Eq (6.11)] if $K = F$, and in [Lev16, Def 4.1.1] otherwise; here we denote it $\mathrm{Gr}_{\tilde{\mathcal{G}}}$, see §4.4.1.

iii) *Equal characteristic.* Let F be a field, and let C be a smooth affine F-curve. Let $\mathcal{O} = \Gamma(C, \mathcal{O}_C)$ be the global sections, and let $X = C \times_F C = C_{\mathcal{O}}$. Let $\mathcal{G}_0$ be a smooth affine $\mathcal{O}$-group scheme of finite presentation, and let $\mathcal{G} = \mathcal{G}_0 \otimes_{\mathcal{O}} X$. Let $D := \Delta(C)$ be the diagonal divisor in X. If $C = \mathbb{A}^1_F$, then $\mathrm{Gr}_{\mathcal{G}}$ is the ind-scheme defined in [Zhu14, Eq (3.1.1)]. If $x \in C(F)$ is a point, and $\mathcal{O}_x \to \mathcal{O}$ denotes the completed local ring, then $\mathrm{Gr}_{\mathcal{G}} \otimes_{\mathcal{O}} \mathcal{O}_x$ is the ind-scheme defined in [Ri16b, Def 2.3]. Let us remark that this is a special case of the general set-up in [He10, §2].

iv) *Fusion Grassmannians.* Let F be a field, and let C be an affine curve over F. The d-th symmetric product $C^{(d)}$ is by [SGA IV, Exp. XVII, Prop. 6.3.9] the moduli space of degree d effective Cartier divisors on C. Let $\operatorname{Spec}(\mathcal{O}) := C^{(d)}$, and we let $D := C^{(d)}$ be the universal degree d divisor on $X := C \times_F C^{(d)} = C_{\mathcal{O}}$. For a smooth affine F-group scheme G, we let $\mathcal{G} = G \otimes_F X$. Then the ind-scheme $\mathrm{Gr}_{\mathcal{G}} \times_{\operatorname{Spec}(\mathcal{O})} C^d$ is the fusion Grassmannian defined in [BD, 5.3.11].

v) *Generically trivial bundles.* If $X = \mathbb{A}^1_{\mathcal{O}}$ and $\mathcal{G}$ is split reductive, then the functor $\mathrm{Gr}_{\mathcal{G}}$ in (3.1) is the moduli space of objects used in [Fe, Thm 2].

3.1.1. *Loop Groups.* The functor $\mathrm{Gr}_{\mathcal{G}}$ is related to loop groups as follows. For an $\mathcal{O}$-algebra R, let $(X_R/D_R)^\frown$ be the formal affine[1] scheme defined by D_R in X_R, and denote by $R[\![D]\!]$ its ring of regular functions. Explicitly, if $\mathcal{I}_R \subset \mathcal{O}_{X_R}$ is the ideal sheaf for D_R, then $(D_R, \mathcal{O}_{X_R}/\mathcal{I}_R^n)$ is an affine scheme $\mathrm{Spec}(A_n)$ for all $n \geq 1$, and $R[\![D]\!] \overset{\text{def}}{=} \varprojlim A_n = \varprojlim \Gamma(D_R, \mathcal{O}_{X_R}/\mathcal{I}_R^n)$. Let $\hat{D}_R = \mathrm{Spec}(R[\![D]\!])$ be the associated affine (true) scheme. The map $(X_R/D_R)^\frown \to X_R$ uniquely extends to a map $p\colon \hat{D}_R \to X_R$, and $p^{-1}(D_R) \simeq D_R$ defines a relative effective Cartier divisor on $\hat{D}_R$, cf. [BD, §2.12]. Let $\hat{D}_R^o = \hat{D}_R \backslash D_R$. As D_R is a Cartier divisor in $\hat{D}_R$, it is locally principal, and hence the complement $\hat{D}_R^o := \mathrm{Spec}(R(\!(D)\!))$ is an affine scheme. The (*twisted*) *loop group* $L\mathcal{G} = L_D\mathcal{G}$ is the functor on the category of $\mathcal{O}$-algebras

$$L\mathcal{G}\colon R \mapsto \mathcal{G}(R(\!(D)\!)). \tag{3.4}$$

The *positive* (*twisted*) *loop group* $L^+\mathcal{G} = L^+_D\mathcal{G}$ is the functor on the category of $\mathcal{O}$-algebras

$$L^+\mathcal{G}\colon R \mapsto \mathcal{G}(R[\![D]\!]). \tag{3.5}$$

As every Cartier divisor is locally defined by a single non-zero divisor, we see that $L^+\mathcal{G} \subset L\mathcal{G}$ is a subgroup functor. Let us explain why these functors are representable in this generality.

Lemma 3.2. *i*) *The functor $L^+\mathcal{G}$* (*resp. $L\mathcal{G}$*) *is representable by an affine scheme* (*resp. ind-affine ind-scheme*). *In particular, $L^+\mathcal{G}$ and $L\mathcal{G}$ are $\mathcal{O}$-spaces.*

ii) *The scheme $L^+\mathcal{G}$ is a faithfully flat affine $\mathcal{O}$-group scheme which is pro-smooth.*

Proof. Part i) is true for every affine scheme $\mathcal{G}$ of finite presentation over $\mathcal{O}$: Let $\mathcal{G} = \mathbb{A}^1_{\mathcal{O}}$ first. Denote by I_D the invertible ideal defined by D in $\mathcal{O}[\![D]\!]$. By the preceding discussion, the ring $\mathcal{O}[\![D]\!]/I_D$ is isomorphic to the global sections of D and both are finite locally free $\mathcal{O}$-modules, cf. [StaPro, Tag 0B9C]. For any $a \in \mathbb{Z}$, we form I_D^a as an invertible $\mathcal{O}[\![D]\!]$-module. For $a \leq b$, denote by $E_{[a,b]}$ the $\mathcal{O}$-module I_D^a/I_D^b which is also finite locally free (hence reflexive) by an induction argument. As b varies, the set of $\mathcal{O}$-modules $\{E_{[a,b]}\}_{b\geq a}$ forms an inverse system, and $\mathcal{O}[\![D]\!] = \lim_{b\geq 0} E_{[0,b]}$ by definition. It follows that $I_D^a = \lim_{b\geq a} E_{[a,b]}$ for any $a \in \mathbb{Z}$. In particular, we get $\mathcal{O}(\!(D)\!) = \mathrm{colim}_a \lim_{b\geq a} E_{[a,b]}$. As $E_{[a,b]}$ is a reflexive $\mathcal{O}$-module, we get for every $\mathcal{O}$-algebra R that

$$E_{[a,b]} \otimes_{\mathcal{O}} R = \mathrm{Hom}_{\mathcal{O}\text{-Mod}}((E_{[a,b]})^*, R) = \mathrm{Hom}_{\mathcal{O}\text{-Sch}}(\mathrm{Spec}(R), \mathbb{V}_{[a,b]}), \tag{3.6}$$

where $\mathbb{V}_{[a,b]} = \mathrm{Spec}(\mathrm{Sym}^{\otimes}(E_{[a,b]})^*)$ for every pair $b \geq a$. Taking limits shows that

$$\mathbb{A}^1_{\mathcal{O}}(R[\![D]\!]) = R[\![D]\!] = \lim_{b\geq 0}(E_{[0,b]} \otimes_{\mathcal{O}} R)$$

is identified with the R-points of the affine $\mathcal{O}$-scheme $\lim_{b\geq 0}\mathbb{V}_{[0,b]}$. The same argument shows that $R \mapsto \mathbb{A}^1_{\mathcal{O}}(R(\!(D)\!))$ is representable by the ind-affine ind-scheme $\mathrm{colim}_a \lim_{b\geq a}\mathbb{V}_{[a,b]}$. This gives part i) in the case $\mathcal{G} = \mathbb{A}^1_{\mathcal{O}}$. For the general case, one verifies that the L^+-construction (resp. L-construction) commutes with taking finite products and equalizers, and that finite products and equalizers are constructed termwise in the category of ind-schemes. Hence, the lemma follows for $L^+\mathbb{A}^n_{\mathcal{O}}$ (resp. $L\mathbb{A}^n_{\mathcal{O}}$). A finite presentation $\mathcal{G} = \mathrm{Spec}(\mathcal{O}[t_1, \dots, t_n]/(f_1, \dots, f_m))$ realizes $\mathcal{G}$ as the equalizer of the two maps $\varphi, \psi\colon \mathbb{A}^n_{\mathcal{O}} \to \mathbb{A}^m_{\mathcal{O}}$ where φ is given by the functions $f_1, \dots, f_m$ and ψ is the composition of the structure map with the zero section. Hence, $L^+\mathcal{G}$ (resp. $L\mathcal{G}$) is the equalizer of $L^+\varphi$ and $L^+\psi$ (resp. $L\varphi$ and $L\psi$) in the category of schemes (resp. ind-schemes). As equalizers define closed subschemes and $L^+\mathbb{A}^n_{\mathcal{O}}$ is affine (resp. $L\mathbb{A}^n_{\mathcal{O}}$ ind-affine), i) follows.

Part ii) is true for every smooth affine $\mathcal{O}$-scheme $\mathcal{G}$ of finite presentation: For $n \geq 0$, let $D_n = \mathrm{Spec}(\mathcal{O}[\![D]\!]/I_D^{n+1})$ be the n-th infinitesimal neighborhood of D in X. The Weil restriction of scalars $\mathcal{G}_n := \mathrm{Res}_{D_n/\mathcal{O}}(\mathcal{G} \times_X D_n)$ is a smooth affine $\mathcal{O}$-group scheme of finite presentation, cf. [BLR90, §7.6, Thm 4, Prop 5]. For varying n, these groups fit into an inverse system $\mathcal{G}_m \to \mathcal{G}_n$ for $m \geq n$, and the natural map of functors

$$L^+\mathcal{G} \xrightarrow{\simeq} \lim_{n\geq 0}\mathcal{G}_n \tag{3.7}$$

[1]One can show that a formal completion $(X/X')^\frown$ of a scheme X along an *affine* closed subscheme $X' \subset X$ is of the form $\mathrm{Spf}(A)$ for an admissible topological ring A. This is implicit in [BD, 2.12.2].

is an isomorphism. This proves ii), and the lemma follows. □

Remark 3.3. If nD is a positive multiple of D, then there is a canonical isomorphism $\mathcal{O}[\![D]\!] \overset{\simeq}{\to} \mathcal{O}[\![nD]\!]$ (resp. $\mathcal{O}(\!(D)\!) \overset{\simeq}{\to} \mathcal{O}(\!(nD)\!)$). Indeed, as $I_{nD} = I_D^n \subset I_D$, the ring $\mathcal{O}[\![D]\!]$ is complete with respect to the I_{nD}-adic topology, and hence $\mathcal{O}[\![D]\!] \simeq \lim_{k\geq 0} R[\![D]\!]/I_{nD}^k = R[\![nD]\!]$.

Lemma 3.4. *i) The loop group $L\mathcal{G}$ represents the functor on the category of $\mathcal{O}$-algebras which assigns to every R the set of isomorphism classes of triples $(\mathcal{F}, \alpha, \beta)$, where $\mathcal{F}$ is a $\mathcal{G}$-torsor on X_R, $\alpha\colon \mathcal{F}|_{X_R\backslash D_R} \xrightarrow{\simeq} \mathcal{F}_0$ (resp. $\beta\colon \mathcal{F}_0 \xrightarrow{\simeq} \mathcal{F}|_{\hat{D}_R}$) is a trivialization over $X_R\backslash D_R$ (resp. $\hat{D}_R$).*
ii) The projection $L\mathcal{G} \to \mathrm{Gr}_{\mathcal{G}}$, $(\mathcal{F}, \alpha, \beta) \to (\mathcal{F}, \alpha)$ is a right $L^+\mathcal{G}$-torsor in the étale topology, and induces an isomorphism of sheaves $L\mathcal{G}/L^+\mathcal{G} \xrightarrow{\simeq} \mathrm{Gr}_{\mathcal{G}}$.

Proof. Part i) is deduced from the Beauville-Laszlo theorem [BL95], cf. [BD, §2.12] for a further discussion (cf. also [PZ13, Lem 6.1]). For ii), it is enough to prove that the projection $L\mathcal{G} \to \mathrm{Gr}_{\mathcal{G}}$ admits sections étale locally. Let $(R, \mathfrak{m}, F)$ be a Noetherian strictly Henselian local R-algebra. We have to show that every $\mathcal{G}$-torsor $\mathcal{F}$ on $\hat{D}_R$ is trivial, i.e., $\mathcal{F}$ admits a $\hat{D}_R$-section. The torsor $\mathcal{F}$ is smooth affine because $\mathcal{G}$ is so. By applying the lifting criterion for smoothness and an algebraization result for sections (algebraization is easy because $\mathcal{F}$ is affine), it is enough to show that the restriction $\mathcal{F}|_{D_R}$ admits an $R' := \Gamma(D_R, \mathcal{O}_{D_R})$-section. As the ring extension $R \subset R'$ is integral, the pair $(R', \mathfrak{m}R')$ is Henselian, cf. [StaPro, Tag 09XD Lem 15.10.10]. Since $\mathcal{F}$ is smooth, Elkik's approximation theorem [El73, Thm. p.578] applies, and it is enough to construct a section above $R'/\mathfrak{m}R' = \Gamma(D_F, \mathcal{O}_{D_F})$ where $D_F = D_R \otimes_R F$. We write $D_F = \sum_{i=1}^n D_i$, where $D_i \subset X_F$ are irreducible pairwise distinct Cartier divisors (possibly non-reduced). We get an isomorphism of F-algebras

$$\Gamma(D_F, \mathcal{O}_{D_F}) \xrightarrow{\simeq} \Gamma(D_1, \mathcal{O}_{D_1}) \times \ldots \times \Gamma(D_n, \mathcal{O}_{D_n}),$$

and it is enough to construct a section above every single factor. Thus, we may assume that D_F is irreducible (possibly non-reduced), and we denote by $D_{F,\mathrm{red}} \subset D_F$ the reduced locus. Then $D_{F,\mathrm{red}}$ defines a finite field extension $F' := \Gamma(D_{F,\mathrm{red}}, \mathcal{O}_{D_{F,\mathrm{red}}})$ of the separably closed field F. Hence, F' is itself separably closed, and $\mathcal{F}|_{D_{F,\mathrm{red}}}$ admits a section because $\mathcal{F}$ is smooth. As the kernel of the map $\Gamma(D_F, \mathcal{O}_{D_F}) \to F'$ is nilpotent, the section lifts to D_F by the lifting criterion for smoothness. This finishes the proof. □

Lemma 3.4 ii) shows that there is a transitive action map

$$L\mathcal{G} \times_{\mathcal{O}} \mathrm{Gr}_{\mathcal{G}} \longrightarrow \mathrm{Gr}_{\mathcal{G}}. \tag{3.8}$$

Let us look at the fibers of (3.8) over $\mathcal{O}$.

Corollary 3.5. *i) Let F be a field, and let $\mathcal{O} \to F$ be a ring morphism. The underlying reduced subscheme $D_{F,\mathrm{red}} \subset D_F$ is an effective Cartier divisor on X_F, and we write $D_{F,\mathrm{red}} = \sum_{i=1}^n D_i$ where D_i are distinct irreducible, i.e., the D_i are closed points of X_F. There is a canonical isomorphism of F-spaces*

$$\mathrm{Gr}_{(X,\mathcal{G},D)} \otimes_{\mathcal{O}} F \xrightarrow{\simeq} \prod_{i=1}^n \mathrm{Gr}_{(X_F,\mathcal{G}_F,D_i)},$$

compatible with the action of $L\mathcal{G}_{(X,\mathcal{G},D)} \otimes_{\mathcal{O}} F \simeq \prod_{i=1}^n L\mathcal{G}_{(X_F,\mathcal{G}_F,D_i)}$.
ii) Let $\mathcal{O} = F$ be a field, and let $D = [x]$ be the divisor on X defined by a closed point $x \in X$. The residue field $K := \kappa(x)$ is a finite field extension, and we assume that K/F is separable. There is a canonical isomorphism of F-spaces

$$\mathrm{Gr}_{(X,\mathcal{G},D)} \xrightarrow{\simeq} \mathrm{Res}_{K/F}(\mathrm{Gr}_{(X_K,\mathcal{G}_{X_K},D)})$$

compatible with the action of $L\mathcal{G}_{(X,\mathcal{G},D)} \simeq \mathrm{Res}_{K/F}(L\mathcal{G}_{(X_K,\mathcal{G}_{X_K},D)})$.

Proof. For i), we may by (3.2) assume $\mathcal{O} = F$. It is immediate from Remark 3.3 that for any $\mathcal{O}$-algebra R, we have $R[\![D_{\mathrm{red}}]\!] \simeq R[\![D]\!]$ (resp. $R(\!(D_{\mathrm{red}})\!) \simeq R(\!(D)\!)$). Further, there is a canonical isomorphism

$$R[\![D_{\mathrm{red}}]\!] \xrightarrow{\simeq} \prod_{i=1}^n R[\![D_i]\!] \quad (\text{resp. } R(\!(D_{\mathrm{red}})\!) \xrightarrow{\simeq} \prod_{i=1}^n R(\!(D_i)\!))$$

because X is of dimension 1, and hence $D_i \cap D_j = \varnothing$ for $i \neq j$. Part i) follows from Lemma 3.4 ii). For ii), first note that if we consider D as the divisor on X_K defined by the K-point x, then $\mathrm{Gr}_{(X_K, \mathcal{G}_{X_K}, D)}$ is the twisted affine Grassmannian over K, cf. Example 3.1 i). Let $\tilde{K}/F$ be the splitting field of K which is a finite Galois extension with Galois group $\tilde{\Gamma}$. There is a canonical isomorphism of $\tilde{K}$-algebras

$$K \otimes_F \tilde{K} \xrightarrow{\simeq} \prod_{\psi\colon K \hookrightarrow \tilde{K}} \tilde{K}, \qquad a \otimes b \longmapsto (\psi(a)\cdot b)_\psi,$$

which is $\tilde{\Gamma}$-equivariant for the action $\gamma * (c_\psi)_\psi \mapsto (\gamma(c_\psi))_{\gamma\psi}$ on the target. Applying this isomorphism to $D \otimes_F \tilde{K}$, we obtain by i) a $\tilde{\Gamma}$-equivariant isomorphism

$$\mathrm{Gr}_{(X,\mathcal{G},D)} \otimes_F \tilde{K} \xrightarrow{\simeq} \prod_\psi \mathrm{Gr}_{(X_K, \mathcal{G}_{X_K}, D)} \otimes_{K,\psi} \tilde{K}, \tag{3.9}$$

compatible with the actions of the loop groups. The canonical descent datum on the source in (3.9) induces a descent datum on the target of (3.9) which implies ii). □

Let us point out some useful compatibility with Weil restriction of scalars.

Corollary 3.6. *Let $X' \to X$ be a finite flat surjective map of smooth quasi-projective $\mathcal{O}$-curves, and assume $\mathcal{G} = \mathrm{Res}_{X'/X}(\mathcal{G}')$ for a smooth affine X'-group scheme $\mathcal{G}'$ of finite presentation. If $D' := D \times_X X'$, then the natural map is an isomorphism of $\mathcal{O}$-spaces*

$$\mathrm{Gr}_{(X',\mathcal{G}',D')} \xrightarrow{\simeq} \mathrm{Gr}_{(X,\mathcal{G},D)}, \quad (\mathcal{F}', \alpha') \mapsto (\mathrm{Res}_{X'/X}(\mathcal{F}'), \mathrm{Res}_{X'/X}(\alpha')). \tag{3.10}$$

Proof. Since $X' \to X$ is finite flat surjective, the closed subscheme $D' \subset X'$ is a relative effective Cartier divisor which is finite flat over $\mathcal{O}$. Hence, the functor $\mathrm{Gr}_{(X',\mathcal{G}',D')}$ is well defined. Using Lemma 3.4 ii), the map (3.10) is induced for any $\mathcal{O}$-algebra R by the canonical map of R-algebras

$$\mathrm{Spec}(R[\![D']\!]) \to \mathrm{Spec}(R[\![D]\!]) \times_X X' \quad (\text{resp. } \mathrm{Spec}(R(\!(D')\!)) \to \mathrm{Spec}(R(\!(D)\!)) \times_X X').$$

If R is Noetherian, then the first map (hence the second map) is an isomorphism by [StaPro, 00MA] because $X' \to X$ is finite. In particular, (3.10) is an isomorphism for any Noetherian $\mathcal{O}$-algebra R. As both functors in (3.10) commute with filtered colimits of $\mathcal{O}$-algebras, the corollary follows. □

3.1.2. *Basic representability properties.* The starting point is the following lemma, and we sketch its proof.

Lemma 3.7. *If $\mathcal{G} = \mathrm{Gl}_{n,X}$, then the functor $\mathrm{Gr}_\mathcal{G}$ is representable by an ind-projective $\mathcal{O}$-ind-scheme.*

Proof. Let R be an $\mathcal{O}$-algebra. If $\mathcal{G} = \mathrm{Gl}_{n,X}$, then $\mathrm{Gr}_\mathcal{G}(R)$ classifies rank n vector bundles $\mathcal{E}$ on X_R together with an isomorphism $\mathcal{E}|_{U_R} \simeq \mathcal{O}^n_{U_R}$ where $U_R := (X \backslash D)_R$. Let $\mathcal{I}_{D_R} \subset \mathcal{O}_{X_R}$ be the invertible ideal sheaf defined by $D_R \subset X_R$. For $N \geq 1$, let $\mathrm{Gr}_{\mathcal{G},N}$ be the $\mathcal{O}$-space whose R-valued points are rank n vector bundles $\mathcal{E}$ on X_R such that as $\mathcal{O}_{X_R}$-modules

$$\left(\mathcal{I}^N_{D_R}\right)^n \subset \mathcal{E} \subset \left(\mathcal{I}^{-N}_{D_R}\right)^n.$$

Every vector bundle is locally free and by bounding the poles (resp. zeros) of basis elements, one gets as $\mathcal{O}$-spaces

$$\mathrm{colim}_{N \geq 1} \mathrm{Gr}_{\mathcal{G},N} \xrightarrow{\simeq} \mathrm{Gr}_\mathcal{G}.$$

We claim that each $\mathrm{Gr}_{\mathcal{G},N}$ is representable by a Quot-scheme as follows. The $\mathcal{O}_{X_R}$-module $\mathcal{E}_{N,R} := (\mathcal{I}_D^{-N}/\mathcal{I}_D^N)^n \otimes_\mathcal{O} R$ is coherent and locally free over R. Let Quot_N be the $\mathcal{O}$-space whose R-points are coherent $\mathcal{O}_{X_R}$-module quotients $\mathcal{E}_{N,R} \twoheadrightarrow \mathcal{Q}$ which are locally free R-modules. The functor Quot_N is representable by a projective $\mathcal{O}$-scheme by the theory of Quot-schemes applied to the finite flat $\mathcal{O}$-scheme $2ND$, and the coherent $\mathcal{O}_{2ND} = \mathcal{O}_X/\mathcal{I}_D^{2N}$-module $\mathcal{E}_{N,\mathcal{O}}$. More precisely, in the notation of [FGA, §5.1.4], one has a finite disjoint union

$$\mathrm{Quot}_N = \coprod_{r \in \mathbb{Z}_{\geq 0}} \mathrm{Quot}^{r,\mathcal{O}_{2ND}}_{\mathcal{E}_{N,\mathcal{O}}/2ND/\mathrm{Spec}(\mathcal{O})},$$

and the representability result is then a theorem of Grothendieck [FGA, §5.5.2, Thm 5.14]. Note that the structure sheaf $\mathcal{O}_{2ND}$ is relatively ample for $2ND \to \operatorname{Spec}(\mathcal{O})$ because the map is finite (cf. [StaPro, Tag 01VG, 28.35.6]). Concretely, Quot_N is the closed subscheme of the Grassmannian

$$\mathrm{Quot}_N \hookrightarrow \mathrm{Grass}(\mathcal{E}_{N,\mathcal{O}}),$$

which is cut out by the condition that the quotients are stable under the finitely many nilpotent operators $u_1, \dots, u_n$ on $\mathcal{E}_{N,\mathcal{O}}$ induced by some presentation $2ND = \operatorname{Spec}(\mathcal{O}[u_1, \dots, u_n]/J)$. Hence, to prove the lemma it is enough to show that as functors

$$\mathrm{Gr}_{\mathcal{G},N} \xrightarrow{\simeq} \mathrm{Quot}_N, \quad \mathcal{E} \longmapsto \left(\mathcal{I}_{D_R}^{-N}\right)^n/\mathcal{E}. \tag{3.11}$$

We need to check that $\mathcal{Q} := \left(\mathcal{I}_{D_R}^{-N}\right)^n/\mathcal{E}$ is a locally free R-module. This follows from the isomorphism as R-modules $\mathcal{O}_{U_R}^n/\mathcal{E} \simeq \oplus_{k\geq 0}\mathcal{I}_{D_R}^{-k-1}\mathcal{E}/\mathcal{I}_{D_R}^{-k}\mathcal{E}$, and the short exact sequence

$$0 \to \left(\mathcal{I}_{D_R}^{-N}\right)^n/\mathcal{E} \to \mathcal{O}_{U_R}^n/\mathcal{E} \to \mathcal{O}_{U_R}^n/\left(\mathcal{I}_{D_R}^{-N}\right)^n \to 0,$$

cf. also the argument in [Zhu, Lem 1.1.5]. Conversely, let $\mathcal{Q} \in \mathrm{Quot}_N(R)$, and define the coherent $\mathcal{O}_{X_R}$-module

$$\mathcal{E} \overset{\text{def}}{=} \ker\left(\left(\mathcal{I}_{D_R}^{-N}\right)^n \to \mathcal{E}_{N,R} \to \mathcal{Q}\right).$$

We need to show that $\mathcal{E}$ is a rank n vector bundle on X_R. Covering X_R with affine schemes, we may assume $X_R = \operatorname{Spec}(S)$ is affine. Let $\mathfrak{p} \subset S$ be a prime ideal lying over a prime ideal $\mathfrak{m} := \mathfrak{p} \cap R \subset R$. By [StaPro, Tag 00M] applied to the map of local rings $R_\mathfrak{m} \to S_\mathfrak{p}$ and the module $\mathcal{E}_\mathfrak{p}$ (note that $\mathcal{E}_\mathfrak{p}$ is still $R_\mathfrak{m}$-flat), to prove $\mathcal{E}_\mathfrak{p}$ is free over $S_\mathfrak{p}$ we are reduced to the case where R is a field. In the case where R is a field, $\mathcal{E} \subset \left(\mathcal{I}_{D_R}^{-N}\right)^n$ is a torsion-free rank n submodule, and since $X_R \to \operatorname{Spec}(R)$ is a smooth curve, $\mathcal{E}$ is a vector bundle. □

Remark 3.8. Using Lemma 3.4 ii), the set $\mathrm{Gr}_{\mathrm{Gl}_{n,X}}(\mathcal{O})$ can be identified with the set of $\mathcal{O}[\![D]\!]$-lattices in $\mathcal{O}(\!(D)\!)$, i.e., in the notation of Lemma 3.2, the set of $\mathcal{O}[\![D]\!]$-submodules $M \subset \mathcal{O}(\!(D)\!)$ such that for some $N \gg 0$, $\left(I_D^N\right)^n \subset M \subset \left(I_D^{-N}\right)^n$ and $\left(I_D^{-N}\right)^n/M$ is a locally free $\mathcal{O}$-module.

Proposition 3.9. *If $\mathcal{G} \hookrightarrow G$ is a monomorphism of smooth X-group schemes of finite presentation such that the fppf-quotient $G/\mathcal{G}$ is a quasi-affine scheme (resp. affine scheme), then the map $\mathrm{Gr}_\mathcal{G} \to \mathrm{Gr}_G$ is representable by a quasi-compact immersion (resp. closed immersion).*

Proof. Following the proof of [Zhu, Prop 1.2.6], it is enough to establish the analogue of [Zhu, Lem 1.2.7]. Let R an $\mathcal{O}$-algebra, and let $p\colon V \to \hat{D}_R$ be an affine scheme of finite presentation. Let s be a section of p over $\hat{D}_R^o$. We need to prove that the presheaf assigning to any R-algebra R', the set of sections s' of p over $\hat{D}_{R'}$ such that $s'|_{\hat{D}_{R'}^o} = s|_{\hat{D}_{R'}^o}$ is representable by a closed subscheme of $\operatorname{Spec}(R)$. Indeed, choosing a closed embedding $V \subset \mathbb{A}^n_{\hat{D}_R}$ for some $n \gg 0$ and using that $R[\![D]\!] \subset R(\!(D)\!)$ is injective, we reduce to the case $V = \mathbb{A}^n_{\hat{D}_R}$. The presheaf in question is representable by the locus on $\operatorname{Spec}(R)$ where the class $\bar{s}$ of the section $s \in V(\hat{D}_R^o) = R(\!(D)\!)^n$ in $(R(\!(D)\!)/R[\![D]\!])^n$ vanishes. With the notation of Lemma 3.2, we have $\bar{s} \in E_{[-N,0]} \otimes_\mathcal{O} R$ for some $N \gg 0$. As $E_{[-N,0]}$ is a reflexive $\mathcal{O}$-module, we see that giving an element of $E_{[-N,0]} \otimes_\mathcal{O} R$ is equivalent to giving a map of R-schemes $\operatorname{Spec}(R) \to \mathbb{V}(E_{[-N,0]} \otimes_\mathcal{O} R)$. Then the presheaf in question is representable by the equalizer of the two maps corresponding to the elements $\bar{s}, 0 \in E_{[-N,0]} \otimes_\mathcal{O} R$ which is a closed subscheme of $\operatorname{Spec}(R)$. □

Corollary 3.10. *i) If there exists a monomorphism $\mathcal{G} \hookrightarrow \mathrm{Gl}_{n,X}$ such that the fppf-quotient is a quasi-affine scheme (resp. an affine scheme), then $\mathrm{Gr}_\mathcal{G} = \operatorname{colim}_i \mathrm{Gr}_{\mathcal{G},i}$ is representable by a separated $\mathcal{O}$-ind-scheme of ind-finite type (resp. separated ind-proper $\mathcal{O}$-ind-scheme). Each $\mathrm{Gr}_{\mathcal{G},i}$ can be chosen to be $L^+\mathcal{G}$-stable.*

ii) If in i) the representation $\mathcal{G} \hookrightarrow \mathrm{Gl}_{n,X}$ exists étale locally on $\mathcal{O}$, then $\mathrm{Gr}_\mathcal{G} = \operatorname{colim}_i \mathrm{Gr}_{\mathcal{G},i}$ is a separated $\mathcal{O}$-ind-algebraic space of ind-finite presentation (resp. separated ind-proper $\mathcal{O}$-ind-algebraic space). Each $\mathrm{Gr}_{\mathcal{G},i}$ can be chosen to be $L^+\mathcal{G}$-stable.

iii) If $\mathcal{G} = G \otimes_\mathcal{O} X$ is constant and G is a reductive $\mathcal{O}$-group scheme, then $\mathrm{Gr}_\mathcal{G}$ is representable by an ind-proper $\mathcal{O}$-ind-algebraic space.

Proof. Part i) is immediate from Lemma 3.7 and Proposition 3.9. For ii), we use part i) together with Lemma 3.11 below. Note that the diagonal of $\mathrm{Gr}_{\mathcal{G}}$ being representable by a closed immersion follows from the same property of $\mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}}}$ and the effectivity of descent for closed immersions. Further, if $\mathcal{O} \to \mathcal{O}'$ is étale, then the method of Lemma 3.11 shows that an $L^+\mathcal{G} \otimes_{\mathcal{O}} \mathcal{O}'$-stable presentation of $\mathrm{Gr}_{\mathcal{G}} \otimes_{\mathcal{O}} \mathcal{O}'$ induces an $L^+\mathcal{G}$-stable presentation of $\mathrm{Gr}_{\mathcal{G}}$ (because $L^+\mathcal{G}$ is affine and flat, and taking the scheme theoretic image commutes with flat base change). For iii), note that after an étale cover $\mathcal{O} \to \mathcal{O}'$, the group scheme $G_{\mathcal{O}'} := G \otimes_{\mathcal{O}} \mathcal{O}'$ is split reductive, and in particular linearly reductive. If we choose a closed immersion $G_{\mathcal{O}'} \hookrightarrow \mathrm{Gl}_{n,\mathcal{O}'}$, then the quotient $\mathrm{Gl}_{n,\mathcal{O}'}/G_{\mathcal{O}'}$ is representable by an affine scheme by [Al14, Cor 9.7.7], and iii) follows from ii). □

Lemma 3.11. *Let X be an $\mathcal{O}$-space with schematic diagonal, and such that there exists a étale surjective (as sheaves) map of $\mathcal{O}$-spaces $U \to X$ with U an $\mathcal{O}$-ind-scheme. If either $U \to X$ is quasi-compact or U is quasi-separated, then X is an $\mathcal{O}$-ind-algebraic space.*

Proof. Given a presentation $U = \mathrm{colim}_{i\in I} U_i$ with U_i being schemes, we need to construct a presentation $X = \mathrm{colim}_{i\in I} X_i$ with X_i being algebraic spaces. For each i, consider $U_i \subset U \to X$. We define X_i' to be the scheme theoretic image of the map

$$U_i \times_X U \subset U \times_X U \xrightarrow{p_2} U. \tag{3.12}$$

This well defined for the following reason: Since $U_i \times_X U$ is a quasi-compact scheme, the map (3.12) factors through $U_j \subset U$ for some $j \gg i$. In either case, $U \to X$ quasi-compact or U quasi-separated, the map (3.12) is quasi-compact. By [StaPro, 01R8], the scheme theoretic image behaves well for quasi-compact maps, and $X_i' \subset U_j$ is a quasi-compact closed subscheme. As scheme theoretic images of quasi-compact maps commute with flat base change [StaPro, Tag 081I], the scheme X_i' is equipped with a descent datum relative to $U \to X$, and defines a closed $\mathcal{O}$-subspace $X_i \subset X$ together with an étale surjective map $X_i' \to X_i$. As $X_i \subset X$ is closed, the diagonal of X_i is schematic, and X_i is a quasi-compact algebraic space. By construction the X_i form a filtered direct system indexed by the poset I, and the canonical map $\mathrm{colim}_{i\in I} X_i \to X$ is an isomorphism (because $U \to X$ is a sheaf surjection, and $\mathrm{colim}_i X_i' = U$ by construction). □

Remark 3.12. It would be nice to give a proof of representability of $\mathrm{Gr}_{\mathcal{G}}$ which does not refer to the choice of an embedding $\mathcal{G} \hookrightarrow \mathrm{Gl}_{n,X}$.

3.2. **The open cell.** In the following two subsections, we apply our methods to prove Theorem 3.16, a generalization of Theorem A from the introduction. The results are not used in the proof of our Main Theorem.

We specialize to the case where $X = \mathbb{A}^1_{\mathcal{O}}$, and where $\mathcal{G} = G \otimes_{\mathcal{O}} X$ is constant, i.e., the base change of a smooth affine $\mathcal{O}$-group scheme G of finite presentation. In this case, we denote $L_D\mathcal{G}$ (resp. $L_D^+\mathcal{G}$; resp. $\mathrm{Gr}_{(X,\mathcal{G},D)}$) by $LG = L_DG$ (resp. $L^+G = L^+_DG$; resp. $\mathrm{Gr}_G = \mathrm{Gr}_{(X,G,D)}$).

Since $D \subset \mathbb{A}^1_{\mathcal{O}}$ is assumed to be finite over $\mathcal{O}$, the subscheme $D \subset \mathbb{P}^1_{\mathcal{O}}$ is closed and defines a relative effective Cartier divisor. In particular, Lemma 3.4 ii) (the Beauville-Laszlo lemma) implies that $\mathrm{Gr}_{(\mathbb{A}^1_{\mathcal{O}},G,D)} = \mathrm{Gr}_{(\mathbb{P}^1_{\mathcal{O}},G,D)}$ by extending torsors trivially to ∞.

The *negative loop group* is the functor on the category of $\mathcal{O}$-algebras

$$L^-G\colon R \mapsto G(\mathbb{P}^1_R \backslash D_R). \tag{3.13}$$

Then L^-G is an $\mathcal{O}$-space which is a subgroup functor $L^-G \subset LG$.

Lemma 3.13. *The functor L^-G is representable by an ind-affine ind-scheme locally of ind-finite presentation over $\mathcal{O}$.*

Proof. That the affine schemes are of finite presentation follows from the fact that L^-G commutes with filtered colimits (because G is of finite presentation). One verifies that L^- commutes with finite products and equalizers, and hence the proof of representability is reduced to the case $G = \mathbb{A}^1_{\mathcal{O}}$, cf. the proof of Lemma 3.2. We have to show that the functor on the category of $\mathcal{O}$-algebras R given by the global sections $R \mapsto \Gamma(\mathcal{O}_{\mathbb{P}^1_R \backslash D_R})$ is representable by an ind-affine ind-scheme. But as R-modules $\Gamma(\mathcal{O}_{\mathbb{P}^1_R \backslash D_R}) = \mathrm{colim}_n \Gamma(\mathcal{O}_{\mathbb{P}^1_R}(nD_R))$, and we claim that $\Gamma(\mathcal{O}_{\mathbb{P}^1_R}(nD_R))$ is finite locally free: Indeed, this follows from the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^1_R} \to \mathcal{O}_{\mathbb{P}^1_R}(nD_R) \to \mathcal{I}^{-1}_{nD_R}/\mathcal{O}_{\mathbb{P}^1_R} \to 0,$$

and the vanishing of $H^1_{\mathrm{Zar}}(\mathbb{P}^1_R, \mathcal{O}_{\mathbb{P}^1_R})$. This proves the lemma. □

Now define $L^{--}G = \ker(L^-G \to G)$ for $g \mapsto g(\infty)$. Then the intersection $L^{--}G \cap L^+G$ is trivial inside LG, and we consider the orbit map

$$L^{--}G \longrightarrow \mathrm{Gr}_G, \quad g^- \longmapsto g^- \cdot e_0, \tag{3.14}$$

where $e_0 \in \mathrm{Gr}_G$ denotes the base point.

Lemma 3.14. *The map (3.14) is representable by an open immersion, and identifies $L^{--}G$ with those pairs $(\mathcal{F}, \alpha)$ where $\mathcal{F}$ is the trivial torsor.*

Proof. The argument is the same as the deformation argument given in [HaRi, Lem 3.1], and we do not repeat it here. □

3.3. **Geometry of $\mathbb{G}_m$-actions on Gr_G.** We assume $X = \mathbb{A}^1_{\mathcal{O}}$, and $\mathcal{G} = G \otimes_{\mathcal{O}} X$ with G being a reductive $\mathcal{O}$-group scheme with connected (and hence geometrically connected) fibers. Let $\chi\colon \mathbb{G}_{m,\mathcal{O}} \to G$ be an $\mathcal{O}$-rational cocharacter. The cocharacter χ induces via the composition

$$\mathbb{G}_{m,\mathcal{O}} \subset L^+\mathbb{G}_{m,\mathcal{O}} \xrightarrow{L^+\chi} L^+G \subset LG \tag{3.15}$$

a left $\mathbb{G}_m$-action on the affine Grassmannian $\mathrm{Gr}_G \to \mathrm{Spec}(\mathcal{O})$. As in (2.2), we obtain maps of $\mathcal{O}$-spaces

$$(\mathrm{Gr}_G)^0 \leftarrow (\mathrm{Gr}_G)^{\pm} \to \mathrm{Gr}_G. \tag{3.16}$$

Let us mention the following lemma which implies the ind-representability of the spaces (3.16), in light of Theorem 2.1 and Corollary 3.10.

Lemma 3.15. *The $\mathbb{G}_m$-action on Gr_G is étale locally linearizable.*

Proof. After an étale cover $\mathcal{O} \to \mathcal{O}'$, there exists a closed immersion $\mathrm{Gr}_{G'_{\mathcal{O}}} \to \mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}'}}$ (cf. Proposition 3.10 iii)) which is $\mathbb{G}_m$-equivariant with respect to the action on $\mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}'}}$ given by the cocharacter $\mathbb{G}_{m,\mathcal{O}'} \xrightarrow{\chi} G_{\mathcal{O}'} \to \mathrm{Gl}_{n,\mathcal{O}'}$. The proof of Lemma 3.11 shows that an $L^+G_{\mathcal{O}'}$-stable presentation of $\mathrm{Gr}_{G_{\mathcal{O}'}}$ by quasi-compact schemes induces an L^+G-stable presentation of Gr_G by quasi-compact algebraic spaces. To prove the lemma it is enough to show that the $\mathbb{G}_m$-action on $\mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}'}}$ is Zariski locally linearizable, and we reduce to the case $\mathcal{O} = \mathcal{O}'$, $G = \mathrm{Gl}_{n,\mathcal{O}}$. By [Co14, Prop 6.2.11; Prop 3.1.9], Zariski locally on $\mathcal{O}$ the cocharacter χ lies in a split maximal torus in $\mathrm{Gl}_{n,\mathcal{O}}$ which is $\mathcal{O}$-conjugate to the diagonal matrices in $\mathrm{Gl}_{n,\mathcal{O}}$, and hence is after conjugation with a permutation matrix dominant. In this way, we reduce to the case where χ is a standard dominant cocharacter given by $\lambda \mapsto \mathrm{diag}(\lambda^{a_1}, \dots, \lambda^{a_n})$ for some integers $a_1 \geq \ldots \geq a_n$. With the notation of Lemma 3.7, it is now immediate that the $\mathbb{G}_m$-action on $\mathrm{Quot}_N \subset \mathrm{Grass}(\mathcal{E}_{N,\mathcal{O}})$ is linear, and compatible with the transition maps for varying N. The lemma follows. □

Our aim is to express (3.16) in terms of group theoretical data related to the cocharacter χ, cf. Theorem 3.16 below.

Let χ act on G via conjugation $(\lambda, g) \mapsto \chi(\lambda) \cdot g \cdot \chi(g)^{-1}$. The fixed points $M = G^0$ (resp. the attractor $P^+ = G^+$; resp. the repeller $P^- = G^-$) defines a closed subgroup of G which is smooth of finite presentation over $\mathcal{O}$, cf. [Mar15]. The group M is the centralizer of χ, and is by the classical theory over a field a reductive $\mathcal{O}$-group scheme which is fiberwise connected (hence fiberwise geometrically connected). By (2.2) we have natural maps of $\mathcal{O}$-groups

$$M \leftarrow P^{\pm} \to G. \tag{3.17}$$

Theorem 3.16. *The maps (3.17) induce a commutative diagram of $\mathcal{O}$-ind-algebraic spaces*

$$\begin{tikzcd} \mathrm{Gr}_M \arrow[d, "\iota^0"'] & \mathrm{Gr}_{P^\pm} \arrow[l] \arrow[r] \arrow[d, "\iota^\pm"'] & \mathrm{Gr}_G \arrow[d, "\mathrm{id}"'] \\ (\mathrm{Gr}_G)^0 & (\mathrm{Gr}_G)^\pm \arrow[l] \arrow[r] & \mathrm{Gr}_G, \end{tikzcd} \tag{3.18}$$

where the vertical maps ι^0 and $\iota^\pm$ are isomorphisms.

Remark 3.17. i) An interesting example to which Theorem 3.16 applies is the case of fusion Grassmannians $\mathrm{Gr}_G \to \mathbb{A}^n_F$, cf. Example 3.1 iv) with[2] $C = \mathbb{A}^1_F$. Hence, Theorem 3.16 implies Theorem A from the introduction. Note that the group G need not be defined over F, but can be a general reductive group scheme over the n-th symmetric power $(\mathbb{A}^1_F)^{(n)}$. Changing the set up slightly, the group G could even be a general reductive group scheme over $\mathbb{A}^n_F$ (take $D = \mathrm{Spec}(\mathcal{O}) = \mathbb{A}^n_F$, $X = \mathbb{A}^n_F \times_F \mathbb{A}^1_F$ and consider the divisor $\mathbb{A}^n_F \to \mathbb{A}^n_F \times \mathbb{A}^1_F$, $(x_i)_i \mapsto ((x_i)_i, \sum_i x_i)$ for $i = 1, \dots, n$).
ii) Note that Theorem 3.16 also generalizes [HaRi, Lem 3.6] and justifies [HaRi, sentence containing (3.33)].

3.3.1. *Construction of ι^0 and $\iota^\pm$.* The strategy of construction is the same as in [HaRi] which we recall for readability.

As the $\mathbb{G}_m$-action on Gr_M is trivial, the natural map $\mathrm{Gr}_M \to \mathrm{Gr}_G$ factors as $\mathrm{Gr}_M \to (\mathrm{Gr}_G)^0 \to \mathrm{Gr}_G$ which defines ι^0. For the construction of the map $\iota^\pm$, we use the Rees construction explained in Heinloth [He18, 1.6.2]. The $\mathbb{G}_m$-action $P^\pm \times \mathbb{G}_{m,\mathcal{O}} \to P^\pm, (p,\lambda) \mapsto \chi(\lambda^\pm)\cdot p \cdot \chi(\lambda^\pm)^{-1}$ via conjugation extends via the monoid action of $\mathbb{A}^1$ on $(\mathbb{A}^1_{\mathcal{O}})^\pm$ in (2.1) to a monoid action

$$m_\chi \colon P^\pm \times \mathbb{A}^1_{\mathcal{O}} \longrightarrow P^\pm \tag{3.19}$$

such that $m_\chi(p,0) \in M$. We let $\mathrm{gr}_\chi : P^\pm \times \mathbb{A}^1_{\mathcal{O}} \to P^\pm \times \mathbb{A}^1_{\mathcal{O}}$, $(p,\lambda) \mapsto (m_\chi(p,\lambda),\lambda)$ viewed as an $\mathbb{A}^1_{\mathcal{O}}$-group homomorphism. Then the restriction $\mathrm{gr}_\chi|_{\{1\}}$ is the identity whereas $\mathrm{gr}_\chi|_{\{0\}}$ is the composition $P^\pm \to M \to P^\pm$. For a point $(\mathcal{F}^\pm, \alpha^\pm) \in \mathrm{Gr}_{P^\pm}(R)$, the Rees bundle is

$$\mathrm{Rees}_\chi(\mathcal{F}^\pm, \alpha^\pm) \overset{\mathrm{def}}{=} \mathrm{gr}_{\chi,*}(\mathcal{F}^\pm_{\mathbb{A}^1_R}, \alpha^\pm_{\mathbb{A}^1_R}) \in \mathrm{Gr}_{P^\pm}(\mathbb{A}^1_R), \tag{3.20}$$

where $\mathrm{gr}_{\chi,*}$ denotes the push forward under the $\mathbb{A}^1$-group homomorphism. Then the restriction $\mathrm{Rees}_\chi(\mathcal{F}^\pm, \alpha^\pm)|_{\{1\}_R}$ is equal to $(\mathcal{F}^\pm, \alpha^\pm)$ whereas $\mathrm{Rees}_\chi(\mathcal{F}^\pm, \alpha^\pm)|_{\{0\}_R}$ is the image of $(\mathcal{F}^\pm, \alpha^\pm)$ under the composition $\mathrm{Gr}_{P^\pm} \to \mathrm{Gr}_M \to \mathrm{Gr}_{P^\pm}$. One checks that $\mathrm{Rees}_\chi(\mathcal{F}^\pm, \alpha^\pm)$ is $\mathbb{G}_m$-equivariant, and hence defines an R-point of $(\mathrm{Gr}_{P^\pm})^\pm$. As the Rees construction is functorial, we obtain a map of $\mathcal{O}$-spaces

$$\mathrm{Rees}_\chi : \mathrm{Gr}_{P^\pm} \to (\mathrm{Gr}_{P^\pm})^\pm, \tag{3.21}$$

which is inverse to the map $(\mathrm{Gr}_{P^\pm})^\pm \to \mathrm{Gr}_{P^\pm}$ given by evaluating at the unit section. We define the map $\mathrm{Gr}_{P^\pm} \to (\mathrm{Gr}_G)^\pm$ to be the composition $\mathrm{Gr}_{P^\pm} \simeq (\mathrm{Gr}_{P^\pm})^\pm \to (\mathrm{Gr}_G)^\pm$ where the latter map is deduced from the natural map $\mathrm{Gr}_{P^\pm} \to \mathrm{Gr}_G$. This constructs the commutative diagram (3.18).

We claim that the map ι^0 (resp. $\iota^\pm$) is representable by a quasi-compact immersion. By [Co14, Thm 2.4.1], the fppf quotient G/M is quasi-affine, and hence ι^0 is representable by a quasi-compact immersion by Proposition 3.9. Note that since M is reductive, the space Gr_M is ind-proper and hence ι^0 is even a closed immersion. For $\iota^\pm$, we use that quasi-compact immersions are of effective descent (cf. [StaPro, Tag 0247, 02JR]), and after passing to an étale ring extension of $\mathcal{O}$, we reduce to the case where G is linearly reductive. As in the proof of Corollary 3.10, we choose $G \hookrightarrow \mathrm{Gl}_{n,\mathcal{O}}$ such that $\mathrm{Gl}_{n,\mathcal{O}}/G$ is quasi-affine (or even affine). Let $Q^+ \subset \mathrm{Gl}_{n,\mathcal{O}}$ (resp. $Q^- \subset \mathrm{Gl}_{n,\mathcal{O}}$) be the attractor (resp. repeller) subgroup defined by the cocharacter $\mathbb{G}_{m,\mathcal{O}} \xrightarrow{\chi} G \to \mathrm{Gl}_{n,\mathcal{O}}$. Then we have $P^\pm = Q^\pm \times_{\mathrm{Gl}_{n,\mathcal{O}}} G$. The quotient $Q^\pm/P^\pm$ is an algebraic space of finite presentation over $\mathcal{O}$, and the map $i \colon Q^\pm/P^\pm \hookrightarrow \mathrm{Gl}_{n,\mathcal{O}}/G$ is a monomorphism of finite type (hence separated and quasi-finite, by [StaPro, Tag 0463, 59.27.10]). Thus, $Q^\pm/P^\pm$ is a scheme, and the map i is quasi-affine by Zariski's main theorem. In particular, $Q^\pm/P^\pm$ is quasi-affine as well. Now there is a commutative diagram of $\mathcal{O}$-spaces

$$\begin{tikzcd} \mathrm{Gr}_{P^\pm} \arrow[dr, dotted] \arrow[drr, bend left] \arrow[dd] & & \\ & (\mathrm{Gr}_G)^\pm \arrow[r] \arrow[d] & \mathrm{Gr}_G \arrow[d] \\ \mathrm{Gr}_{Q^\pm} \arrow[r, "\simeq"] & (\mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}}})^\pm \arrow[r] & \mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}}}, \end{tikzcd} \tag{3.22}$$

[2] The case of general smooth F-curves C can be reduced to the special case of $\mathbb{A}^1_F$, but we do not need this in the present manuscript.

constructed as follows. The map $\mathrm{Gr}_G \to \mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}}}$ is a quasi-compact immersion by Proposition 3.9, and as Gr_G is ind-proper, it is a closed immersion. Hence, the square is Cartesian by general properties of attractor (resp. repeller) ind-schemes. This also constructs the dotted arrow in (3.22) which is the map $\iota^\pm$. Further, the map $\mathrm{Gr}_{Q^\pm} \to (\mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}}})^\pm$ is an isomorphism by Lemma 3.18 below. The map $\mathrm{Gr}_{P^\pm} \to \mathrm{Gr}_{Q^\pm}$ is a quasi-compact immersion because $Q^\pm/P^\pm$ is quasi-affine. Since $(\mathrm{Gr}_G)^\pm \to (\mathrm{Gr}_{\mathrm{Gl}_{n,\mathcal{O}}})^\pm$ is a closed immersion, the map $\iota^\pm$ is a quasi-compact immersion.

Lemma 3.18. *If $G = \mathrm{Gl}_{n,\mathcal{O}}$, then the maps ι^0 and $\iota^\pm$ are isomorphisms.*

Proof. As in the proof of Lemma 3.15, we reduce to the case where χ is a standard dominant cocharacter. Then χ corresponds to a $\mathbb{Z}$-grading on $V := \mathcal{O}^n$, say $V = \oplus_{i\in\mathbb{Z}} V_i$, compatible with the standard $\mathcal{O}$-basis of V. The group M (resp. P^+/P^-) is a standard Levi (resp. standard parabolic) of automorphisms of V preserving the grading (resp. the ascending/descending filtration induced from the grading). In the description of Lemma 3.7, the subfunctor Gr_M (resp. $\mathrm{Gr}_{P^\pm}$) are those vector bundles $\mathcal{E} \in \mathrm{Gr}_G(R)$ compatible with the grading (resp. filtration induced by the grading) on $V \otimes_{\mathcal{O}} \mathcal{O}_{U_R}$. Likewise, the grading on V induces in the notation of Lemma 3.7 gradings on $\mathcal{E}_{N,\mathcal{O}} = V \otimes_{\mathcal{O}} (\mathcal{I}_D^{-N}/\mathcal{I}_D^N)$ for each $N \geq 1$. As in Lemma 3.15, we have a closed $\mathbb{G}_m$-equivariant immersion, and hence the diagram of $\mathcal{O}$-schemes

$$\begin{array}{ccc} \mathrm{Quot}_N^0 & \longrightarrow & \mathrm{Grass}(\mathcal{E}_{N,\mathcal{O}})^0 \\ \downarrow & & \downarrow \\ \mathrm{Quot}_N & \longrightarrow & \mathrm{Grass}(\mathcal{E}_{N,\mathcal{O}}), \end{array}$$

is cartesian, and likewise on attractor (resp. repeller) schemes. The equality $\mathrm{Grass}(\mathcal{E}_{N,\mathcal{O}})^0 = \prod_{i\in\mathbb{Z}} \mathrm{Grass}(V_i \otimes_{\mathcal{O}} (\mathcal{I}_D^{-N}/\mathcal{I}_D^N))$ is immediate, and one checks that $\mathrm{Grass}(V \otimes_{\mathcal{O}} (\mathcal{I}_D^{-N}/\mathcal{I}_D^N))^\pm$ is the subfunctor of those subspaces in $\mathcal{E}_{N,\mathcal{O}}$ compatible with the filtration. The lemma follows. $\square$

3.3.2. *Proof of Theorem 3.16.* We need a lemma first. By functoriality of the loop group construction, the $\mathbb{G}_m$-action on G via χ-conjugation gives an $\mathbb{G}_m$ on LG (resp. L^+G; resp. L^-G). There are natural monomorphisms on negative loop groups

$$L^-M \longrightarrow (L^-G)^0; \tag{3.23}$$

$$L^-P^\pm \longrightarrow (L^-G)^\pm. \tag{3.24}$$

Lemma 3.19. *The maps (3.23) and (3.24) are isomorphisms.*

Proof. Replacing $\mathcal{O}$ by an étale cover, we may assume that there exists a closed embedding $G \hookrightarrow \mathrm{Gl}_{n,\mathcal{O}}$. By the proof of Lemma 3.13 (resp. Lemma 3.2 i)), the induced map $L^-G \to L^-\mathrm{Gl}_{n,\mathcal{O}}$ is a closed immersion.

Let $\chi' \colon \mathbb{G}_{m,\mathcal{O}} \xrightarrow{\chi} G \to \mathrm{Gl}_{n,\mathcal{O}}$, and denote the fixed point group (resp. attractor/repeller group) by L (resp. $Q^\pm$). It is straight forward to check $L^-M = L^-G \cap L^-L$ (resp. $L^-P^\pm = L^-G \cap L^-Q^\pm$) and $(L^-G)^0 = L^-G \cap (L^-\mathrm{Gl}_{n,\mathcal{O}})^0$ (resp. $(L^-G)^\pm = L^-G \cap (L^-\mathrm{Gl}_{n,\mathcal{O}})^\pm$). Hence, we may assume $G = \mathrm{Gl}_{n,\mathcal{O}}$.

After passing to a Zariski cover of $\mathcal{O}$, we may assume that χ is a standard dominant cocharacter, cf. proof of Lemma 3.15. We have for every $\mathcal{O}$-algebra R,

$$(L^-\mathrm{Gl}_{n,\mathcal{O}})^0(R) = \{g \in G(\mathbb{P}^1_R \backslash D_R) \mid \forall S \in (\text{R-Alg}), \lambda \in \mathbb{G}_m(S)\colon\ \chi(\lambda)\cdot g \cdot \chi(\lambda)^{-1} = g\}.$$

Let $g \in (L^-\mathrm{Gl}_{n,\mathcal{O}})^0(R)$. To show $g \in (L^-M)(R)$, we can take $S = R[t,t^{-1}]$ to see that the desired entries in the matrix g vanish. The case of $(L^-G)^\pm$ is similar, and the lemma follows. $\square$

First case. Let $\mathcal{O} = F$ be a field. By fpqc-descent, we may assume that F is algebraically closed. Then $D_{\mathrm{red}} = \sum_{i=1}^d [x_i]$ for pairwise distinct points $x_i \in X(F)$. If $d = 1$, the maps ι^0 and $\iota^\pm$ are isomorphisms in light of Example 3.1 i) and [HaRi, Prop 3.4]. In general, by Corollary 3.5 each ind-scheme in (3.18) is a direct product of d copies (compatible with the maps) of classical affine Grassmannians formed using local parameters at x_i. The $\mathbb{G}_m$-action on the product via

$$\mathbb{G}_m \subset L^+_D\mathbb{G}_m \simeq L^+_{[x_1]}\mathbb{G}_m \times_F \ldots \times_F L^+_{[x_n]}\mathbb{G}_m$$

is the diagonal action, and we conclude using Lemma 2.2 and the case $d = 1$.

Second case. Let $\mathcal{O}$ be an Artinian local ring with maximal ideal $\mathfrak{m}$, and residue field F. Passing to the strict Heselization, we may assume that F is separably closed. The restriction of ι^0 (resp. $\iota^\pm$) to the open cell $L^{--}M$ (resp. $L^{--}P^\pm$) is an isomorphism by Lemma 3.19. By Lemma 3.14, there is the open subset

$$V_M \overset{\text{def}}{=} \bigcup_m m \cdot L^{--}M \cdot e_0 \quad (\text{resp. } V_{P^\pm} \overset{\text{def}}{=} \bigcup_p p \cdot L^{--}P^\pm \cdot e_0),$$

of Gr_M (resp. $\mathrm{Gr}_{P^\pm}$), where the union runs over all $m \in LM(\mathcal{O})$ (resp. $p \in LP^\pm(\mathcal{O})$). The LM-equivariance (resp. $LP^\pm$-equivariance) of ι^0 (resp. $\iota^\pm$) implies that $\iota^0|_{V_M}$ (resp. $\iota^\pm|_{V_{P^\pm}}$) is an isomorphism. As Gr_M (resp. $\mathrm{Gr}_{P^\pm}$) is a nilpotent thickening of $\mathrm{Gr}_M \otimes_{\mathcal{O}} F$ (resp. $\mathrm{Gr}_{P^\pm} \otimes_{\mathcal{O}} F$), it is enough to show that V_M (resp. $V_{P^\pm}$) contains the special fiber. As G splits over F (because separably closed), the points $\mathrm{Gr}_M(F) \subset \mathrm{Gr}_M$ (resp. $\mathrm{Gr}_{P^\pm}(F) \subset \mathrm{Gr}_{P^\pm}$) are dense which follows from the density of $\mathbb{A}^n_F(F) \subset \mathbb{A}^n_F$ and the cellular structure of these spaces. Thus, it suffices to show that $\mathrm{Gr}_M(F) \subset V_M$ (resp. $\mathrm{Gr}_{P^\pm}(F) \subset V_{P^\pm}$). In view of Lemma 3.4 ii), it suffices to show that the reduction map $LM(\mathcal{O}) \to LM(F)$ (resp. $LP^\pm(\mathcal{O}) \to LP^\pm(F)$) is surjective. As $\mathcal{O}$ is Artinian, the ring $\mathcal{O}(\!(D)\!)$ is (semi-local) Artinian, and the reduction map $\mathcal{O}(\!(D)\!) \to F(\!(D)\!)$ is surjective with nilpotent kernel $\mathfrak{m}(\!(D)\!)$. Hence, the desired surjectivity follows from the formal lifting criterion using the smoothness of M (resp. $P^\pm$). This handles the second case.

The general case. Passing to an étale extension of $\mathcal{O}$, we may assume that (3.18) is a diagram of ind-schemes, cf. Corollary 3.10. In view of (3.2), the closed immersion ι^0 (resp. quasi-compact immersion $\iota^\pm$) is fiberwise bijective, and hence bijective. Now Theorem 3.16 follows from Lemma 3.20 below using the second case.

Lemma 3.20. *Let $\mathcal{O}$ be a Noetherian ring, and let $\iota\colon Y \to Z$ be a quasi-compact immersion of finite type $\mathcal{O}$-schemes. If ι is set-theoretically bijective, and if for every maximal ideal $\mathfrak{m} \subset \mathcal{O}$ and every $n \geq 1$, the reduction $\iota \otimes \mathcal{O}/\mathfrak{m}^n$ is an isomorphism, then ι is an isomorphism.*

Proof. By [StaPro, Tag 01QV], the map ι factors as an open immersion followed by a closed immersion: $Y \to \bar{Y} \to Z$. As ι is bijective, we have $Y = \bar{Y}$ and ι is a bijective closed immersion. Being an isomorphism is local on the target, and we may assume that $Z = \mathrm{Spec}(A)$ and hence $Y = \mathrm{Spec}(B)$ are affine. The map of $\mathcal{O}$-algebras $\iota^\#\colon A \to B$ is surjective (because closed immersion), and each element in $I := \ker(\iota^\#)$ is nilpotent (because $\iota^\#$ is bijective on spectra). It is enough to show that for the localization $I_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m} \subset \mathcal{O}$. Without loss of generality, we may assume that $\mathcal{O}$ is local with maximal ideal $\mathfrak{m}$. If $\mathfrak{m}A = A$, i.e., the fiber of Z over $\mathfrak{m}$ is empty, there is nothing to prove, and we may assume that $\mathfrak{m}A \subset A$ is a proper ideal. As $A/\mathfrak{m}^nA \to B/\mathfrak{m}^nB$ is an isomorphism for all $n \geq 1$, we have $I \subset \cap_{n\geq 1}\mathfrak{m}^nA$. But since $\mathfrak{m}^nA = (\mathfrak{m}A)^n$ and $\mathfrak{m}A \subset A$ is a proper ideal in a Noetherian ring, we have $\cap_{n\geq 1}\mathfrak{m}^nA = 0$ by Krull's intersection theorem. The lemma follows. □

4. Recollection on local models for Weil-restricted groups

In this section, we collect a few properties of the Weil-restricted affine Grassmannians as constructed in [Lev16]. We provide proofs for several statements which appear to be well-known but for which we could not find proofs in the literature.

4.1. **Notation.** Let $F/\mathbb{Q}_p$ be a finite extension with ring of integers $\mathcal{O}_F$, and residue field k with q elements. Let K/F be a finite, totally ramified extension with ring of integers $\mathcal{O}_K$ and the same residue field k. Fix a uniformizer ϖ of K, and denote by $Q \in F[u]$ the minimal polynomial, i.e. Q is the unique irreducible normalized polynomial with $Q(\varpi) = 0$. Note that $Q \in \mathcal{O}_F[u]$, and that $Q \equiv u^{[K:F]} \mod \varpi$.

Let $\breve{F}$ denote the completion of the maximal unramified extension of F inside a fixed algebraic closure $\bar{F}$, and let $\sigma \in \mathrm{Aut}(\breve{F}/F)$ denote the Frobenius generator.

In §4.4 below, we specialize the general set-up of §3 to the case where $\mathcal{O} = \mathcal{O}_F$, $X = \mathbb{A}^1_{\mathcal{O}}$ and where D is the relative effective Cartier on X defined by $\{Q = 0\}$. We first summarize some properties of parahoric groups for Weil-restricted groups (cf. §4.2), and the group schemes $\mathcal{G}$ over $X = \mathbb{A}^1_{\mathcal{O}_F}$ constructed in [PZ13, Lev16] (cf. §4.3).

4.2. **Parahoric Group Schemes for Weil-restricted groups.** Let G be a connected reductive K-group. Fix a maximal K-split torus A, a maximal $\breve{K}$-split torus S containing A and defined over K. Let $M = Z_G(A)$ denote the centralizer of A which is a minimal K-Levi subgroup of G, and let $T = Z_G(S)$ be the centralizer of S. Then T is a maximal torus because $G_{\breve{K}}$ is quasi-split by Steinberg's theorem.

We are interested in parahoric subgroups of the Weil restriction of scalars $\tilde{G} := \mathrm{Res}_{K/F}(G)$. We will first need to classify the maximal F-split tori in $\tilde{G}$.

Lemma 4.1. *Suppose T is any K-torus, so that $\tilde{T} = \mathrm{Res}_{K/F}T$ is an F-torus. Then there is a canonical isomorphism of groups*

$$X_*(\tilde{T})_{\Gamma_F} = X_*(T)_{\Gamma_K}. \tag{4.1}$$

In particular, the F-split rank of $\tilde{T}$ is the K-split rank of T.

Proof. Recall that $\tilde{T}$ represents the functor on F-tori which sends the F-torus T' to

$$\mathrm{Hom}_{K-\mathrm{tori}}(T' \otimes_F K, T) = \mathrm{Hom}_{\Gamma_K-\mathrm{Mod}}(X_*(T'), X_*(T)) = \mathrm{Hom}_{\Gamma_F-\mathrm{Mod}}(X_*(T'), \mathrm{Ind}_{\Gamma_K}^{\Gamma_F}(X_*(T)).$$

We deduce that $X_*(\mathrm{Res}_{K/F}(T)) = \mathrm{Ind}_{\Gamma_K}^{\Gamma_F}(X_*(T)) \cong X_*(T) \otimes_{\mathbb{Z}[\Gamma_K]} \mathbb{Z}[\Gamma_F]$ (since $[\Gamma_F : \Gamma_K] < \infty$). Then the H_0-version of Shapiro's lemma gives $(X_*(T) \otimes_{\mathbb{Z}[\Gamma_K]} \mathbb{Z}[\Gamma_F])_{\Gamma_F} = X_*(T)_{\Gamma_K}$, which implies the lemma. □

Under the bijection

$$\mathrm{Hom}_F(T', \mathrm{Res}_{K/F}G) = \mathrm{Hom}_K(T'_K, G), \tag{4.2}$$

$T' \to \mathrm{Res}_{K/F}G$ is injective if and only if the corresponding morphism $T'_K \to G$ is injective. Since any K-split torus is of the form T'_K for a unique F-split torus T', this shows that the rank of a maximal F-split torus in $\mathrm{Res}_{K/F}G$ is the same as the rank of a maximal K-split torus in G. Suppose now $A \subset G$ is a maximal K-split torus. Write $A = A_{0,K}$ for a unique F-split torus A_0. Using the canonical embedding $A_0 \hookrightarrow \mathrm{Res}_{K/F}A_{0,K} = \mathrm{Res}_{K/F}A$, we see that A_0 is the F-split component of $\mathrm{Res}_{K/F}A$ and also a maximal F-split torus in $\tilde{G}$.

From now on, we will abuse notation and denote by $\tilde{A}$ the image of $A_0 \hookrightarrow \mathrm{Res}_{K/F}A \hookrightarrow \mathrm{Res}_{K/F}G$ (even though $\tilde{A}$ is not a Weil restriction of a torus). The discussion following (4.2) shows that $A \mapsto \tilde{A}$ gives a bijection between maximal K-split tori in G and maximal F-split tori in $\tilde{G}$.

Let us note that since S is $\breve{K}$-split (and K and F have the same residue field), in our notation $\tilde{S} \hookrightarrow \mathrm{Res}_{K/F}S$ is a maximal $\breve{F}$-split torus which is defined over F.

Lemma 4.2. *Letting $\tilde{M} = \mathrm{Res}_{K/F}(M)$ and $\tilde{T} = \mathrm{Res}_{K/F}(T)$, we have $\tilde{M} = Z_{\tilde{G}}(\tilde{A})$ and $\tilde{T} = Z_{\tilde{G}}(\tilde{S})$ as subgroups of $\tilde{G} = \mathrm{Res}_{K/F}(G)$.*

Proof. The torus $\tilde{A}_{\bar{F}}$ (resp. $\tilde{S}_{\bar{F}}$) is the diagonal torus inside $\prod_{K \hookrightarrow \bar{F}} A \otimes_{K,\psi} \bar{F}$ (resp. $\prod_{K \hookrightarrow \bar{F}} S \otimes_{K,\psi} \bar{F}$). By considering their centralizers inside $\prod_{K \hookrightarrow \bar{F}} G \otimes_{K,\psi} \bar{F}$, the lemma is obvious. □

The correspondence $\tilde{A} \leftrightarrow A$ induces a correspondence between the apartments in the (extended) Bruhat-Tits buildings $\mathscr{B}(\tilde{G}, F)$ and $\mathscr{B}(G, K)$. We will show that there is a canonical isomorphism

$$\mathscr{B}(\tilde{G}, F) \simeq \mathscr{B}(G, K), \tag{4.3}$$

equivariant for the action of $\tilde{G}(F) = G(K)$, and compatible with an identification of apartments $\mathscr{A}(\tilde{G}, \tilde{A}, F) = \mathscr{A}(G, A, K)$.

The *Iwahori-Weyl group* $W = W(\tilde{G}, \tilde{A}, F)$ is the group

$$W \overset{\mathrm{def}}{=} \mathrm{Norm}_{\tilde{G}}(\tilde{A})(F)/\tilde{M}_1, \tag{4.4}$$

where $\tilde{M}_1$ is the unique parahoric subgroup of the minimal Levi $\tilde{M}$, cf. [HR08, Ri16a]. (By Lemma 4.2, $\tilde{M}$ is a minimal F-Levi subgroup of $\tilde{G}$.) We define $\breve{W} = W(\tilde{G}, \tilde{S}, \breve{F})$ analogously.

Lemma 4.3. *There is a canonical identification of Iwahori-Weyl groups*

$$W(\tilde{G}, \tilde{A}, F) = W(G, A, K) \quad \text{and} \quad W(\tilde{G}, \tilde{S}, \breve{F}) = W(G, S, \breve{K}).$$

Proof. As in Lemma 4.2, one shows $\mathrm{Norm}_{\tilde{G}}(\tilde{A}) = \mathrm{Res}_{K/F}(\mathrm{Norm}_G(A))$, and hence $\mathrm{Norm}_{\tilde{G}}(\tilde{A})(F) = \mathrm{Norm}_G(A)(K)$. By Lemma 4.4 below, $\tilde{M}_1 = M_1$. The first equality follows and the second is similar. □

Lemma 4.4. *Let $G(F)_1 \subset G(F)$ denote the Kottwitz kernel, i.e., $G(F)_1 = G(F) \cap G(\breve{F})_1$ where*

$$G(\breve{F})_1 = \ker[\kappa_G \,:\, G(\breve{F}) \to X^*(Z(G^\vee)^{I_F})]$$

where κ_G is the Kottwitz homomorphism of [Ko97, §7]. *Then $\tilde{G}(\breve{F})_1 = G(\breve{K})_1$ and $\tilde{G}(F)_1 = G(K)_1$.*

Proof. The result is clear when G is an induced torus: $\tilde{G}(\breve{F})_1$ and $G(\breve{K})_1$ coincide with the unique maximal bounded subgroup of $\tilde{G}(\breve{F}) = G(\breve{K})$, thus thanks to Lemma 4.1, $\kappa_{\tilde{G}} : \tilde{G}(\breve{F}) \to X_*(\tilde{G})_{I_F}$ is $\kappa_G : G(\breve{K}) \to X_*(G)_{I_K}$. If G is any torus, then taking a presentation by induced tori as in the construction of κ_G (cf. [Ko97, §7.2]), the same assertion holds for G. Clearly the result holds for $G = G_{\mathrm{sc}}$ and hence for $G_{\mathrm{der}} = G_{\mathrm{sc}}$ by reduction to the torus case. Finally the general case follows by the method of z-extensions as in the construction of κ_G ([Ko97, §7.4]). □

Lemma 4.5. *There is a canonical isomorphism of apartments $\mathscr{A}(\tilde{G}, \tilde{S}, \breve{F}) = \mathscr{A}(G, S, \breve{K})$ compatible with the action of the Iwahori-Weyl groups $W(\tilde{G}, \tilde{S}, \breve{F}) = W(G, S, \breve{K})$ and the action of a geometric Frobenius element $\Phi \in \Gamma_F$.*

Proof. Let $\breve{\Sigma}_{\tilde{G}}$ (resp. $\breve{\Sigma}_G$) denote the Bruhat-Tits échelonnage root system attached to $(\tilde{G}, \tilde{S})$ (resp. (G, S)). Taking $T = T_{\mathrm{sc}}$ in Lemma 4.1 and using [HR08, Lem. 15], we obtain

$$Q^\vee(\breve{\Sigma}_{\tilde{G}}) = X_*(\tilde{T}_{\mathrm{sc}})_{I_F} = X_*(T_{\mathrm{sc}})_{I_K} = Q^\vee(\breve{\Sigma}_G).$$

By considering minimal positive generators of these lattices, we deduce that $\breve{\Sigma}_{\tilde{G}} = \breve{\Sigma}_G$. This isomorphism is compatible with the action of Φ on both sides, noting Φ is a common geometric Frobenius element in Γ_F and in Γ_K. This gives the identification of affine root hyperplanes needed to prove the isomorphism of apartments

$$\mathscr{A}(\tilde{G}, \tilde{S}, \breve{F}) = \mathscr{A}(G, S, \breve{K}).$$

The isomorphism is equivariant for $W(\tilde{G}, \tilde{S}, \breve{F}) = W(G, S, \breve{K})$ and Φ. □

Proposition 4.6. *There is a canonical isomorphism $\mathscr{B}(\tilde{G}, F) \simeq \mathscr{B}(G, K)$, equivariant for the action of $\tilde{G}(F) = G(K)$, and compatible with an identification of apartments $\mathscr{A}(\tilde{G}, \tilde{A}, F) = \mathscr{A}(G, A, K)$.*

Proof. By construction $\mathscr{B}(G, \breve{K}) = (G(\breve{K}) \times \mathscr{A}(G, S, \breve{K}))/\sim$, where $(g, x) \sim (g', x')$ if there exists $n \in \mathrm{Norm}_G(S)(\breve{K})$ such that $n \cdot x = x'$ and $g^{-1}g'n \in U_x$. Here U_x is the subgroup of $G(\breve{K})$ generated by the affine root groups $U_{\alpha+r}$ associated to $\alpha + r$ with $\alpha(x) + r \geq 0$, for $(\alpha, r) \in \breve{\Sigma}_G \times \mathbb{Z}$. Because $\breve{\Sigma}_{\tilde{G}} = \breve{\Sigma}_G$, U_x is the same for $\tilde{G}$ and G, and so the equivalence relation is the same for $\tilde{G}$ and G. Using Lemma 4.5, this proves $\mathscr{B}(\tilde{G}, \breve{F}) = \mathscr{B}(G, \breve{K})$, equivariantly for Φ, and the proposition follows by étale descent. □

Let $\tilde{\mathbf{f}}$ be a facet of $\mathscr{A}(\tilde{G}, \tilde{A}, F)$, and denote by $\mathbf{f}$ the corresponding facet in $\mathscr{A}(G, A, K)$. Let $\mathcal{G}_{\tilde{\mathbf{f}}}$ (resp. $\mathcal{G}_{\mathbf{f}}$) be the associated parahoric group scheme over $\mathcal{O}_F$ (resp. over $\mathcal{O}_K$).

Proposition 4.7. *There is a canonical isomorphism of $\mathcal{O}_F$-group schemes $\mathcal{G}_{\tilde{\mathbf{f}}} \simeq \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}(\mathcal{G}_{\mathbf{f}})$ inducing the identity on generic fibers.*

Proof. By the defining property of parahoric group schemes, it suffices to check that the group $\mathcal{H} := \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}(\mathcal{G}_{\mathbf{f}})$ is a smooth affine $\mathcal{O}_F$-group scheme of finite type with (geometrically) connected special fiber, with the property that $\mathcal{H}(\mathcal{O}_{\breve{F}})$ is the intersection of the Kottwitz kernel $\tilde{G}(\breve{F})_1$ with the pointwise fixer in $\tilde{G}(\breve{F})$ of $\tilde{\mathbf{f}}$ (which we view as a subset of the building over $\breve{F}$). The $\mathcal{O}_F$-group $\mathcal{H}$ is smooth affine and of finite type by general properties of Weil restriction of scalars, cf. [BLR90, §7.6, Thm 4, Prop 5]. If $R = \mathcal{O}_K/\varpi^{[K:F]}$, then the special fiber is given by

$$\mathcal{H} \otimes_{\mathcal{O}_F} k \;=\; \mathrm{Res}_{R/k}(\mathcal{G}_{\mathbf{f}} \otimes_{\mathcal{O}_K} R),$$

which is a successive extension of smooth (geometrically) connected groups, and hence (geometrically) connected. As $\breve{K} = K \otimes_F \breve{F}$ we have $\mathcal{H}(\mathcal{O}_{\breve{F}}) = \mathcal{G}_{\mathbf{f}}(\mathcal{O}_{\breve{K}})$. But $\mathcal{G}_{\mathbf{f}}(\mathcal{O}_{\breve{K}})$ is the intersection of

$G(\breve{K})_1 = \tilde{G}(\breve{F})_1$ (Lemma 4.4) with the pointwise fixer in $G(\breve{K})$ of $\tilde{\mathbf{f}}$, by Proposition 4.6 applied over the field extension $\breve{K}/\breve{F}$. The proposition follows. □

Corollary 4.8. *Every parahoric $\mathcal{O}_F$-group scheme of $\tilde{G}$ is of the form $\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}(\mathcal{G}_{\mathbf{f}})$ for a unique facet $\mathbf{f} \subset \mathscr{B}(G, K)$.*

□

The subgroup $W_{\tilde{\mathbf{f}}} = W_{\tilde{\mathbf{f}}}(\tilde{G}, \tilde{A}, F)$ of W associated with $\tilde{\mathbf{f}}$ is the group

$$W_{\tilde{\mathbf{f}}} \overset{\mathrm{def}}{=} \left(\mathrm{Norm}_{\tilde{G}}(\tilde{A})(F) \cap \mathcal{G}_{\tilde{\mathbf{f}}}(\mathcal{O}_F)\right)/\tilde{M}_1.$$

The isomorphism $W(\tilde{G}, \tilde{A}, F) = W(G, A, K)$ induces an isomorphism $W_{\tilde{\mathbf{f}}}(\tilde{G}, \tilde{A}, F) = W_{\mathbf{f}}(G, A, K)$.

Let us point out a consequence of Proposition 4.6 which is used later.

Corollary 4.9. *There is a canonical identification $\mathcal{Z}(\tilde{G}(F), \mathcal{G}_{\tilde{\mathbf{f}}}(\mathcal{O}_F)) = \mathcal{Z}(G(K), \mathcal{G}_{\mathbf{f}}(\mathcal{O}_K))$ of centers of parahoric Hecke algebras compatible with the Bernstein isomorphism of* [Hai14, Thm 11.10.1]*, where the Haar measures are normalized to give $\mathcal{G}_{\tilde{\mathbf{f}}}(\mathcal{O}_F) = \mathcal{G}_{\mathbf{f}}(\mathcal{O}_K)$ volume 1.*

Proof. In view of $\mathcal{G}_{\tilde{\mathbf{f}}}(\mathcal{O}_F) = \mathcal{G}_{\mathbf{f}}(\mathcal{O}_K)$, the equality of the centers is clear, and it remains to show the compatibility with the Bernstein isomorphism. This follows from the equality

$$\Lambda_{\tilde{M}} := \tilde{M}(F)/\tilde{M}_1 = M(K)/M_1 =: \Lambda_M,$$

combined with the definition of Bernstein isomorphisms given by the integration formula (e.g. [Hai14, 11.11]) and the isomorphism of finite relative Weyl groups $W_0(\tilde{G}, \tilde{A}, F) = W_0(G, A, K)$ consistent with Lemma 4.3. □

4.3. **Group schemes over $\mathbb{A}^1_{\mathcal{O}_F}$.** Let G be a connected reductive K-group which splits over a tamely ramified extension, and fix a chain of subgroups $A \subset S \subset T \subset M$ as in §4.2. Further, fix a parabolic F-subgroup P containing M.

In [PZ13, §3], a reductive $\mathcal{O}_K[u^{\pm}]$-group scheme $\underline{G}$ admitting a maximal torus, and with connected fibers is constructed. As observed in [Lev16, §3.1; Prop 3.3], the group scheme $\underline{G}$ is defined over $\mathcal{O}_F[u^{\pm}]$ in the following sense.

Proposition 4.10. *i) There exists a connected reductive $\mathcal{O}_F[u^{\pm}]$-group $\underline{G}$ together with a tuple of smooth closed $\mathcal{O}_F[u^{\pm}]$-subgroups $(\underline{A}, \underline{S}, \underline{T}, \underline{M}, \underline{P})$ and an isomorphism of K-groups*

$$(\underline{G}, \underline{A}, \underline{S}, \underline{T}, \underline{M}, \underline{P}) \otimes_{\mathcal{O}_F[u^{\pm}], u \mapsto \varpi} K \;\simeq\; (G, A, S, T, M, P),$$

where $\underline{A}$ is a maximal $\mathcal{O}_F[u^{\pm}]$-split torus, $\underline{S}$ a maximal $\mathcal{O}_{\breve{F}}[u^{\pm}]$-split torus defined over $\mathcal{O}_F[u^{\pm}]$, $\underline{T}$ its centralizer, $\underline{M}$ the centralizer of $\underline{A}$ (a minimal Levi), and $\underline{P}$ a parabolic $\mathcal{O}_F[u^{\pm}]$-subgroup with Levi $\underline{M}$.

ii) The base change $\underline{G}_{\breve{\mathcal{O}}_{K_0}[u^{\pm}]}$ is quasi-split. In particular, $\underline{T}$ is a maximal torus.

Proof. The result in [Lev16, Prop 3.3] is slightly more general where K/F is not assumed to be totally ramified, but we do not need this more general version in the manuscript. Let us recall some elements of the construction as needed later. Let $\tilde{K}/K$ be a tamely ramified extension which splits G. After possibly enlarging $\tilde{K}$, we may assume:

1) the group G is quasi-split over the maximal unramified subextension $\tilde{F}_0$ of $\tilde{K}/F$;
2) there is a uniformizer $\tilde{\varpi} \in \tilde{K}$ and an integer $\tilde{e} \geq 1$ such that $\varpi = \tilde{\varpi}^{\tilde{e}}$, and therefore $\tilde{K} \xrightarrow{\sim} \tilde{F}_0[v]/Q(v^{\tilde{e}})$ via $\tilde{\varpi} \mapsto v$;
3) $\tilde{F}_0$ contains a primitive $\tilde{e}$-th root of unity, cf. [PZ13, §3.1].

There is a cocartesian diagram[3] of $\mathcal{O}_F$-algebras

$$\begin{array}{ccc} \mathcal{O}_{\tilde{F}_0}[v] & \xrightarrow{v \mapsto \tilde{\varpi}} & \tilde{K} \\ {\scriptstyle u \mapsto v^{\tilde{e}}}\uparrow & & \uparrow \\ \mathcal{O}_F[u] & \xrightarrow[u \mapsto \varpi]{} & K \end{array} \tag{4.5}$$

[3]This differs from [Lev16, §3.1] in that Levin uses instead of $\tilde{F}_0$ the maximal unramified subxtension $\tilde{K}_0/K$ of $\tilde{K}/K$; this seems to be a mistake, e.g., the diagram corresponding to (4.5) is not cocartesian.

One can prove that $\mathcal{O}_{\tilde{F}_0}[v]/\mathcal{O}_F[u]$ is a ramified Galois cover with group isomorphic to $\tilde{\Gamma} := \mathrm{Gal}(\tilde{K}/K)$; for this we use that $\tilde{F}_0$ contains a primitive $\tilde{e}$-th root of unity. As in [PZ13, §3], the $\mathcal{O}_F[u^{\pm}]$-group scheme $\underline{G}$ is constructed in [Lev16, §3.1] by descending a suitable choice of Chevalley model for $G_{\tilde{K}}$ along the (étale) ring extension $\mathcal{O}_{\tilde{F}_0}[v^{\pm}]/\mathcal{O}_F[u^{\pm}]$, cf. [PZ13, §3] and [Lev16, §3.1] for details. See also Example 4.14. □

Let us denote

$$(G', A', S', T', M', P') \overset{\text{def}}{=} (\underline{G}, \underline{A}, \underline{S}, \underline{T}, \underline{M}, \underline{P}) \otimes_{\mathcal{O}_F[u^{\pm}]} k(\!(u)\!). \tag{4.6}$$

Then G' is a connected reductive $F' := k(\!(u)\!)$-group, and (A', S', T', M', P') are analogous to the corresponding groups above, cf. the discussion in [PZ13, 4.1.2; 4.1.3], [Lev16, 3.3]. Further, we obtain a canonical identification of the apartments

$$\mathscr{A}(G, A, K) = \mathscr{A}(G', A', F'), \tag{4.7}$$

cf. [PZ13, 4.1.3], [Lev16, Prop 3.3.1 ff.]. We shall use the following two results in §6 below.

Lemma 4.11. *There is an identification of Iwahori-Weyl groups $W(G, A, K) = W(G', A', F')$ which is compatible with the action on the apartments under the identification (4.7).*

Proof. Over $\breve{F}$ we obtain a σ-equivariant isomorphism according to [PZ13, 4.1.2], [Lev16, 3.3.0.1] compatible with the action on the apartments. The general case follows by taking σ-fixed points from [Ri16a, §1.2] (cf. also [PZ13, 4.1.3], [Lev16, Prop 3.3.1 ii)]). □

Now let $\mathcal{G}_{\mathbf{f}}$ be a parahoric $\mathcal{O}_K$-group scheme of G whose facet $\mathbf{f}$ is contained in $\mathscr{A}(G, A, K)$. Then under (4.7) we obtain a unique facet $\mathbf{f}' \in \mathscr{A}(G', A', F')$, and hence a parahoric $k[\![u]\!]$-group scheme $\mathcal{G}_{\mathbf{f}'}$ of G'.

Lemma 4.12. *There is a canonical identification $\mathcal{Z}(G(K), \mathcal{G}_{\mathbf{f}}(\mathcal{O}_K)) = \mathcal{Z}(G'(F'), \mathcal{G}_{\mathbf{f}'}(\mathcal{O}_{F'}))$ of centers of parahoric Hecke algebras, where the Haar measures are normalized to give $\mathcal{G}_{\mathbf{f}}(\mathcal{O}_K)$ (resp. $\mathcal{G}_{\mathbf{f}'}(\mathcal{O}_{F'})$) volume 1.*

Proof. Applying Lemma 4.11 for M, we obtain an identification of abelian groups

$$\Lambda_M := M(K)/M_1 = M'(k(\!(t)\!))/M'_1 =: \Lambda_{M'}, \tag{4.8}$$

where M_1 (resp. M'_1) is the unique parahoric group scheme of $M(K)$ (resp. $M'(k(\!(t)\!))$). The result follows via the Bernstein isomorphisms [Hai14, Thm 11.10.1]

$$\mathcal{Z}(G'(F'), \mathcal{G}'(\mathcal{O}_{F'})) \simeq \bar{\mathbb{Q}}_\ell[\Lambda_{M'}]^{W_0(G',A',F')} = \bar{\mathbb{Q}}_\ell[\Lambda_M]^{W_0(G,A,K)} \simeq \mathcal{Z}(G(K), \mathcal{G}(\mathcal{O}_K)),$$

noting that the finite relative Weyl groups of (G, A, K) and (G', A', F') are isomorphic (compatible with the action on $\Lambda_M = \Lambda_{M'}$), and that $k_K = k_F$ because K/F is totally ramified. □

Theorem 4.13. *Fix $(\underline{G}, \underline{A}, \underline{S}, \underline{T})$ and $\mathcal{G}_{\mathbf{f}}$ with $\mathbf{f} \in \mathscr{A}(G, A, K)$ as above. There exists a unique (up to unique isomorphism) tuple of smooth affine $\mathcal{O}_F[u]$-group schemes $(\underline{\mathcal{G}}, \underline{\mathcal{A}}, \underline{\mathcal{S}}, \underline{\mathcal{T}})$ with geometrically connected fibers satisfying the following properties:*

i) The restriction $(\underline{\mathcal{G}}, \underline{\mathcal{A}}, \underline{\mathcal{S}}, \underline{\mathcal{T}})|_{\mathcal{O}_F[u^{\pm}]}$ is $(\underline{G}, \underline{A}, \underline{S}, \underline{T})$ as $\mathcal{O}_F[u^{\pm}]$-groups.

ii) The base change of $\underline{\mathcal{G}}$ under $\mathcal{O}_F[u] \to \mathcal{O}_K$, $u \mapsto \varpi$ is the parahoric group $\mathcal{G} = \mathcal{G}_{\mathbf{f}}$.

iii) The base change of $\underline{\mathcal{G}}$ under $\mathcal{O}_F[u] \to k[\![u]\!]$, $u \mapsto u \mod \mathfrak{m}_F$ is the parahoric group scheme $\mathcal{G}' = \mathcal{G}_{\mathbf{f}'}$.

iv) The group $\underline{\mathcal{A}}$ is a split $\mathcal{O}_F[u]$-torus, $\underline{\mathcal{S}}$ a $\mathcal{O}_F[u]$-torus which splits over $\mathcal{O}_{\breve{F}}[u]$ and $\underline{\mathcal{T}}$ is a smooth affine $\mathcal{O}_F[u]$-group scheme such that $\underline{\mathcal{T}} \otimes \mathcal{O}_K$ (resp. $\underline{\mathcal{T}} \otimes k[\![u]\!]$) is the neutral component of the lft Néron model of T (resp. T').

Proof. This is [Lev16, Thm 3.3.3, Prop 3.3.4], cf. also [PZ13, Thm 4.1]. □

Example 4.14. Suppose $G = T$ is a tamely ramified torus over K. Let T_H be the split torus over $\mathcal{O}_F$ such that T is given by a 1-cocycle

$$[\tau] \in H^1(\tilde{\Gamma}, \mathrm{Aut}(T_H \otimes_{\mathcal{O}_F} \tilde{K})).$$

Explicitly,

$$T = \Big(\mathrm{Res}_{\tilde{K}/K}(T_H \otimes_{\mathcal{O}_F} \tilde{K})\Big)^{\tilde{\Gamma}}.$$

We let $T_H \otimes_{\mathcal{O}_F} \tilde{\mathcal{O}}_0[v]$ be the split torus over $\tilde{\mathcal{O}}_0 := \mathcal{O}_{\tilde{F}_0}$ (cf. (4.5)), which is endowed with Galois actions $\tau(\gamma) \otimes \gamma$ for $\gamma \in \tilde{\Gamma}$ which we view as Galois descent data used to give a torus over $\mathcal{O}_F[u]$. Explicitly, we define $\underline{T}/\mathcal{O}_F[u^{\pm}]$ and $\underline{\mathcal{T}}/\mathcal{O}_F[u]$ by

$$\underline{T} = \Big(\mathrm{Res}_{\tilde{\mathcal{O}}_0[v^{\pm}]/\mathcal{O}_F[u^{\pm}]}(T_H \otimes_{\mathcal{O}_F} \tilde{\mathcal{O}}_0[v^{\pm}])\Big)^{\tilde{\Gamma}}.$$

and $\underline{\mathcal{T}}$ as the (fiberwise) neutral component of

$$\Big(\mathrm{Res}_{\tilde{\mathcal{O}}_0[v]/\mathcal{O}_F[u]}(T_H \otimes_{\mathcal{O}_F} \tilde{\mathcal{O}}_0[v])\Big)^{\tilde{\Gamma}}.$$

Write $\mathrm{Gal}(\tilde{K}/K) = \langle\gamma\rangle \rtimes \langle\sigma\rangle$ where γ generates the inertia subgroup and σ is a lift of a generator of $\mathrm{Gal}(\tilde{K}_0/K) \cong \mathrm{Gal}(\tilde{F}_0/F)$ for $\tilde{K}_0/K$ the maximal unramified subextension of $\tilde{K}/K$. Then $\underline{T}$ is realized as a $\mathrm{Gal}(\breve{F}/F)$-descent of

$$\big(\mathrm{Res}_{\mathcal{O}_{\breve{F}}[v^{\pm}]/\mathcal{O}_{\breve{F}}[u^{\pm}]}\,(T_H \otimes_{\mathcal{O}_F} \mathcal{O}_{\breve{F}}[v^{\pm}])\big)^{\gamma}.$$

This shows that the formation of $\underline{T}$ commutes with base change $\mathbb{A}^1_{E_0} \to \mathbb{A}^1_F$, where E_0/F is any unramified extension. Similar remarks apply to $\underline{\mathcal{T}}$.

4.4. **Affine Grassmannians and Local Models.** Fix K/F finite totally ramified, $\varpi \in K$ a uniformizer, and $Q \in \mathcal{O}_F[u]$ its Eisenstein polynomial as in §4.1. Let (G, A, S, T) be tamely ramified over K, and fix a spreading $(\underline{G}, \underline{A}, \underline{S}, \underline{T})$ defined over $\mathcal{O}_F[u^{\pm}]$ as in Proposition 4.10. Choose a facet $\mathbf{f} \in \mathscr{A}(G, A, K)$, and let $\mathcal{G}_{\mathbf{f}}$ the parahoric $\mathcal{O}_K$-group scheme. Associated with these data, we have the tuple $(\underline{\mathcal{G}}, \underline{\mathcal{A}}, \underline{\mathcal{S}}, \underline{\mathcal{T}})$ of smooth affine $X := \mathrm{Spec}(\mathcal{O}_F[u])$-groups constructed in Theorem 4.13. Let $D \subset X$ be the relative effective Cartier divisor over $\mathcal{O}_F$ defined by $\{Q = 0\}$. We are interested in local models for the group $\tilde{G} = \mathrm{Res}_{K/F}(G)$ with level structure given by the parahoric $\mathcal{O}_F$-group $\tilde{\mathcal{G}} := \mathcal{G}_{\tilde{\mathbf{f}}} = \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}(\mathcal{G})$, cf. Proposition 4.7.

4.4.1. *Affine Grassmannians for Weil-restricted groups.* The Beilinson-Drinfeld Grassmannian

$$\mathrm{Gr}_{\tilde{\mathcal{G}}} \overset{\text{def}}{=} \mathrm{Gr}_{(X, \underline{\mathcal{G}}, D)} \tag{4.9}$$

from (3.1) specializes to [Lev16, Def 4.1.1], and we think about (4.9) as being the Beilinson-Drinfeld Grassmannian associated with the parahoric $\mathcal{O}_F$-group scheme $\tilde{\mathcal{G}}$. Explicitly, $\mathrm{Gr}_{\tilde{\mathcal{G}}}$ is the the functor on the category of $\mathcal{O}_F$-algebras R given by the isomorphism classes of tuples $(\mathcal{F}, \alpha)$ with

$$\begin{cases} \mathcal{F} \text{ a } \underline{\mathcal{G}}\text{-torsor on } \mathrm{Spec}(R[u]); \\ \alpha\colon \mathcal{F}|_{\mathrm{Spec}(R[u][1/Q])} \simeq \mathcal{F}^0|_{\mathrm{Spec}(R[u][1/Q])} \text{ a trivialization,} \end{cases} \tag{4.10}$$

where $\mathcal{F}^0$ denotes the trivial torsor. If $Q = u - \varpi$, i.e., $K = F$, then $\mathrm{Gr}_{\tilde{\mathcal{G}}}$ is the BD-Grassmannian defined in [PZ13, 6.2.3; (6.11)].

For an $\mathcal{O}_F$-algebra R, we have the regular functions on the completion of X_R along D_R, namely the $\mathcal{O}_F[u]$-algebra $R[\![D]\!] = \lim_N R[u]/(Q^N)$, and likewise $R(\!(D)\!) = R[\![Q]\!][1/Q]$. With the notation of §3.1.1, we have the loop group

$$L\tilde{\mathcal{G}}(R) \overset{\text{def}}{=} L_D\underline{\mathcal{G}}(R) = \underline{\mathcal{G}}(R(\!(D)\!)),$$

and the positive loop group

$$L^+\tilde{\mathcal{G}} \overset{\text{def}}{=} L^+_D\underline{\mathcal{G}}(R) = \underline{\mathcal{G}}(R[\![D]\!]).$$

By Lemma 3.4, there is a natural isomorphism $L\tilde{\mathcal{G}}/L^+\tilde{\mathcal{G}} \simeq \mathrm{Gr}_{\tilde{\mathcal{G}}}$, and thus a transitive action morphism

$$L\tilde{\mathcal{G}} \times_{\mathcal{O}_F} \mathrm{Gr}_{\tilde{\mathcal{G}}} \longrightarrow \mathrm{Gr}_{\tilde{\mathcal{G}}}. \tag{4.11}$$

The following proposition is [Lev16, Prop 4.1.6, 4.1.8].

Proposition 4.15. *i) The generic fiber of* (4.11) *is canonically isomorphic to*

$$L_z\tilde{G} \times_F \mathrm{Gr}_{\tilde{G}} \longrightarrow \mathrm{Gr}_{\tilde{G}}, \tag{4.12}$$

where $L_z\tilde{G}(R) = \tilde{G}(R(\!(z)\!)) = G((K\otimes_F R)(\!(z)\!))$ is the loop group for $\tilde{G} = \mathrm{Res}_{K/F}(G)$ formed using the parameter $z := u - \varpi \in K[u]$, and $\mathrm{Gr}_{\tilde{G}}$ is as in Example 3.1 i) the affine Grassmannian for the group $\tilde{G}\otimes_F F[\![z]\!]$, i.e., the fpqc-sheaf associated with the functor on F-algebras $R \mapsto \tilde{G}(R(\!(z)\!))/\tilde{G}(R[\![z]\!])$.
ii) The special fiber of (4.11) *is canonically isomorphic to*

$$L\mathcal{G}' \times_{k_F} \mathcal{F}\ell_{\mathcal{G}'} \longrightarrow \mathcal{F}\ell_{\mathcal{G}'}, \tag{4.13}$$

where $L\mathcal{G}'(R) = \mathcal{G}'(R(\!(u)\!))$ is the twisted affine loop group associated with $\mathcal{G}' := \underline{\mathcal{G}} \otimes_{\mathcal{O}_F[u]} k_F[\![u]\!]$, and $\mathcal{F}\ell_{\mathcal{G}'}$ is the twisted affine flag variety for $\mathcal{G}'/k[\![u]\!]$ defined in [PR08], i.e., the fpqc-sheaf associated with the functor on k_F-algebras $R \mapsto \mathcal{G}'(R(\!(u)\!))/\mathcal{G}'(R[\![u]\!])$.

Proof. Part ii) is Corollary 3.5 i). For i), note the natural maps $\mathrm{Res}_{K/F}(L_zG) \to L_z\mathrm{Res}_{K/F}(G)$ and $\mathrm{Res}_{K/F}(\mathrm{Gr}_G) \to \mathrm{Gr}_{\mathrm{Res}_{K/F}(G)}$ are isomorphisms, cf. [PR08, (1.2)] and [Lev, §2.6]. Note that $Q(z+\varpi) \in zK[z]$. Hence by induction on $n \geq 1$, the map $u \mapsto z + \varpi$ sets up an isomorphism $F[u]/(Q^n) \xrightarrow{\sim} K[z]/(z^n)$, and hence $F[\![u]\!] \xrightarrow{\sim} K[\![z]\!]$. Similarly, we remark that for any F-algebra R $u \mapsto z+\varpi$ gives an isomorphism $R[\![u]\!] \cong (R\otimes_F K)[\![z]\!]$. Let $\underline{\mathcal{G}}_{K[\![z]\!]} := \underline{\mathcal{G}} \otimes_{\mathcal{O}_F[u]} K[\![z]\!]$, and denote by $\mathrm{Gr}_{\underline{\mathcal{G}}_{K[\![z]\!]}}$ the twisted affine Grassmannian for $\underline{\mathcal{G}}_{K[\![z]\!]}$, cf. Example 3.1 i). In view of Corollary 3.5, or the above remark, the generic fiber of (4.11) is canonically isomorphic to the action morphism

$$\mathrm{Res}_{K/F}(L\underline{\mathcal{G}}_{K[\![z]\!]}) \times_F \mathrm{Res}_{K/F}(\mathrm{Gr}_{\underline{\mathcal{G}}_{K[\![z]\!]}}) \longrightarrow \mathrm{Res}_{K/F}(\mathrm{Gr}_{\underline{\mathcal{G}}_{K[\![z]\!]}}).$$

Hence, as in [PZ13, §6.2.6] and [Lev16, Prop 4.1.6] it suffices to give an isomorphism of $K[\![z]\!]$-groups $\underline{\mathcal{G}}_{K[\![z]\!]} \simeq G\otimes_K K[\![z]\!]$. But as u is invertible in $K[\![z]\!]$, we have $\underline{\mathcal{G}}_{K[\![z]\!]} = \underline{G} \otimes_{\mathcal{O}_F[u^\pm]} K[\![z]\!]$. With the notation of (4.5), the group scheme $\underline{G}$ is constructed by descent from $\mathcal{O}_{\tilde{F}_0}[v^\pm]$ where it is a constant Chevalley group scheme. As in [PZ13, (6.9)], it is enough to give a commutative diagram of $\tilde{\Gamma}$-covers

$$\begin{tikzcd}
\mathrm{Spec}(\mathcal{O}_{\tilde{F}_0}[v^\pm] \otimes_{\mathcal{O}_F[u^\pm]} K[\![z]\!]) \arrow[dr, "\mathrm{pr}_2"'] & & \mathrm{Spec}(\tilde{K}[\![z]\!]) \arrow[ll, "\simeq"'] \arrow[dl, "\tau"] \\
& \mathrm{Spec}(K[\![z]\!]) &
\end{tikzcd} \tag{4.14}$$

which matches the $\tilde{\Gamma}$-action on $\mathcal{O}_{\tilde{F}_0}[v^\pm]/\mathcal{O}_F[u^\pm]$ via (4.5) with the $\tilde{\Gamma}$-action on the coefficients in $\tilde{K}[\![z]\!]$ (see below for why this is enough). As in [PZ13, (6.9)], the isomorphism is given on rings by $v \mapsto \tilde{\varpi}\cdot(1+z)$ and $z \mapsto b\cdot z$ with

$$b := \frac{\varpi\cdot(1+z)^{\tilde{e}} - \varpi}{z} \in K[\![z]\!]^\times.$$

The map τ is the K-algebra morphism given by $z \mapsto b\cdot z$. (To see that the horizontal morphism is an isomorphism, observe that $K[\![z]\!] = K[\![bz]\!]$, and let $f(z) \in K[\![z]\!]$ be such that $f(bz) = (1+z)^{-1}$; then $v\otimes f(z) \mapsto \tilde{\varpi}$ and the morphism is surjective. One sees it is injective using an $\mathcal{O}_F[u]$-basis for $\mathcal{O}_{\tilde{F}_0}[v]$ of the form a_iv^j for $a_i \in \mathcal{O}_{\tilde{F}_0}$ to write any element in the source uniquely in the form $\sum_{i,j} a_iv^j\otimes f_{ij}$ for $f_{ij} \in K[\![z]\!]$. To see that diagram (4.14) suffices, note that the right oblique arrow is isomorphic via $\tilde{K}[\![z]\!] \xrightarrow{\sim} \tilde{K}[\![z]\!]$, $z \mapsto b\cdot z$, to the arrow $\mathrm{Spec}(\tilde{K}[\![z]\!]) \to \mathrm{Spec}(K[\![z]\!])$ induced by the inclusion $K[\![z]\!] \hookrightarrow \tilde{K}[\![z]\!]$.) Since we fixed $\underline{G}_K \simeq G$ in the beginning, the isomorphism $\underline{\mathcal{G}}_{K[\![z]\!]} \simeq G\otimes_K K[\![z]\!]$ is canonical. □

Recall from [PZ13, Cor 11.7] that there exists a closed immersion of X-groups $\underline{\mathcal{G}} \hookrightarrow \mathrm{Gl}_{n,X}$ such that the quotient $\mathrm{Gl}_{n,X}/\underline{\mathcal{G}}$ is quasi-affine. Thus, the $\mathcal{O}_F$-space $\mathrm{Gr}_{\tilde{\mathcal{G}}} = \mathrm{Gr}_{(X,\underline{\mathcal{G}},D)}$ is representable by a separated $\mathcal{O}_F$-ind-scheme of ind-finite type, cf. Corollary 3.10. We need the following stronger statement.

Theorem 4.16. *The BD-Grassmannian $\mathrm{Gr}_{\tilde{\mathcal{G}}} = \mathrm{colim}_i\, \mathrm{Gr}_{\tilde{\mathcal{G}},i}$ is representable by an ind-projective $\mathcal{O}_F$-ind-scheme, and for each i, the projective $\mathcal{O}_F$-scheme $\mathrm{Gr}_{\tilde{\mathcal{G}},i}$ can be choosen to be $L^+\tilde{\mathcal{G}}$-stable compatible with the transition maps.*

Proof. The ind-projectivity is proven in [Lev16, Thm 4.2.11, Prop 5.1.5]. If G is unramified, the proof is considerably simpler, cf. [Lev16, Prop 2.2.8]. The proof relies on the existence and properties of specialization morphisms $\mathrm{sp} : \mathrm{Gr}_{\tilde{\mathcal{T}}}(\bar{F}) \longrightarrow \mathrm{Gr}_{\tilde{\mathcal{T}}}(\bar{k})$, cf. Lemma 4.17 below. Levin constructs this map "by hand" in [Lev16, Prop. 4.2.8]. We will follow a more conceptual approach which avoids constructing sp ahead of time and the calculations that entails. Our outline is the following:

(a) Prove $\mathrm{Gr}_{\tilde{\mathcal{T}}}$ is ind-finite, using the method of [Ri16b, Lem. 2.20], cf. §4.4.2.
(b) Deduce existence of the specialization maps for $\tilde{\mathcal{T}}$ via the valuative criterion of properness, and prove the required compatibility with Kottwitz homomorphisms, cf. §4.4.3.
(c) Use (b) to show that each local model has non-empty special fiber and deduce by [Ri16b, Lem. 2.22] that each local model is proper, cf. §4.4.5.
(d) Conclude that $\mathrm{Gr}_{\tilde{\mathcal{G}}}$ is ind-proper, cf. §4.4.6.

In view of Lemma 3.7 and Corollary 3.10, the ind-properness of $\mathrm{Gr}_{\tilde{\mathcal{G}}}$ implies the theorem. The steps (a)-(d) are explicated in the next several subsections, and with them the proof is concluded. □

4.4.2. $\mathrm{Gr}_{\tilde{\mathcal{T}}}$ *is ind-finite.* Without loss of generality, we assume that $F = \breve{F}$, $\mathcal{O}_F = \mathcal{O}_{\breve{F}}$. Here we use that the formation of the affine Grassmannian (4.9) and the group scheme $\underline{\mathcal{T}}$ in Example 4.14 is compatible with unramified base change. We show that $\mathrm{Gr}_{\tilde{\mathcal{T}}}$ is ind-proper. It is then ind-finite, since this holds fiberwise by Proposition 4.15. We proceed in two steps as follows.

Step 1): First assume that $\tilde{T} = \mathrm{Res}_{K/F}(T)$ where T is an *induced* K-torus which splits over a tamely ramified extension. Then T is isomorphic to a finite product of K-tori of the form $T_1 := \mathrm{Res}_{K_1/K}(\mathbb{G}_m)$ where K_1/K is a Galois tamely ramified finite field extension. Note that K_1/K is totally ramified by our assumption $F = \breve{F}$. Accordingly, the $\mathbb{A}^1_{\mathcal{O}_F}$-group scheme $\underline{\mathcal{T}}$ is isomorphic to a finite product of $\mathbb{A}^1_{\mathcal{O}_F}$-group schemes of the form

$$\underline{\mathcal{T}}_1 := \mathrm{Res}_{\mathcal{O}_F[v]/\mathcal{O}_F[u]}(\mathbb{G}_m),$$

where $v^{[K_1:K]} = u$. After fixing a uniformizer $\varpi_1 \in K_1$ with $(\varpi_1)^{[K_1:K]} = \varpi$ (possible because $F = \breve{F}$), this can be verified using Example 4.14 (use that, in this case, $T_H \otimes \tilde{\mathcal{O}}_0[v] \cong (\mathbb{G}_{m,\tilde{\mathcal{O}}_0[v]})^{[K_1:K]}$ with $\mathrm{Gal}(K_1/K)$ acting via the permutation of the factors). Likewise, the affine Grassmannian $\mathrm{Gr}_{\tilde{\mathcal{T}}}$ is a finite $\mathcal{O}_F$-product of the affine Grassmannians $\mathrm{Gr}_{(X,\underline{\mathcal{T}}_1,D)}$, where $X = \mathbb{A}^1_{\mathcal{O}_F}$ and $D = \{Q(u) = 0\}$ as in (4.9). Hence, we reduce to the case where $\underline{\mathcal{T}} = \underline{\mathcal{T}}_1$, i.e., $T = \mathrm{Res}_{K_1/K}(\mathbb{G}_m)$. By Corollary 3.6, there is an equality of ind-schemes

$$\mathrm{Gr}_{(X,\underline{\mathcal{T}},D)} \;=\; \mathrm{Gr}_{(X',\mathbb{G}_m,D')},$$

where $X' = \mathbb{A}^1_{\mathcal{O}_F} = \mathrm{Spec}(\mathcal{O}_F[v])$ and $D' = \{Q(v^{[K_1:K]}) = 0\}$. We reduce to the case $X = X'$, $\underline{\mathcal{T}} = \mathbb{G}_m$ and $D = D'$. Then $\mathrm{Gr}_{(X,\mathbb{G}_m,D)}$ is ind-projective (hence ind-proper) by Lemma 3.7.

Step 2): Now let $\tilde{T} = \mathrm{Res}_{K/F}(T)$ where T is an K-torus which splits over a tamely ramified extension. As in [Ko97, §7], we choose a surjection of K-tori $T_1 \to T$ where T_1 is induced, and where the kernel $T_2 := \ker(T_1 \to T)$ is a K-torus. Note that T_1 can be chosen to split over a tamely ramified extension (and so does T_2 as well). The proof of [KP, Prop 2.2.2] adapts to our set-up, and the map $T_1 \to T$ extends to a map of X-groups $\underline{\mathcal{T}}_1 \to \underline{\mathcal{T}}$ with kernel $\underline{\mathcal{T}}_2$ an X-group scheme extending T_2. (Instead of using [KP], one can also deduce this making use of the prescription given in Example 4.14.) We claim that the resulting map of $\mathcal{O}_F$-ind-schemes

$$\mathrm{Gr}_{\tilde{\mathcal{T}}_1} = \mathrm{Gr}_{(X,\underline{\mathcal{T}}_1,D)} \longrightarrow \mathrm{Gr}_{(X,\underline{\mathcal{T}},D)} = \mathrm{Gr}_{\tilde{\mathcal{T}}} \tag{4.15}$$

is surjective on the underlying topological spaces. Clearly, this can be tested on the fibers of (4.15) over $\mathcal{O}_F$ which are determined by Proposition 4.15. The geometric generic fiber of (4.15) is isomorphic (on the underlying topological spaces) to the map of discrete groups $X_*(\mathrm{Res}_{K/F}(T_1)) \to X_*(\mathrm{Res}_{K/F}(T))$ which is surjective because $T_1 \to T$ is surjective and its kernel T_2 is a torus (i.e., connected). The geometric special fiber of (4.15) is under the Kottwitz map isomorphic to $X_*(T_1')_{I_{k(\!(u)\!)}} \to X_*(T')_{I_{k(\!(u)\!)}}$ which is induced by $T_1' := \underline{\mathcal{T}}_1 \otimes k(\!(u)\!) \to \underline{\mathcal{T}} \otimes k(\!(u)\!) =: T'$. This map is isomorphic to $X_*(T_1)_{I_K} \to X_*(T)_{I_K}$ which follows by applying the Kottwitz map to the identification (4.8). As in [Ko97, §7 (7.2.5)] the desired surjectivity now follows from T_2 being a K-torus.

By Step 1), the $\mathcal{O}_F$-scheme $\mathrm{Gr}_{\tilde{\mathcal{T}}_1}$ is ind-proper and maps surjectively onto the separated ind-scheme $\mathrm{Gr}_{\tilde{\mathcal{T}}}$ which is therefore ind-proper as well. This concludes §4.4.2.

4.4.3. *The specialization map.* Once $\mathrm{Gr}_{\tilde{\mathcal{G}}}$ is known to be ind-proper, by the valuative criterion for properness there exists a specialization map

$$\mathrm{sp}: \mathrm{Gr}_{\tilde{G}}(\bar{F}) = \mathrm{Gr}_{\tilde{\mathcal{G}}}(\bar{F}) \longrightarrow \mathrm{Gr}_{\tilde{\mathcal{G}}}(\bar{k}) = \mathcal{F}\ell_{\mathcal{G}'}(\bar{k}). \tag{4.16}$$

In case $G = T$ is a maximal torus, and hence $\underline{\mathcal{G}} = \underline{\mathcal{T}}$ is as in Theorem 4.13 iv), we therefore know the existence of the specialization map. It is made explicit in [PZ13, Lem 9.8], [Lev16, Prop 4.2.8]. Recall the following result for later use (which compared to *loc. cit.* is proved in more conceptual way here).

Lemma 4.17. *There is a commutative diagram of abelian groups*

$$\begin{tikzcd}
\mathrm{Gr}_{\mathrm{Res}_{K/F}(T)}(\bar{F}) \arrow[rrr, "\simeq"] \arrow[d, "\mathrm{sp}"'] & & & X_*(\mathrm{Res}_{K/F}(T)) \arrow[d, "\mathrm{pr}"'] \\
\mathcal{F}\ell_{\mathcal{T}'}(\bar{k}) \arrow[r, "\simeq"'] & X_*(T')_{I_{k(\!(u)\!)}} \arrow[r, "\simeq"'] & X_*(T)_{I_K} & X_*(\mathrm{Res}_{K/F}(T))_{I_F}, \arrow[l, "\Sigma", "\simeq"']
\end{tikzcd} \tag{4.17}$$

which is Galois equivariant for the Γ_F-action on the top covering the $\mathrm{Gal}(\bar{k}/k)$-action on the bottom.

Proof. Let us construct the diagram. The map pr is the canonical projection to the coinvariants. Note that $X_*(\mathrm{Res}_{K/F}(T)) = \mathrm{Hom}^{I_K}(I_F, X_*(T))$ is an induced Galois module, and the map $\Sigma\colon f \mapsto \sum_{\gamma\in I_K\backslash I_F} f(\dot{\gamma})$ is well defined on coinvariants and an $\mathrm{Gal}(\bar{k}/k)$-equivariant isomorphism of abelian groups. See also the proof of Lemma 4.1. The isomorphism

$$\mathcal{F}\ell_{\mathcal{T}'}(\bar{k}) \;=\; T'(\bar{k}(\!(u)\!))/\mathcal{T}'(\bar{k}[\![u]\!]) \;\xrightarrow{\simeq}\; X_*(T')_{I_{k(\!(u)\!)}},$$

is given by the Kottwitz map, cf. [Ko97, §7], which is $\mathrm{Gal}(\bar{k}/k)$-equivariant as well. Finally, $X_*(T')_{I_{k(\!(u)\!)}} \simeq X_*(T)_{I_K}$ is also given by the Kottwitz map applied to (4.8) in the case of $T(\breve{K})$ (resp. $T'(\bar{k}(\!(u)\!))$). The $\mathrm{Gal}(\bar{k}/k)$-equivariance follows from Lemma 4.5. This constructs (4.17).

It remains to prove the commutativity which is a reformulation of [Lev16, Prop 4.2.8]: the composition $\Sigma\circ\mathrm{pr}$ is the map given by $\mu' \mapsto \bar{\lambda}_{\mu'}$ in the notation of *loc. cit.*. We show the commutativity as follows. Changing notation, we may assume that $F = \breve{F}$, $k = \bar{k}$. The diagram (4.17) is functorial in the tamely ramified K-torus T. Arguing as in §4.4.2 Step 2), we choose an induced tamely ramified K-torus $T_1 \twoheadrightarrow T$ with kernel being a torus. Each item in the diagram for T_1 maps surjectively onto each item in the diagram for T, and we reduce to the case where $T = T_1$ is an induced tamely ramified K-torus. Arguing as in §4.4.2 Step 1), the torus T is a product of K-tori of the form $\mathrm{Res}_{K_1/K}(\mathbb{G}_m)$ with K_1/K being totally (tamely) ramified. Accordingly, each item in the diagram (4.17) splits as a product compatible with the maps, and we reduce to the case where $T = \mathrm{Res}_{K_1/K}(\mathbb{G}_m)$. Replacing the pair (X, D) with the pair (X', D') as in §4.4.2 Step 1), we reduce further to the case where $T = \mathbb{G}_m$. In this case, we have for the (global) loop group

$$L\mathbb{G}_m(\mathcal{O}_{\bar{F}}) = (L\mathbb{G}_m)_{(X,\mathbb{G}_m,D)}(\mathcal{O}_{\bar{F}}) \;=\; \mathcal{O}_{\bar{F}}(\!(Q)\!)^\times,$$

where $Q \in \mathcal{O}_F[u]$ is the minimal polynomial of $\varpi \in K$ over F. Writing $Q = (u - a_1)\cdot\ldots\cdot(u - a_d)$ for $d = [K : F]$ and pairwise distinct elements $a_1, \ldots, a_d \in \mathcal{O}_{\bar{F}}$, we compute for the generic fiber

$$(L\mathbb{G}_m)_{(X,\mathbb{G}_m,D)}(\bar{F}) \;=\; \prod_{i=1,\ldots,n} \bar{F}(\!(u - a_i)\!)^\times.$$

For $i = 1, \ldots, d$, let v_i be the $(u - a_i)$-adic valuation of $\bar{F}(\!(u - a_i)\!)$. The specialization map (4.16) is explicitly given by the map

$$\prod_{i=1,\ldots,d} \bar{F}(\!(u-a_i)\!)^\times/\bar{F}[\![u-a_i]\!]^\times \to k(\!(u)\!)^\times/k[\![u]\!]^\times, \quad (x_1,\ldots,x_d) \mapsto u^{\sum_{i=1}^d v_i(x_i)},$$

where we use that $Q \equiv u^{[K:F]} \mod \varpi$. One immediately checks that (4.17) commutes for $T = \mathbb{G}_m$ which finishes the proof of the lemma.

□

4.4.4. *Local Models for Weil-restricted groups.* We now recall the definition of local models for the pair $(\tilde{G},\tilde{\mathcal{G}}) = (\mathrm{Res}_{K/F}(G), \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}(\mathcal{G}_{\mathbf{f}}))$. Let $\{\mu\}$ be a $\tilde{G}(\bar{F})$-conjugacy class of geometric cocharacters with reflex field E/F. For a representative $\mu \in \{\mu\}$, the associated *Schubert variety* is the reduced $L_z^+\tilde{G}_{\bar{F}}$-orbit closure

$$\mathrm{Gr}_{\tilde{G}}^{\leq\{\mu\}} \overset{\mathrm{def}}{=} \overline{L_z^+\tilde{G}_{\bar{F}}\cdot z^\mu \cdot e_0} \subset \mathrm{Gr}_{\tilde{G},\bar{F}}. \tag{4.18}$$

The $\bar{F}$-scheme $\mathrm{Gr}_{\tilde{G}}^{\leq\{\mu\}}$ is defined over the reflex field $E = E(\{\mu\})$, i.e., the field of definition of $\{\mu\}$ which is a finite extension of F, and is a (geometrically irreducible) projective E-variety.

The following definition is [PZ13, Def 7.1] if K/F is tamely ramified, and [Lev16, Def 4.2.1] in general, cf. [Lev16, Prop 4.2.4]).

Definition 4.18. The local model $M_{\{\mu\}} = M(\underline{G}, \mathcal{G}_{\mathbf{f}}, \{\mu\}, \varpi)$ is the scheme theoretic closure of the locally closed subscheme

$$\mathrm{Gr}_{\tilde{G}}^{\leq\{\mu\}} \hookrightarrow (\mathrm{Gr}_{\tilde{G}}\otimes_F E)_{\mathrm{red}} \hookrightarrow (\mathrm{Gr}_{\tilde{\mathcal{G}}}\otimes_{\mathcal{O}_F}\mathcal{O}_E)_{\mathrm{red}},$$

where $\mathrm{Gr}_{\tilde{G}}^{\leq\{\mu\}}$ is as in (4.18).

By definition, the local model $M_{\{\mu\}}$ is a closed flat $L^+\tilde{\mathcal{G}}_{\mathcal{O}_E}$-invariant subscheme of $(\mathrm{Gr}_{\tilde{\mathcal{G}}}\otimes_{\mathcal{O}_F}\mathcal{O}_E)_{\mathrm{red}}$ which is uniquely determined up to unique isomorphism by the data $(\underline{G}, \mathcal{G}_{\mathbf{f}}, \{\mu\}, \varpi)$. Its generic fiber $M_{\{\mu\}}\otimes E = \mathrm{Gr}_{\tilde{G},E}^{\leq\{\mu\}}$ is a (geometrically irreducible) variety, and the special fiber $M_{\{\mu\}}\otimes k_E$ is equidimensional, cf. [GW10, Thm 14.114]. By Proposition 4.15, the map $\mathrm{Gr}_{\tilde{\mathcal{G}}} \to \mathrm{Spec}(\mathcal{O}_F)$ is fiberwise ind-proper, and hence the map $M_{\{\mu\}} \to \mathrm{Spec}(\mathcal{O}_E)$ is fiberwise proper. Note that there is a closed embedding into the flag variety

$$M_{\{\mu\}}\otimes k_E \hookrightarrow \mathrm{Gr}_{\tilde{\mathcal{G}}}\otimes_{\mathcal{O}_F} k_E = \mathcal{F}\ell_{\mathcal{G}',k_E}, \tag{4.19}$$

which identifies the reduced locus $(M_{\{\mu\}}\otimes k_E)_{\mathrm{red}}$ with a union of Schubert varieties in $\mathcal{F}\ell_{\mathcal{G}',k_E}$.

Remark 4.19. The local model $M_{\{\mu\}}$ should up to unique isomorphism only depend on the data $(\tilde{G},\tilde{\mathcal{G}},\{\mu\})$. The uniqueness of $M_{\{\mu\}}$ is a separate question, and not of importance for the present article. We refer the reader to [PZ13, Rmk 3.2] for remarks on the uniqueness of $\underline{G}$, and to [Lev16, Rmk 4.2.5] for remarks on the independence of $M_{\{\mu\}}$ on the choice of the uniformizer $\varpi \in K$. In the recent preprint [HPR, Thm 2.7], it is shown the ind-scheme $\mathrm{Gr}_{\tilde{\mathcal{G}}}$ for $K = F$ depends up to equivariant isomorphism only on the data $(\tilde{G},\tilde{\mathcal{G}})$. So $M_{\{\mu\}}$ for $K = F$ depends up to equivariant isomorphism only on the data $(\tilde{G},\tilde{\mathcal{G}},\{\mu\})$. Note that [HPR, Conj 2.12] uniquely characterizes $M_{\{\mu\}}$ for $K = F$ in the case where $\{\mu\}$ is minuscule.

4.4.5. *Each local model is proper.* For every conjugacy class $\{\mu\}$, we need to show that the local model $M_{\{\mu\}}$ is proper over $\mathcal{O}_E$ where $E = E(\{\mu\})$ is the reflex field. In view of [Ri16b, Lem 2.20] and the discussion after Definition 4.18, it remains to show that the special fiber of $M_{\{\mu\}}$ is non-empty. The inclusion $\underline{\mathcal{T}} \subset \underline{\mathcal{G}}$ induces a map of $\mathcal{O}_F$-ind-schemes

$$\mathrm{Gr}_{\tilde{\mathcal{T}}} = \mathrm{Gr}_{(X,\underline{\mathcal{T}},D)} \to \mathrm{Gr}_{(X,\underline{\mathcal{G}},D)} = \mathrm{Gr}_{\tilde{\mathcal{G}}}. \tag{4.20}$$

In the notation of Proposition 4.15, the geometric generic fiber $M_{\{\mu\}}(\bar{F})$ contains the element

$$\mu \in \mathrm{Gr}_{\tilde{T}}(\bar{F}) = \mathrm{Gr}_{\tilde{\mathcal{T}}}(\bar{F}),$$

for any representative $\mu \in X_*(\tilde{T})$ of $\{\mu\}$. As $\mathrm{Gr}_{\tilde{\mathcal{T}}}$ is ind-finite (hence ind-proper) by §4.4.2, the element $\mu \in \mathrm{Gr}_{\tilde{\mathcal{T}}}(\bar{F})$ uniquely extends to a point $\tilde{\mu} \in \mathrm{Gr}_{\tilde{\mathcal{T}}}(\mathcal{O}_{\bar{F}})$ by the valuative criterion for properness. Composed with (4.20), this defines a point (still denoted) $\tilde{\mu} \in \mathrm{Gr}_{\tilde{\mathcal{G}}}(\mathcal{O}_{\bar{F}})$. Since $M_{\{\mu\}} \subset \mathrm{Gr}_{\tilde{\mathcal{G}},\mathcal{O}_E}$ is a closed subscheme, we have

$$\tilde{\mu} \in M_{\{\mu\}}(\bar{F}) \cap \mathrm{Gr}_{\tilde{\mathcal{G}}}(\mathcal{O}_{\bar{F}}) = M_{\{\mu\}}(\mathcal{O}_{\bar{F}}), \tag{4.21}$$

and its special fiber $\bar{\mu} := \tilde{\mu}_{\bar{k}} \in M_{\{\mu\}}(\bar{k})$ is non-empty. This concludes §4.4.5.

4.4.6. *Conclusion of Proof of Theorem 4.16.* We need to show that $\mathrm{Gr}_{\tilde{\mathcal{G}}} \to \mathrm{Spec}(\mathcal{O}_F)$ is ind-proper. It suffices to prove that the map $(\mathrm{Gr}_{\tilde{\mathcal{G}}} \otimes \mathcal{O}_{\bar{F}})_{\mathrm{red}} \to \mathrm{Spec}(\mathcal{O}_{\bar{F}})$ is ind-proper. In view of §4.4.5, we have to show that the closed immersion

$$\bigcup_{\{\mu\}} (M_{\{\mu\},\mathcal{O}_{\bar{F}}})_{\mathrm{red}} \subset (\mathrm{Gr}_{\tilde{\mathcal{G}}} \otimes \mathcal{O}_{\bar{F}})_{\mathrm{red}} \tag{4.22}$$

is an equality. Here $\{\mu\}$ ranges over all $\tilde{G}(\bar{F})$-conjugacy classes of geometric cocharacters. As both ind-schemes in (4.22) are reduced, one can check the equality on the underlying topological spaces. As in [Ri16b, §2.5] (resp. [Lev16, Thm 4.2.11]), this follows from Lemma 4.17 combined with (4.19) and (4.21). This concludes §4.4.6, and hence the proof of Theorem 4.16.

5. $\mathbb{G}_m$-actions on affine Grassmannians for Weil-restricted groups

5.1. Geometry of $\mathbb{G}_m$-actions on affine Grassmannians.

Fix the data and notation as in §4.4. In particular, we denote the group schemes over $X = \mathbb{A}^1_{\mathcal{O}_F}$ by $(\underline{\mathcal{G}}, \underline{\mathcal{A}}, \underline{\mathcal{S}}, \underline{\mathcal{T}})$.

5.1.1. *Main geometric result.* Let $\chi\colon \mathbb{G}_{m,K} \to A \subset G$ be a cocharacter which acts on G by conjugation. As in (3.17), the centralizer is a Levi subgroup $M \subset G$, and the attractor (resp. repeller) subgroup P^+ (resp. P^-) is a parabolic subgroup with $P^+ \cap P^- = M$. Further, we have semidirect product decompositions $P^\pm = M \rtimes N^\pm$ defined over K.

Via the fixed isomorphism $\underline{\mathcal{G}}_K \simeq G$ compatible with $\underline{\mathcal{A}}_K \simeq A$, we may view χ as a cocharacter of $\underline{\mathcal{A}}_K$. As X is connected and $\underline{\mathcal{A}}$ a split torus, χ extends uniquely to a cocharacter also denoted

$$\chi\colon \mathbb{G}_{m,X} \longrightarrow \underline{\mathcal{A}} \subset \underline{\mathcal{G}}. \tag{5.1}$$

Hence, the cocharacter χ acts by conjugation on $\underline{\mathcal{G}}$ via the rule $\mathbb{G}_{m,X} \times_X \underline{\mathcal{G}} \to \underline{\mathcal{G}}$, $(\lambda, g) \mapsto \chi(\lambda)\cdot g \cdot \chi(\lambda)^{-1}$. Using the dynamic method promulgated in [CGP10], the functors (2.1) define X-subgroup schemes of $\underline{\mathcal{G}}$ given by the fixed points $\underline{\mathcal{M}} = \underline{\mathcal{G}}^{0,\chi}$, and the attractor $\underline{\mathcal{P}}^+ = \underline{\mathcal{G}}^{+,\chi}$ (resp. the repeller $\underline{\mathcal{P}}^- = \underline{\mathcal{G}}^{-,\chi}$). Note that $\underline{\mathcal{M}}$ is by definition the schematic centralizer of χ in $\underline{\mathcal{G}}$.

Lemma 5.1. *i) The X-group schemes $\underline{\mathcal{M}}$ and $\underline{\mathcal{P}}^\pm$ are smooth closed subgroup schemes of $\underline{\mathcal{G}}$ with geometrically connected fibers.*

ii) The centralizer $\underline{\mathcal{M}}$ is a parahoric X-group scheme for M in the sense of Theorem 4.13.

iii) There is a semidirect product decomposition as X-group schemes $\underline{\mathcal{P}}^\pm = \underline{\mathcal{M}} \ltimes \underline{\mathcal{N}}^\pm$ where $\underline{\mathcal{N}}^\pm$ is a smooth affine group scheme with geometrically connected fibers.

iv) The fixed isomorphism $\underline{\mathcal{G}}_K \simeq G$ induces isomorphisms of $K[\![z]\!]$-groups $\underline{\mathcal{M}}_{K[\![z]\!]} \simeq M \otimes_K K[\![z]\!]$, and $\underline{\mathcal{P}}^\pm_{K[\![z]\!]} \simeq P^\pm \otimes_K K[\![z]\!]$ compatible with the semidirect product decomposition in iii).

Proof. The method of [HaRi, Lem 5.15] extends to give i), ii) and iii) of the lemma. Part iv) is immediate from the construction of χ, and the proof of Proposition 4.15 i). □

By (2.2), there are natural maps of X-group schemes

$$\underline{\mathcal{M}} \leftarrow \underline{\mathcal{P}}^\pm \rightarrow \underline{\mathcal{G}}. \tag{5.2}$$

The maps (5.2) induce, by functoriality of BD-Grassmannians, maps of $\mathcal{O}_F$-spaces

$$\mathrm{Gr}_{\tilde{\mathcal{M}}} \leftarrow \mathrm{Gr}_{\tilde{\mathcal{P}}^\pm} \rightarrow \mathrm{Gr}_{\tilde{\mathcal{G}}}, \tag{5.3}$$

where $\mathrm{Gr}_{\tilde{\mathcal{G}}} := \mathrm{Gr}_{(X,\underline{\mathcal{G}},D)}$ (resp. $\mathrm{Gr}_{\tilde{\mathcal{M}}} := \mathrm{Gr}_{(X,\underline{\mathcal{M}},D)}$; resp. $\mathrm{Gr}_{\tilde{\mathcal{P}}^\pm} := \mathrm{Gr}_{(X,\underline{\mathcal{P}}^\pm,D)}$) by notational convention. In light of [PZ13, Cor 11.7] and Corollary 3.10 i), the functors in (5.3) are representable by separated $\mathcal{O}_F$-ind-schemes of ind-finite type. Note that by Theorem 4.16 i) and Lemma 5.1 ii), the $\mathcal{O}_F$-ind-schemes $\mathrm{Gr}_{\tilde{\mathcal{G}}}$ and $\mathrm{Gr}_{\tilde{\mathcal{M}}}$ are even ind-projective. The $\mathcal{O}_F$-ind-scheme $\mathrm{Gr}_{\tilde{\mathcal{P}}}$ is never ind-projective besides the trivial cases.

By functoriality of the loop group, we obtain via the composition

$$\mathbb{G}_{m,\mathcal{O}_F} \subset L^+_D\mathbb{G}_{m,X} \xrightarrow{L^+_D\chi} L^+_D\underline{\mathcal{A}} \subset L^+_D\underline{\mathcal{G}} \tag{5.4}$$

a $\mathbb{G}_{m,\mathcal{O}_F}$-action on $\mathrm{Gr}_{\tilde{\mathcal{G}}} \to \mathrm{Spec}(\mathcal{O}_F)$.

Lemma 5.2. *The $\mathbb{G}_m$-action on $\mathrm{Gr}_{\tilde{\mathcal{G}}}$ is Zariski locally linearizable.*

Proof. By [PZ13, Cor 11.7] there exists an monomorphism of X-groups $\underline{\mathcal{G}} \hookrightarrow \mathrm{Gl}_{n,X}$ such that the fppf-quotient $\mathrm{Gl}_{n,X}/\underline{\mathcal{G}}$ is quasi-affine. Hence, the induced monomorphism $\iota\colon \mathrm{Gr}_{\tilde{\mathcal{G}}} \hookrightarrow \mathrm{Gr}_{\mathrm{Gl}_{n,X}}$ is representable by a quasi-compact immersion (cf. Proposition 3.9) which is even a closed immersion because $\mathrm{Gr}_{\tilde{\mathcal{G}}}$ is ind-proper, cf. Theorem 4.16. The map ι is $\mathbb{G}_m$-equivariant for the cocharacter $\mathbb{G}_{m,X} \overset{\chi}{\to} \underline{\mathcal{G}} \to \mathrm{Gl}_{n,X}$, and we reduce to the case $\underline{\mathcal{G}} = \mathrm{Gl}_{n,X}$. By [Co14, Prop 6.2.11] (use $\mathrm{Pic}(X) = 0$), the cocharacter $\chi\colon \mathbb{G}_{m,X} \to \mathrm{Gl}_{n,X}$ is conjugate to a cocharacter with values in the standard diagonal torus, and hence defined over $\mathcal{O}_F$. The lemma follows from the proof of Lemma 3.15. ☐

In light of Theorem 4.16 and Theorem 2.1, we obtain maps of separated $\mathcal{O}_F$-ind-schemes

$$(\mathrm{Gr}_{\tilde{\mathcal{G}}})^0 \leftarrow (\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm} \to \mathrm{Gr}_{\tilde{\mathcal{G}}}. \tag{5.5}$$

The following theorem compares (5.3) with (5.5).

Theorem 5.3. *The maps induce a commutative diagram of $\mathcal{O}_F$-ind-schemes*

$$\begin{tikzcd} \mathrm{Gr}_{\tilde{\mathcal{M}}} \arrow[d, "\iota^0"'] & \mathrm{Gr}_{\tilde{\mathcal{P}}^{\pm}} \arrow[l] \arrow[r] \arrow[d, "\iota^{\pm}"'] & \mathrm{Gr}_{\tilde{\mathcal{G}}} \arrow[d, "\mathrm{id}"'] \\ (\mathrm{Gr}_{\tilde{\mathcal{G}}})^0 & (\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm} \arrow[l] \arrow[r] & \mathrm{Gr}_{\tilde{\mathcal{G}}}, \end{tikzcd} \tag{5.6}$$

where the maps ι^0 and $\iota^{\pm}$ satisfy the following properties:

i) In the generic fiber, the diagram is isomorphic to (5.7) below, and the maps ι^0_F and $\iota^{\pm}_F$ are isomorphisms.

ii) In the special fiber, the diagram is isomorphic to (5.8) below, and the maps ι^0_k and $\iota^{\pm}_k$ are closed immersions which are open immersions on the underlying reduced loci.

iii) The maps ι^0 and $\iota^{\pm}$ are closed immersions which are open immersions on the underlying reduced loci.

The diagram is constructed as follows. The fppf-quotient $\underline{\mathcal{G}}/\underline{\mathcal{M}}$ is quasi-affine by [Co14, Thm 2.4.1], which implies that the map $\mathrm{Gr}_{\tilde{\mathcal{M}}} \to \mathrm{Gr}_{\tilde{\mathcal{G}}}$ as in the proof of Lemma 5.2 is representable by a closed immersion. Since the $\mathbb{G}_m$-action on $\mathrm{Gr}_{\tilde{\mathcal{M}}}$ is trivial, the map factors as $\mathrm{Gr}_{\tilde{\mathcal{M}}} \to (\mathrm{Gr}_{\tilde{\mathcal{G}}})^0 \to \mathrm{Gr}_{\tilde{\mathcal{G}}}$, and we obtain the closed immersion ι^0.

The map $\iota^{\pm}$ is given via a Rees construction in terms of the moduli description (4.10), cf. §3.3.1. Alternatively, if we choose a monomorphism of X-groups $\underline{\mathcal{G}} \hookrightarrow \mathrm{Gl}_{n,X}$ such that $\mathrm{Gl}_{n,X}/\underline{\mathcal{G}}$ is quasi-affine (cf. [PZ13, Cor 11.7]), then the same argument as in (3.22) applies, and we conclude that $\iota^{\pm}$ is representable by a quasi-compact immersion. We do not repeat the argument here, but instead refer the reader to §3.3.1 for details. This constructs the commutative diagram (5.6).

Proof of Theorem 5.3. Part i). In the generic fiber, (5.6) is by (4.12) and Lemma 5.1 iv), the commutative diagram of F-ind-schemes

$$\begin{tikzcd} \mathrm{Gr}_{\tilde{M}} \arrow[d, "\iota^0_F"'] & \mathrm{Gr}_{\tilde{P}^{\pm}} \arrow[l] \arrow[r] \arrow[d, "\iota^{\pm}_F"'] & \mathrm{Gr}_{\tilde{G}} \arrow[d, "\mathrm{id}"'] \\ (\mathrm{Gr}_{\tilde{G}})^0 & (\mathrm{Gr}_{\tilde{G}})^{\pm} \arrow[l] \arrow[r] & \mathrm{Gr}_{\tilde{G}}, \end{tikzcd} \tag{5.7}$$

where $\tilde{G} = \mathrm{Res}_{K/F}(G)$ (resp. $\tilde{M} = \mathrm{Res}_{K/F}(M)$; resp. $\tilde{P}^{\pm} = \mathrm{Res}_{K/F}(P^{\pm})$). The $\mathbb{G}_m$-action on the diagram is induced by the L^+_z-construction applied to the cocharacter

$$\tilde{\chi}\colon \mathbb{G}_{m,F} \subset \mathrm{Res}_{K/F}(\mathbb{G}_{m,K}) \overset{\mathrm{Res}_{K/F}(\chi)}{\longrightarrow} \mathrm{Res}_{K/F}(A) \subset \mathrm{Res}_{K/F}(G) = \tilde{G},$$

combined with the inclusion $\mathbb{G}_{m,F} \subset L^+_z\mathbb{G}_{m,F}$. We claim that the conjugation action of $\tilde{\chi}$ on $\tilde{G}$ gives the group of fixed points $\tilde{M} = \tilde{G}^{0,\tilde{\chi}}$ and the attractor (resp. repeller) group $\tilde{P}^+ = \tilde{G}^{+,\tilde{\chi}}$ (resp. $\tilde{P}^- = \tilde{G}^{-,\tilde{\chi}}$). Indeed, the canonical maps of F-subgroups of $\tilde{G}$,

$$\mathrm{Res}_{K/F}(M) \hookrightarrow \tilde{G}^{0,\tilde{\chi}}$$
$$\mathrm{Res}_{K/F}(P^{\pm}) \hookrightarrow \tilde{G}^{\pm,\tilde{\chi}}$$

are isomorphisms. By descent, it is enough to prove this after passing to $\bar{F}$. But $\tilde{G}\otimes_F\bar{F}\simeq\prod_{K\hookrightarrow\bar{F}}G\otimes_{K,\psi}\bar{F}$, where the $\mathbb{G}_m$-action induced by $\tilde{\chi}$ is the diagonal action on the product. Lemma 2.2 implies the claim. Part i) follows from [HaRi, Prop 3.4] applied to the pair $(\tilde{G},\tilde{\chi})$.

Part ii). In the special fiber, (5.6) is the commutative diagram of k-ind-schemes

$$\begin{array}{ccccc} \mathcal{F}\ell_{\mathcal{M}'} & \longleftarrow & \mathcal{F}\ell_{\mathcal{P}'^{\pm}} & \longrightarrow & \mathcal{F}\ell_{\mathcal{G}'} \\ {\scriptstyle \iota^0_k}\big\downarrow & & {\scriptstyle \iota^{\pm}_k}\big\downarrow & & {\scriptstyle \mathrm{id}}\big\downarrow \\ (\mathcal{F}\ell_{\mathcal{G}'})^0 & \longleftarrow & (\mathcal{F}\ell_{\mathcal{G}'})^{\pm} & \longrightarrow & \mathcal{F}\ell_{\mathcal{G}'}. \end{array} \tag{5.8}$$

The $\mathbb{G}_m$-action on the diagram is given as follows. Base changing (5.1) along $\mathcal{O}_F[u]\to k[\![u]\!]$, we obtain the cocharacter

$$\chi'\colon\mathbb{G}_{m,k[\![u]\!]}\to\mathcal{A}'\subset\mathcal{G}',$$

which acts on the diagram after applying the L^+-construction combined with the inclusion $\mathbb{G}_{m,k}\subset L^+\mathbb{G}_{m,k[\![u]\!]}$. Since taking fixed points (resp. attractors; resp. repellers) commutes with base change [Ri, (1.3)], we have $\mathcal{M}'=(\mathcal{G}')^{0,\chi'}$ and $\mathcal{P}'^{\pm}=(\mathcal{G}')^{\pm,\chi'}$. Part ii) follows from [HaRi, Prop 4.7] applied to the pair $(\mathcal{G}',\chi')$.

Part iii). This follows as in [HaRi, Thm 5.5, 5.17] using Proposition 5.5 below, and we sketch the argument for convenience. With the notation of Proposition 5.5, the map ι^0 (resp. $\iota^{\pm}$) factors as a set-theoretically bijective quasi-compact immersion

$$\iota^{0,c}\colon\mathrm{Gr}_{\tilde{\mathcal{M}}}\to(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{0,c}\qquad(\text{resp. }\iota^{\pm,c}\colon\mathrm{Gr}_{\tilde{\mathcal{P}}^{\pm}}\to(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm,c}),$$

where $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{0,c}$ (resp. $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm,c}$) is an open and closed $\mathcal{O}_F$-sub-ind-scheme of $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^0$ (resp. $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm}$). But any such map $\iota^{0,c}$ (resp. $\iota^{\pm,c}$) is a closed immersion which is an isomorphism on the underlying reduced loci, cf. [HaRi, Lem 5.7]. □

We record the following properties.

Lemma 5.4. *i) The map $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm}\to\mathrm{Gr}_{\tilde{\mathcal{G}}}$ is schematic.*

ii) The map $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm}\to(\mathrm{Gr}_{\tilde{\mathcal{G}}})^0$ is ind-affine with geometrically connected fibers, and induces an isomorphism on the group of connected components $\pi_0((\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm})\simeq\pi_0((\mathrm{Gr}_{\tilde{\mathcal{G}}})^0)$.

Proof. These are general properties of attractors in ind-schemes endowed with étale locally linearizable $\mathbb{G}_m$-actions, cf. Lemma 5.2, and Theorem 2.1 ii) or [HaRi, Thm. 2.1 ii)]. □

The following proposition decomposes the image of the maps ι^0 and $\iota^{\pm}$ into connected components, and will be important in what follows.

Proposition 5.5. *Let either $N=N^+$ or $N=N^-$. There exists an open and closed $\mathcal{O}_F$-ind-subscheme $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{0,c}$ (resp. $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm,c}$) of $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^0$ (resp. $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm}$) together with a disjoint decomposition, depending up to sign on the choice of N, as $\mathcal{O}_F$-ind-schemes*

$$(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{0,c}=\coprod_{m\in\mathbb{Z}}(\mathrm{Gr}_{\tilde{\mathcal{G}}})^0_m\quad(\textit{resp. }(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm,c}=\coprod_{m\in\mathbb{Z}}(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm}_m),$$

which has the following properties.

i) The map $\iota^0\colon\mathrm{Gr}_{\tilde{\mathcal{M}}}\to(\mathrm{Gr}_{\tilde{\mathcal{G}}})^0$ (resp. $\iota^{\pm}\colon\mathrm{Gr}_{\tilde{\mathcal{P}}^{\pm}}\to(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm}$) factors through $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{0,c}$ (resp. $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm,c}$) inducing a closed immersion $\iota^{0,c}\colon\mathrm{Gr}_{\tilde{\mathcal{M}}}\to(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{0,c}$ (resp. $\iota^{\pm,c}\colon\mathrm{Gr}_{\tilde{\mathcal{P}}^{\pm}}\to(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm,c}$) which is an isomorphism on reduced loci.

ii) The complement $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^0\backslash(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{0,c}$ (resp. $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm}\backslash(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm,c}$) has empty generic fiber, i.e., is concentrated in the special fiber.

Proof. The proof follows closely [HaRi, Prop 5.6, 5.19]. We recall some steps of the construction. Let us denote $\mathcal{O}:=\mathcal{O}_F$, and $\breve{\mathcal{O}}:=\mathcal{O}_{\breve{F}}$. Let $\pi_1(M)=X_*(T)/X_*(T_{M_{\mathrm{sc}}})$ be the algebraic fundamental group of M in the sense of [Bo98], and denote by $\pi_1(M)_{I_K}$ the coinvariants. By [PR08, Thm 5.1], the group of connected components is given by

$$\pi_0(\mathcal{F}\ell_{\mathcal{M}',\bar{k}})\;=\;\pi_1(M')_{I_{k(\!(u)\!)}}\;=\;\pi_1(M)_{I_K},$$

where the last equality follows from the proof of Lemma 4.11. Since $\mathrm{Gr}_{\tilde{\mathcal{M}},\breve{\mathcal{O}}} \to \mathrm{Spec}(\breve{\mathcal{O}})$ is ind-proper and $\breve{\mathcal{O}}$ is Henselian, the natural map

$$\pi_0(\mathrm{Gr}_{\tilde{\mathcal{M}},\breve{\mathcal{O}}}) \xrightarrow{\simeq} \pi_0(\mathcal{F}\ell_{\mathcal{M}',\bar{k}})$$

is an isomorphism by [SGA4$\frac{1}{2}$, Arcata; IV-2; Prop 2.1]. This shows that there is a decomposition into connected components

$$\mathrm{Gr}_{\tilde{\mathcal{M}},\breve{\mathcal{O}}} = \coprod_{\bar{\nu}\in\pi_1(M)_{I_K}} (\mathrm{Gr}_{\tilde{\mathcal{M}},\breve{\mathcal{O}}})_{\bar{\nu}} \tag{5.9}$$

such that $(\mathrm{Gr}_{\tilde{\mathcal{M}},\breve{\mathcal{O}}})_{\bar{\nu}} \otimes \bar{k} \simeq (\mathcal{F}\ell_{\mathcal{M}',\bar{k}})_{\bar{\nu}}$. By Lemma 4.17, the generic fiber decomposes as $(\mathrm{Gr}_{\tilde{\mathcal{M}},\breve{\mathcal{O}}})_{\bar{\nu}} \otimes \bar{F} \simeq \coprod_{\nu\mapsto\bar{\nu}} (\mathrm{Gr}_{\tilde{M},\bar{F}})_{\nu}$ where $\nu \in \pi_1(\tilde{M})$ runs over the elements which map to $\bar{\nu}$ under the reduction map $\pi_1(\tilde{M}) \to \pi_1(\tilde{M})_{I_F} \simeq \pi_1(M)_{I_K}$.

By Theorem 5.3 i) and ii), it is easy to see that the closed immersion $\iota^0\colon \mathrm{Gr}_{\tilde{\mathcal{M}},\breve{\mathcal{O}}} \to (\mathrm{Gr}_{\tilde{\mathcal{G}},\breve{\mathcal{O}}})^0$ is open on the underlying topological spaces (e.g., its image is closed under generization), i.e., the image identifies each connected component of $\mathrm{Gr}_{\tilde{\mathcal{M}},\breve{\mathcal{O}}}$ with a connected component of $(\mathrm{Gr}_{\tilde{\mathcal{G}},\breve{\mathcal{O}}})^0$. Using Lemma 5.4 ii), we get an inclusion

$$\pi_1(M)_{I_K} \;=\; \pi_0(\mathrm{Gr}_{\tilde{\mathcal{M}},\breve{\mathcal{O}}}) \;\subset\; \pi_0\left((\mathrm{Gr}_{\tilde{\mathcal{G}},\breve{\mathcal{O}}})^0\right) \;=\; \pi_0\left((\mathrm{Gr}_{\tilde{\mathcal{G}},\breve{\mathcal{O}}})^{\pm}\right).$$

For $\bar{\nu} \in \pi_1(M)_{I_K}$, we denote the corresponding connected component of $(\mathrm{Gr}_{\tilde{\mathcal{G}},\breve{\mathcal{O}}})^0$ (resp. $(\mathrm{Gr}_{\tilde{\mathcal{G}},\breve{\mathcal{O}}})^{\pm}$) by $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^0_{\bar{\nu}}$ (resp. $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm}_{\bar{\nu}}$).

Let ρ denote the half-sum of the roots in $\mathrm{Res}_{K/F}(N)_{\bar{F}}$ with respect to $\mathrm{Res}_{K/F}(T)_{\bar{F}}$. For $\pi_1(\tilde{M}) \ni \nu \mapsto \bar{\nu} \in \pi_1(\tilde{M})_{I_F} = \pi_1(M)_{I_K}$, and $\dot{\nu} \in X_*(\mathrm{Res}_{K/F}(T))$ a lift of ν, we define the integer $n_\nu := \langle 2\rho, \dot{\nu}\rangle$ (resp. $n_{\bar{\nu}} := \langle 2\rho, \dot{\nu}\rangle$) which is well-defined independent of the choice of $\dot{\nu}$, cf. [HaRi, (3.19)]. Note that we have $n_\nu = n_{\bar{\nu}}$ for all $\nu \mapsto \bar{\nu}$ by definition. For fixed $m \in \mathbb{Z}$, we consider the disjoint union

$$(\mathrm{Gr}_{\tilde{\mathcal{G}}})^0_m \;\overset{\mathrm{def}}{=}\; \coprod_{\bar{\nu}} (\mathrm{Gr}_{\tilde{\mathcal{G}}})^0_{\bar{\nu}} \quad \text{(resp. } (\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm}_m \;\overset{\mathrm{def}}{=}\; \coprod_{\bar{\nu}} (\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm}_{\bar{\nu}}\text{)},$$

where the disjoint sum is indexed by all $\bar{\nu} \in \pi_1(M)_{I_K}$ such that $n_{\bar{\nu}} = m$. The Galois action preserves the integers $n_{\bar{\nu}}$, and hence the ind-scheme $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^0_m$ (resp. $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm}_m$) is defined over $\mathcal{O}$. Note that $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm}_m$ is the preimage of $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^0_m$ along $(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm} \to (\mathrm{Gr}_{\tilde{\mathcal{G}}})^0$. We obtain a decomposition as $\mathcal{O}$-ind-schemes

$$(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{0,c} \;\overset{\mathrm{def}}{=}\; \coprod_{m\in\mathbb{Z}} (\mathrm{Gr}_{\tilde{\mathcal{G}}})^0_m \quad \text{(resp. } (\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm,c} \;\overset{\mathrm{def}}{=}\; \coprod_{m\in\mathbb{Z}} (\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm}_m\text{)}.$$

Properties i) and ii) are immediate from the construction. □

5.2. **Cohomology of $\mathbb{G}_m$-actions on affine Grassmannians.** The conventions are the same as in [HaRi, §3.4]. We fix a prime $\ell \neq p$, and an algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$. We fix once and for all $q^{1/2} \in \bar{\mathbb{Q}}_\ell$, and the square root of the cyclotomic character $\mathrm{cycl}\colon \Gamma_F \to \mathbb{Z}_\ell^\times$ which maps any lift of the geometric Frobenius Φ_F to $q^{-1/2}$. The Tate twists are normalized such that the geometric Frobenius Φ_F acts on $\bar{\mathbb{Q}}_\ell(-1/2)$ by $q^{1/2}$.

For a separated ind-scheme $X = \mathrm{colim}_i X_i$ of finite type over a field (e.g. F) or a discrete valuation ring (e.g. $\mathcal{O}_F$), we denote the bounded derived category $D^b_c(X) = D^b_c(X, \bar{\mathbb{Q}}_\ell)$ of $\bar{\mathbb{Q}}_\ell$-complexes with constructible cohomologies by

$$D^b_c(X) \;\overset{\mathrm{def}}{=}\; \mathrm{colim}_i\, D^b_c(X_i, \bar{\mathbb{Q}}_\ell).$$

There is the full abelian subcategory $\mathrm{Perv}(X) \subset D^b_c(X)$ of perverse sheaves, cf. e.g. [Zhu, A.1] in the setting of ind-schemes. For a complex $\mathcal{A} \in D^b_c(X)$, we denote for any $n \in \mathbb{Z}$ the shifted and twisted complex by

$$\mathcal{A}\langle n\rangle \;\overset{\mathrm{def}}{=}\; \mathcal{A}[n](n/2).$$

Let us briefly recall the nearby cycles functor. Let $S = \mathrm{Spec}(\mathcal{O}_F)$ with open (resp. closed) point $\eta = \mathrm{Spec}(F)$ (resp. $s = \mathrm{Spec}(k)$). Let $\bar{\eta} := \mathrm{Spec}(\bar{F}) \to \eta$ (resp. $\bar{s} := \mathrm{Spec}(\bar{k}) \to s$) denote the geometric point with Galois group $\Gamma = \mathrm{Gal}(\bar{\eta}/\eta)$. Let $\bar{S}$ denote the integral closure of S in $\bar{\eta}$. This gives rise to the seven tuple $(S, \eta, s, \bar{S}, \bar{\eta}, \bar{s}, \Gamma)$. Now if X is an $\mathcal{O}_F$-ind-scheme of ind-finite type, there is by [SGA7, Exp. XIII] (cf. also [Il94, App]) the functor of nearby cycles

$$\Psi_X\colon D^b_c(X_\eta) \longrightarrow D^b_c(X_s \times_S \eta), \tag{5.10}$$

where $D^b_c(X_s \times_S \eta)$ denotes the bounded derived category of $\bar{\mathbb{Q}}_\ell$-sheaves on $X_{\bar{s}}$ with constructible cohomologies, and with a continuous action of Γ compatible with the action on $X_{\bar{s}}$. The nearby cycles preserve perversity and restrict to a functor $\Psi_X\colon \mathrm{Perv}(X_\eta) \to \mathrm{Perv}(X_s \times_S \eta)$. We refer the reader to [PZ13, §10] for the extension to ind-schemes.

For a map of $\mathcal{O}_F$-ind-schemes $f\colon X \to Y$, the nearby cycles are functorial in the obvious way, cf. [SGA7, Exp. XIII, 1.2.7-1.2.9]. Further if f is a nilpotent thickening, i.e., a closed immersion defined by an nilpotent ideal sheaf, then $\Psi_X \simeq \Psi_Y$.

5.2.1. *Geometric Satake for Weil-restrictions.* Recall the geometric Satake equivalence from [Gi, Lu81, BD, MV07, Ri14a, RZ15, Zhu]. We work under the same conventions as in [HaRi, §3.4], and we refer the reader to this reference for more details.

Let G be a connected reductive group over K. We are interested in the geometric Satake isomorphism for the group $\tilde{G} = \mathrm{Res}_{K/F}(G)$. For a conjugacy class $\{\mu\}$ of geometric cocharacters in $\tilde{G}$, denote the inclusion of the open $L^+_z\tilde{G}_{\bar{F}}$ by

$$j\colon \mathrm{Gr}^{\{\mu\}}_{\tilde{G}} \hookrightarrow \mathrm{Gr}^{\leq\{\mu\}}_{\tilde{G}},$$

cf. (4.18). The map j is defined over the reflex field $E = E(\{\mu\})$. We define the normalized intersection complex by

$$\mathrm{IC}_{\{\mu\}} \overset{\text{def}}{=} j_{!*}\bar{\mathbb{Q}}_\ell\langle d_\mu\rangle \in P(\mathrm{Gr}_{\tilde{G},E}), \tag{5.11}$$

where d_μ denotes the dimension of $\mathrm{Gr}^{\leq\{\mu\}}_{\tilde{G}}$. The category $P_{L^+_z\tilde{G}}(\mathrm{Gr}_{\tilde{G}})$ of $L^+_z\tilde{G}$-equivariant perverse sheaves (cf. e.g. [Zhu, A.1] for equivariant perverse sheaves on ind-schemes) is generated by the intersection complexes (5.11) and local systems concentrated on the base point $e_0 \in \mathrm{Gr}_{\tilde{G}}(F)$. More precisely, every indecomposable object in $P_{L^+_z\tilde{G}}(\mathrm{Gr}_{\tilde{G}})$ is of the form

$$\left(\oplus_{\gamma\in\Gamma_F/\Gamma_E}\mathrm{IC}_{\gamma\cdot\{\mu\}}\right) \otimes \mathcal{L}, \tag{5.12}$$

where $\mathcal{L}$ is a $\bar{\mathbb{Q}}_\ell$-local system on $e_0 = \mathrm{Spec}(F)$. The Satake category $\mathrm{Sat}_{\tilde{G}}$ is the full subcategory of $P_{L^+_z\tilde{G}}(\mathrm{Gr}_{\tilde{G}})$ generated by objects (5.12) where the local system $\mathcal{L}$ is trivial over a finite field extension $\tilde{F}/F$.

We view Γ_F as a pro-algebraic group, and we let $\mathrm{Rep}^{\mathrm{alg}}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F)$ be the category of algebraic $\bar{\mathbb{Q}}_\ell$-representations of Γ_F, i.e., finite dimensional representations which factor through a finite quotient of Γ_F. There is the Tate twisted global cohomology functor

$$\begin{aligned} \omega\colon \mathrm{Sat}_{\tilde{G}} &\longrightarrow \mathrm{Rep}^{\mathrm{alg}}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F) \\ \mathcal{A} &\longmapsto \bigoplus_{i\in\mathbb{Z}} \mathrm{H}^i(\mathrm{Gr}_{\tilde{G},\bar{F}}, \mathcal{A}_{\bar{F}})(i/2). \end{aligned} \tag{5.13}$$

By the geometric Satake equivalence, the functor ω can be upgraded to an equivalence of abelian tensor categories

$$\omega\colon \mathrm{Sat}_{\tilde{G}} \xrightarrow{\simeq} \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^L\tilde{G}) \tag{5.14}$$

where ${}^L\tilde{G} = \tilde{G}^\vee \rtimes \Gamma_F$ denotes the L-group viewed as a pro-algebraic group over $\bar{\mathbb{Q}}_\ell$. The tensor structure on $\mathrm{Sat}_{\tilde{G}}$ is given by the convolution of perverse sheaves, cf. §5.5 below. The normalized intersection complex $\mathrm{IC}_{\{\mu\}}$ is an object in the category $\mathrm{Sat}_{\tilde{G}_E}$, and its cohomology $\omega(\mathrm{IC}_{\{\mu\}})$ is under the geometric Satake equivalence (5.14) the ${}^L\tilde{G}_E := \tilde{G}^\vee \rtimes \Gamma_E$-representation $V_{\{\mu\}}$ of highest weight $\{\mu\}$ defined in [Hai14, §6.1], cf. [HaRi, Cor 3.12].

Let us describe the dual group $\tilde{G}^\vee = \mathrm{Res}_{K/F}(G)^\vee$ and the representation $V_{\{\mu\}}$ explicitly in terms of $G^\vee$. Of course, $\tilde{G}^\vee$ is canonically isomorphic to the product $\prod_{K\hookrightarrow\bar{F}} G^\vee$, but the Galois action does not respect the factors in general.

Let $\underline{\mathrm{Hom}}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F, G^\vee)$ be the sheaf of $\bar{\mathbb{Q}}_\ell$-scheme morphisms where again Γ_F is viewed as a pro-algebraic group. Then $\underline{\mathrm{Hom}}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F, G^\vee)$ is a group functor, and the pro-algebraic group Γ_K acts on

$\underline{\mathrm{Hom}}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F, G^\vee)$ via $\bar{\mathbb{Q}}_\ell$-group automorphisms by the rule $(\gamma * f)(g) = \gamma(f(\gamma^{-1}g))$. Following [Bo79, I.5], we define the induced group as the Γ_K-fixed point sheaf

$$I_{\Gamma_K}^{\Gamma_F}(G^\vee) \overset{\text{def}}{=} \underline{\mathrm{Hom}}_{\bar{\mathbb{Q}}_\ell}^{\Gamma_K}(\Gamma_F, G^\vee), \tag{5.15}$$

which is a group functor. Note that choosing any finite extension $\tilde{K}/K$ which is Galois over F and splits G, we get an isomorphism of $\bar{\mathbb{Q}}_\ell$-groups

$$\underline{\mathrm{Hom}}_{\bar{\mathbb{Q}}_\ell}^{\Gamma_{\tilde{K}/K}}(\Gamma_{\tilde{K}/F}, G^\vee) \xrightarrow{\simeq} I_{\Gamma_K}^{\Gamma_F}(G^\vee), \tag{5.16}$$

where $\Gamma_{\tilde{K}/K} = \mathrm{Gal}(\tilde{K}/K)$ (resp. $\Gamma_{\tilde{K}/F} = \mathrm{Gal}(\tilde{K}/F)$). In particular, $I_{\Gamma_K}^{\Gamma_F}(G^\vee)$ is an algebraic group, and is the colimit indexed by the filtered direct system (5.16) indexed by the splitting fields $\tilde{K}$. In this way, we get as in [Bo79, I.5] an Γ_F-equivariant isomorphism of algebraic $\bar{\mathbb{Q}}_\ell$-groups

$$\tilde{G}^\vee \simeq I_{\Gamma_K}^{\Gamma_F}(G^\vee). \tag{5.17}$$

Let us turn to the representation $V_{\{\mu\}}$. We write the conjugacy class as $\{\mu\} = (\{\mu_\psi\})_\psi$ according to $\tilde{G}_{\bar{F}} \simeq \prod_{\psi\colon K\hookrightarrow \bar{F}} G \otimes_{\psi,K} \bar{F}$. The reflex field E of $\{\mu\}$ is the intersection (inside $\bar{F}$) of the reflex fields E_ψ of $\{\mu_\psi\}$. For each ψ, let $V_{\{\mu_\psi\}}$ the representation of $G^\vee$ of highest weight $\{\mu_\psi\}$ where we view $\{\mu_\psi\}$ as a Weyl orbit in the dual torus $X^*(T^\vee)$. The following lemma is immediate from the construction, and left to the reader.

Lemma 5.6. *The $\prod_\psi G^\vee$-representation $\boxtimes_\psi V_{\{\mu_\psi\}}$ uniquely extends to the ${}^L\tilde{G}_E = \tilde{G}^\vee \rtimes \Gamma_E$ representation $V_{\{\mu\}}$ defined above.*

□

5.2.2. *Constant terms commute with nearby cycles.* We proceed with the notation as in §5.1, and view the cocharacter χ as in (5.1). Combining Theorem 5.3 and Proposition 5.5 from the previous section, we have constructed a commutative diagram of $\mathcal{O}_F$-ind-schemes

$$\begin{tikzcd} \mathrm{Gr}_{\tilde{\mathcal{M}}} \arrow[d, "\iota^{0,c}"'] & \mathrm{Gr}_{\tilde{\mathcal{P}}^\pm} \arrow[l, "q^\pm"'] \arrow[r, "p^\pm"] \arrow[d, "\iota^{\pm,c}"'] & \mathrm{Gr}_{\tilde{\mathcal{G}}} \arrow[d, "\mathrm{id}"'] \\ (\mathrm{Gr}_{\tilde{\mathcal{G}}})^{0,c} & (\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm,c} \arrow[l] \arrow[r] & \mathrm{Gr}_{\tilde{\mathcal{G}}}, \end{tikzcd} \tag{5.18}$$

The generic fiber of (5.18) is (5.7), and the special fiber of (5.18) is (5.8). The maps $\iota^{0,c}\colon \mathrm{Gr}_{\tilde{\mathcal{M}}} \hookrightarrow (\mathrm{Gr}_{\tilde{\mathcal{G}}})^{0,c}$ and $\iota^{\pm,c}\colon \mathrm{Gr}_{\tilde{\mathcal{P}}^\pm} \hookrightarrow (\mathrm{Gr}_{\tilde{\mathcal{G}}})^{\pm,c}$ are nilpotent thickenings by Proposition 5.5, and we may and do identify their derived categories of ℓ-adic complexes. Then there is a natural isomorphism of functors $D_c^b(\mathrm{Gr}_{\tilde{M}}) \to D_c^b(\mathcal{F}\ell_{\mathcal{M}'} \times_S \eta)$,

$$\Psi_{\mathrm{Gr}_{\tilde{\mathcal{M}}}} \simeq \Psi_{(\mathrm{Gr}_{\tilde{\mathcal{G}}})^{0,c}}. \tag{5.19}$$

We write $\Psi_{\tilde{\mathcal{G}}} = \Psi_{\mathrm{Gr}_{\tilde{\mathcal{G}}}}$ (resp. $\Psi_{\tilde{\mathcal{M}}} = \Psi_{\mathrm{Gr}_{\tilde{\mathcal{M}}}}$) in what follows. Since $\iota^{0,c}$ and $\iota^{\pm,c}$ are nilpotent thickenings, Proposition 5.5 gives us a decomposition

$$q^\pm = \coprod_{m\in\mathbb{Z}} q_m^\pm\colon\ \mathrm{Gr}_{\tilde{\mathcal{P}}^\pm} = \coprod_{m\in\mathbb{Z}} \mathrm{Gr}_{\tilde{\mathcal{P}}^\pm,m} \longrightarrow \coprod_{m\in\mathbb{Z}} \mathrm{Gr}_{\tilde{\mathcal{M}},m} = \mathrm{Gr}_{\tilde{\mathcal{M}}},$$

according to the choice of the parabolic $P^\pm$. We use the generic and the special fiber of diagram (5.18) to define normalized geometric constant term functors as follows.

Definition 5.7. We define the functor $\mathrm{CT}_{\tilde{M}} : D_c^b(\mathrm{Gr}_{\tilde{G}}) \to D_c^b(\mathrm{Gr}_{\tilde{M}})$ (resp. $\mathrm{CT}_{\mathcal{M}'} : D_c^b(\mathcal{F}\ell_{\mathcal{G}'} \times_S \eta) \to D_c^b(\mathcal{F}\ell_{\mathcal{M}'} \times_S \eta)$) as the shifted pull-push functor

$$\mathrm{CT}_{\tilde{M}} \overset{\text{def}}{=} \bigoplus_{m\in\mathbb{Z}} (q_{m,\eta}^+)_!(p_\eta^+)^*\langle m\rangle \quad (\text{resp. } \mathrm{CT}_{\mathcal{M}'} \overset{\text{def}}{=} \bigoplus_{m\in\mathbb{Z}} (q_{m,s}^+)_!(p_s^+)^*\langle m\rangle).$$

As in [HaRi, Thm 6.1, (6.11)], the functorialities of nearby cycles give a transformation of functors $D_c^b(\mathrm{Gr}_{\tilde{G}}) \to D_c^b(\mathcal{F}\ell_{\mathcal{M}'} \times_S \eta)$ as

$$\mathrm{CT}_{\mathcal{M}'} \circ \Psi_{\tilde{\mathcal{G}}} \longrightarrow \Psi_{\tilde{\mathcal{M}}} \circ \mathrm{CT}_{\tilde{M}}. \tag{5.20}$$

Theorem 5.8. *The transformation* (5.20) *is an isomorphism of functors* $\mathrm{Sat}_{\tilde{G}} \to D^b_c(\mathcal{F}\ell_{\mathcal{M}'} \times_S \eta)$*. In particular, for every* $\mathcal{A} \in \mathrm{Sat}_{\tilde{G}}$*, the complex* $\mathrm{CT}_{\mathcal{M}'} \circ \Psi_{\tilde{\mathcal{G}}}(\mathcal{A})$ *is naturally an object in the category* $\mathrm{Perv}_{L^+\mathcal{M}'}(\mathcal{F}\ell_{\mathcal{M}'} \times_S \eta)$*.*

Proof. Every object in $\mathrm{Sat}_{\tilde{G}}$ is $\mathbb{G}_m$-equivariant. In view of Theorem 5.3 and (5.18), the extension of the method used in [HaRi, Thm 6.5] to this more general situation is obvious. We do not repeat the arguments. □

5.3. **Constant terms for tori.** We aim to make Theorem 5.8 more explicit in the special case where $\tilde{M} = \tilde{T}$ is a torus, cf. Theorem 5.12 and Corollary 5.13. We keep the notation as in §5.2.

Let $\mathrm{Sat}_{\mathcal{T}'} \subset \mathrm{Perv}_{L^+\mathcal{T}'}(\mathcal{F}\ell_{\mathcal{T}'} \times_S \eta)$ denote the semi-simple full subcategory defined as in [Ri16a, Def 5.10]. By [Ri16a, Thm 5.11], the category $\mathrm{Sat}_{\mathcal{T}'}$ has a Tannakian structure with tensor structure given by the convolution product, and with fiber functor given by the global sections functor $\omega_{\mathcal{T}'}\colon \mathrm{Sat}_{\mathcal{T}'} \to \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\Gamma_F)$, $\mathcal{A} \mapsto \mathrm{H}^0(\mathcal{F}\ell_{\mathcal{T}',\bar{k}}, \mathcal{A}_{\bar{k}})$. Note that $\mathcal{F}\ell_{\mathcal{T}'}$ is ind-finite, and hence there is no higher cohomology and the convolution product is given by the usual tensor product. Further, for every $\mathcal{A} \in \mathrm{Sat}_{\mathcal{T}'}$ the Γ_F-action on $\omega_{\mathcal{T}'}(\mathcal{A})$ factors by definition through a finite quotient.

Lemma 5.9. *The functor* $\omega_{\mathcal{T}'}$ *can be upgraded to an equivalence of Tannakian categories*

$$\mathrm{Sat}_{\mathcal{T}'} \xrightarrow{\simeq} \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^L\tilde{T}_r),$$

where ${}^L\tilde{T}_r = (\tilde{T}^\vee)^{I_F} \rtimes \Gamma_F$ *viewed as a pro-algebraic subgroup of* ${}^L\tilde{T}$*.*

Proof. By Lemma 4.17, there are $\mathrm{Gal}(\bar{k}/k)$-isomorphisms of abelian groups

$$\mathcal{F}\ell_{\mathcal{T}'}(\bar{k}) \simeq X_*(T')_{I_{k(\!(u)\!)}} \simeq X_*(T)_{I_K} \simeq X_*(\tilde{T})_{I_F} \simeq X^*\left((\tilde{T}^\vee)^{I_F}\right),$$

where the equivariance of the last isomorphism holds by construction of the dual torus. This induces a $\mathrm{Gal}(\bar{k}/k)$-equivariant isomorphism of $\bar{k}$-schemes

$$(\mathcal{F}\ell_{\mathcal{T}',\bar{k}})_{\mathrm{red}} \simeq \underline{X^*\left((\tilde{T}^\vee)^{I_F}\right)}. \tag{5.21}$$

By definition, the objects in $\mathrm{Sat}_{\mathcal{T}'}$ are finite dimensional $\bar{\mathbb{Q}}_\ell$-vector spaces on (5.21) (viewed as complexes concentrated in cohomological degree 0) together with an action of Γ_F which is equivariant over the base, and which factors through a finite quotient. The lemma follows from this description. □

The following proposition is the analogue of [PZ13, Thm 10.18, 10.23] in the special case of a torus.

Proposition 5.10. *There is a commutative diagram of Tannakian categories*

$$\begin{tikzcd} \mathrm{Sat}_{\tilde{T}} \arrow[r, "\Psi_{\tilde{\mathcal{T}}}"] \arrow[d, "\omega_{\tilde{T}}"', "\simeq"] & \mathrm{Sat}_{\mathcal{T}'} \arrow[d, "\omega_{\mathcal{T}'}"', "\simeq"] \\ \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^L\tilde{T}) \arrow[r, "\mathrm{res}"] & \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^L\tilde{T}_r), \end{tikzcd}$$

where res *denotes the restriction of representations along the inclusion* ${}^L\tilde{T}_r \subset {}^L\tilde{T}$*.*

Proof. This is a reformulation of Lemma 4.17 as follows. In view of (5.14) and Lemma 5.9 the diagram is well defined, and it suffices to prove the commutativity. Let $f\colon \mathrm{Gr}_{\tilde{\mathcal{T}}} \to \mathrm{Spec}(\mathcal{O}_F)$ denote the structure map. Since f is ind-proper, there is a Γ_F-equivariant isomorphism

$$\Psi_{\mathcal{O}_F} \circ f_{\bar{\eta},*} \xrightarrow{\simeq} f_{\bar{s},*} \circ \Psi_{\tilde{\mathcal{T}}},$$

and passing to the 0-th cohomology defines a Γ_F-equivariant isomorphism $\alpha\colon \mathrm{res} \circ \omega_{\tilde{T}} \simeq \omega_{\mathcal{T}'} \circ \Psi_{\tilde{\mathcal{T}}}$. We have to show that α is a map of ${}^L\tilde{T}_r$-representations. As we already know the Γ_F-equivariance, it is enough to check that α is a map of $(\tilde{T}^\vee)^{I_F}$-representations, i.e., respects the grading by $X^*((\tilde{T}^\vee)^{I_F}) = X_*(\tilde{T})_{I_F}$ on the underlying $\bar{\mathbb{Q}}_\ell$-vector spaces. By (5.9), we have a decomposition into connected components

$$\mathrm{Gr}_{\tilde{\mathcal{T}}} \otimes \mathcal{O}_{\breve{F}} = \coprod_{\bar{\nu} \in X_*(\tilde{T})_{I_F}} (\mathrm{Gr}_{\tilde{\mathcal{T}},\mathcal{O}_{\breve{F}}})_{\bar{\nu}},$$

where $(\mathrm{Gr}_{\tilde{\mathcal{T}},\mathcal{O}_{\breve{F}}})_{\bar\nu}\otimes\bar k=\{\bar\nu\}$ and $(\mathrm{Gr}_{\tilde{\mathcal{T}},\mathcal{O}_{\breve{F}}})_{\bar\nu}\otimes\bar F=\coprod_{\nu\mapsto\bar\nu}\{\nu\}$ on the underlying reduced subschemes, cf. also Lemma 4.17. The proposition follows from the fact that nearby cycles of a disjoint sum are computed as the sum of the single components. □

Remark 5.11. It would be interesting to see whether the analogue of Proposition 5.10 holds true for more general *very* special parahoric group schemes, as in [PZ13, Thm 10.18, 10.23].

Combining Proposition 5.10 with Theorem 5.8, we arrive as in [HaRi, §6.2] at the following theorem which is the analogue of [AB09, Thm 4] in our situation.

Theorem 5.12. *i) For every $\mathcal{A}\in\mathrm{Sat}_{\tilde T}$, one has $\mathrm{CT}_{\mathcal{T}'}\circ\Psi_{\tilde{\mathcal{G}}}(\mathcal{A})\in\mathrm{Sat}_{\mathcal{T}'}$.*
ii) The functor $\mathrm{CT}_{\mathcal{T}'}\circ\Psi_{\tilde{\mathcal{G}}}\colon\mathrm{Sat}_{\tilde G}\to\mathrm{Sat}_{\mathcal{T}'}$ admits a unique structure of a tensor functor together with an isomorphism $\omega_{\mathcal{T}'}\circ\mathrm{CT}_{\mathcal{T}'}\circ\Psi_{\tilde{\mathcal{G}}}\simeq\omega_{\tilde G}$. Under the geometric Satake equivalence, it corresponds to the restriction of representations $\mathrm{res}\colon\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^L\tilde G)\to\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}({}^L\tilde T_r)$ along the inclusion ${}^L\tilde T_r\subset{}^L\tilde G$.

□

We now apply Theorem 5.12 in a special case. For more details, we refer to [HaRi, §6.2.1] which is analogous. Assume $F=\breve F$. Let $\chi\colon\mathbb{G}_{m,K}\to A\subset G$ be regular cocharacter, i.e., its centralizer $M=T$ is a maximal torus, and let the parahoric $\mathcal{O}_K$-group scheme $\mathcal{G}$ be an Iwahori. Hence, $\tilde{\mathcal{G}}=\mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}(\mathcal{G})$ is an Iwahori $\mathcal{O}_F$-group scheme as well, cf. Proposition 4.7. There is a decomposition into connected components

$$(\mathcal{F}\ell_{\mathcal{G}'})^+=\coprod_{w\in W}(\mathcal{F}\ell_{\mathcal{G}'})^+_w,$$

where $W=W(\tilde G,\tilde A,F)=W(G,A,K)$ is the Iwahori-Weyl group, cf. Lemma 4.3. Let $\Lambda_{\tilde T}=\tilde T(F)/\tilde{\mathcal{T}}(\mathcal{O}_F)\subset W$ be the subset of "translation" elements. Let $X_*(\tilde T)_{I_F}\simeq\Lambda_{\tilde T},\bar\lambda\mapsto t^{\bar\lambda}$ be the isomorphism given by the Kottwitz map. Let $2\rho\in X^*(\tilde T)$ be the sum of the positive roots contained in the positive Borel $\tilde B^+$ of $\tilde G_{\bar F}$ determined by χ. Then the integer $\langle 2\rho,\bar\lambda\rangle:=\langle 2\rho,\lambda\rangle$ is well defined independent of the choice of $\lambda\in X_*(\tilde T)$ with $\lambda\mapsto\bar\lambda$.

Corollary 5.13. *Let $V\in\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\tilde G^\vee)$, and denote by $\mathcal{A}_V\in\mathrm{Sat}_{\tilde G,\bar F}$ the object with $\omega_{\tilde G,\bar F}(\mathcal{A}_V)=V$. For the compact cohomology group as $\bar{\mathbb{Q}}_\ell$-vector spaces*

$$\mathbb{H}^i_c((\mathcal{F}\ell_{\mathcal{G}'})^+_w,\Psi_{\tilde{\mathcal{G}}}(\mathcal{A}_V))=\begin{cases}V(\bar\lambda) & \text{if } w=t^{\bar\lambda}\text{ and } i=\langle 2\rho,\bar\lambda\rangle;\\ 0 & \text{else,}\end{cases}$$

where $V(\bar\lambda)$ is the $\bar\lambda$-weight space in $V|_{(\tilde T^\vee)^{I_F}}$.

□

5.4. **Special fibers of local models.** Levin proved in [Lev16, Thm. 2.3.5] the analogue of the following theorem in the special case where $p\nmid|\pi_1(G_{\mathrm{der}})|$. As in [HaRi, §6.2, 6.3], Corollary 5.13 can be used to obtain this result on the special fibers of local models, with no hypothesis on p. We do not need this result for the proof of our Main Theorem, but include it for completeness: together with the corresponding result in [HaRi], we can conclude that the admissible sets $\mathrm{Adm}^{\mathbf{f}}_{\{\mu\}}$ parametrize the strata in the special fiber of $M_{\{\mu\}}$ *for all known local models* $M_{\{\mu\}}$.

The following is precisely the analogue of [HaRi, Thm. 6.12] in the current Weil restriction setting. We may work over $F=\breve F$, so that $K=\breve K$ and $k=\bar k$. The special fiber $M_{\{\mu\},k}$ and the relevant Schubert varieties live in the affine flag variety attached to *equal characteristic* analogues $G'=G'_{k(\!(u)\!)}$, $A'=S'$ defined in (4.6), and by Lemmas 4.3 and 4.11 there is an identification of Iwahori-Weyl groups

$$W=W(\tilde G,\tilde A,F)=W(G,A,K)=W(G',A',k(\!(u)\!)).$$

For $w\in W$, we define the Schubert varietiy $\mathcal{F}l^{\leq w}_{\mathcal{G}'}$ exactly as in [HaRi, §3.2].

Theorem 5.14. *The smooth locus $(M_{\{\mu\}})^{\mathrm{sm}}$ is fiberwise dense in $M_{\{\mu\}}$, and on reduced subschemes*

$$(M_{\{\mu\},k})_{\mathrm{red}}=\bigcup_{w\in\mathrm{Adm}^{\mathbf{f}}_{\{\mu\}}}\mathcal{F}l^{\leq w}_{\mathcal{G}'}.$$

In particular, the special fiber $M_{\{\mu\},k}$ is generically reduced.

Proof. We may imitate the proof of [HaRi, Thm. 6.12]. First we follow the method of [Ri16b, Lem. 3.12] to prove $\mathrm{Adm}^{\mathbf{f}}_{\{\mu\}} \subseteq \mathrm{Supp}^{\mathbf{f}}_{\{\mu\}} := \mathrm{Supp}\,\Psi_{\mathcal{G}_{\mathbf{f}}}(\mathrm{IC}_{\{\mu\}})$, using our Lemma 4.17 in place of [Ri16b, Lem. 2.21].

Also as in [HaRi, Thm. 6.12], we reduce to the case where $\mathbf{f} = \mathbf{a}$. Then is it enough to show that if $w \in \mathrm{Supp}^{\mathbf{a}}_{\{\mu\}}$ is maximal, then $w \in \mathrm{Adm}^{\mathbf{a}}_{\{\mu\}}$. Now we choose a regular cocharacter $\chi\colon \mathbb{G}_{m,K} \to A \subset G$, and use Corollary 5.13 as follows. As $\bar{\mathbb{Q}}_\ell$-vector spaces, we have

$$\mathbb{H}^*_c((\mathcal{F}\ell_{\mathcal{G}'})^+_w, \Psi_{\tilde{\mathcal{G}}}(\mathrm{IC}_{\{\mu\}})) \neq 0,$$

because $\mathcal{F}\ell^{\leq w}_{\tilde{\mathcal{G}}'} \cap (\mathcal{F}\ell_{\mathcal{G}'})^+_w \subset \mathcal{F}\ell^{w}_{\mathcal{G}'}$ is non-empty by [HaRi, Lem 6.10], and because up to shift and twist $\Psi_{\tilde{\mathcal{G}}}(\mathrm{IC}_{\{\mu\}})|_{\mathcal{F}\ell^w_{\mathcal{G}'}} = \bar{\mathbb{Q}}^d_\ell$ for some $d > 0$ since $w \in \mathrm{Supp}^{\mathbf{a}}_{\{\mu\}}$ is maximal. Thus, Corollary 5.13 applies to show $w = t^{\bar{\lambda}}$ for some $\bar{\lambda} \in X_*(\tilde{T})_{I_F}$ which is a weight in $V_{\{\mu\}}|_{(\tilde{G}^\vee)^{I_F}}$. As in [HaRi, Thm 6.12], we can conclude that $w = t^{\bar{\lambda}} \in \mathrm{Adm}^{\mathbf{a}}_{\{\mu\}}$ by citing [Hai18, Thm. 4.2 and (7.11-12)]. □

5.5. **Central sheaves.** We recall some facts on central sheaves which will be used in what follows. We proceed with the notation as in §4.4. Let $\mathrm{Perv}_{L^+\mathcal{G}'}(\mathcal{F}\ell_{\mathcal{G}'} \times_S \eta)$ be the category of $L^+\mathcal{G}'$-equivariant perverse sheaves compatible with a continuous Galois action, cf. [PZ13, Def 10.3].

Recall that for objects in $\mathrm{Perv}_{L^+\mathcal{G}'}(\mathcal{F}\ell_{\mathcal{G}'} \times_S \eta)$ there is the convolution product defined by Lusztig [Lu81]. Consider the convolution diagram

$$\mathcal{F}\ell_{\mathcal{G}'} \times \mathcal{F}\ell_{\mathcal{G}'} \xleftarrow{q} L\mathcal{G}' \times \mathcal{F}\ell_{\mathcal{G}'} \xrightarrow{p} L\mathcal{G}' \times^{L^+\mathcal{G}'} \mathcal{F}\ell_{\mathcal{G}'} =: \mathcal{F}\ell_{\mathcal{G}'} \tilde{\times} \mathcal{F}\ell_{\mathcal{G}'} \xrightarrow{m} \mathcal{F}\ell_{\mathcal{G}'}.$$

For $\mathcal{A}, \mathcal{B} \in \mathrm{Perv}_{L^+\mathcal{G}'}(\mathcal{F}\ell_{\mathcal{G}'} \times_S \eta)$, let $\mathcal{A}\tilde{\times}\mathcal{B}$ be the (unique up to canonical isomorphism) complex on $\mathcal{F}\ell_{\mathcal{G}'} \tilde{\times} \mathcal{F}\ell_{\mathcal{G}'}$ such that $q^*(\mathcal{A} \boxtimes \mathcal{B}) \simeq p^*(\mathcal{A}\tilde{\times}\mathcal{B})$. By definition

$$\mathcal{A} \star \mathcal{B} \overset{\mathrm{def}}{=} m_*(\mathcal{A}\tilde{\times}\mathcal{B}) \in D^b_c(\mathcal{F}\ell_{\mathcal{G}'} \times_S \eta, \bar{\mathbb{Q}}_\ell). \tag{5.22}$$

In the following, we consider $P_{L^+\mathcal{G}'}(\mathcal{F}\ell_{\mathcal{G}'})$ as a full subcategory of $P_{L^+\mathcal{G}'}(\mathcal{F}\ell_{\mathcal{G}'} \times_S \eta)$.

Let $W = W(G, A, K) = W(G', A', F')$ be the associated Iwahori-Weyl group, cf. Lemma 4.11. For each $w \in W$, the associated Schubert variety $\mathcal{F}\ell^{\leq w}_{\tilde{\mathcal{G}}'} \subset \mathcal{F}\ell_{\mathcal{G}'}$ is defined over k_F. Let $j\colon \mathcal{F}\ell^w_{\mathcal{G}'} \hookrightarrow \mathcal{F}\ell^{\leq w}_{\tilde{\mathcal{G}}'}$, and denote by $\mathrm{IC}_w = j_{!*}(\bar{\mathbb{Q}}_\ell[\dim(\mathcal{F}\ell^w_{\mathcal{G}'})])$ the intersection complex. We have the functor of nearby cycles

$$\Psi_{\tilde{\mathcal{G}}}\colon \mathrm{Perv}_{L^+_z\tilde{G}}(\mathrm{Gr}_{\tilde{G}}) \longrightarrow \mathrm{Perv}_{L^+\mathcal{G}'}(\mathcal{F}\ell_{\mathcal{G}'} \times_S \eta).$$

The next theorem follows from [PZ13, Thm 10.5] if K/F is tamely ramified, and from [Lev16, Thm 5.2.10] in general:

Theorem 5.15 (Gaitsgory, Zhu, Pappas-Zhu, Levin). *For each $\mathcal{A} \in \mathrm{Perv}_{L^+_z\tilde{G}}(\mathrm{Gr}_{\tilde{G}})$, and $w \in W$, both convolutions $\Psi_{\tilde{\mathcal{G}}}(\mathcal{A}) \star \mathrm{IC}_w$, $\mathrm{IC}_w \star \Psi_{\tilde{\mathcal{G}}}(\mathcal{A})$ are objects in $P_{L^+\mathcal{G}'}(\mathcal{F}\ell_{\mathcal{G}'} \times_S \eta)$, and as such there is a canonical isomorphism*

$$\Psi_{\tilde{\mathcal{G}}}(\mathcal{A}) \star \mathrm{IC}_w \simeq \mathrm{IC}_w \star \Psi_{\tilde{\mathcal{G}}}(\mathcal{A}).$$

Proof. If $\mathcal{A} = \mathrm{IC}_{\{\mu\}}$ where $\{\mu\}$ is a class which is defined over F, then the theorem is a special case of [Lev16, Thm 5.2.10] which follows the method of [PZ13, Thm 10.5]. However, the proof given there works for general objects $\mathcal{A} \in P_{L^+_z\tilde{G}}(\mathrm{Gr}_{\tilde{G}})$, and only uses that the support $\mathrm{Supp}(\mathcal{A})$ is finite dimensional and defined over F. We do not repeat the arguments here. □

6. Test functions

6.1. **Preliminaries.** Recall we let $\tilde{G} = \mathrm{Res}_{K/F}(G)$ and ${}^L\tilde{G} = \mathrm{Res}_{K/F}(G)^\vee \rtimes \Gamma_F$. Recall that $\{\mu\}$ is defined over a field E, a separable field extension of F, and that E_0/F is the maximal unramified subextension of E/F. We have $V_{\{\mu\}} \in \mathrm{Rep}({}^L\tilde{G}_E)$ and $I(V_{\{\mu\}}) \in \mathrm{Rep}({}^L\tilde{G}_{E_0})$, where $I(V) := \mathrm{Ind}^{{}^L\tilde{G}_{E_0}}_{{}^L\tilde{G}_E}(V)$ for $V \in \mathrm{Rep}({}^L\tilde{G}_E)$. The parahoric group scheme in $\tilde{G} = \mathrm{Res}_{K/F}(G)$ is given by $\tilde{\mathcal{G}} = \mathrm{Res}_{\mathcal{O}_K/\mathcal{O}_F}(\mathcal{G})$ by Corollary 4.8. Let $K' = E_0K$, which is the maximal unramified subextension of KE/K, and let $\mathcal{O}_{K'} = \mathcal{O}_K \otimes_{\mathcal{O}_F} \mathcal{O}_{E_0}$ be its ring of integers.

Lemma 6.1. *We have $\tilde{G}_{E_0} = \mathrm{Res}_{K'/E_0}(G_{K'})$ and $\tilde{\mathcal{G}}_{\mathcal{O}_{E_0}} = \mathrm{Res}_{\mathcal{O}_{K'}/\mathcal{O}_{E_0}}(\mathcal{G}_{\mathcal{O}_{K'}})$.*

Proof. This is a consequence of the compatibility of Weil restriction of scalars with base change along the finite free ring extension $F \to E_0$ (resp. $\mathcal{O}_F \to \mathcal{O}_{E_0}$). The compatibility is immediate from the definition of Weil restriction (see e.g. [Oe84, A.2.7]). □

Let $X = \mathbb{A}^1_{\mathcal{O}_F}$ and $X_{\mathcal{O}_{E_0}} = \mathbb{A}^1_{\mathcal{O}_{E_0}}$ and $D = \{Q = 0\}$. The following lemma helps us to determine $\mathrm{Gr}_{(X,\underline{\mathcal{G}},D)} \otimes_{\mathcal{O}_F} \mathcal{O}_{E_0}$.

Lemma 6.2. *In the notation above, we have identifications*

(i) $\underline{\mathcal{G}} \otimes_{\mathcal{O}_F[u]} \mathcal{O}_{E_0}[u] = \underline{\mathcal{G}_{\mathcal{O}_{K'}}}$;

(ii) $(L_D\underline{\mathcal{G}}) \otimes_{\mathcal{O}_F} \mathcal{O}_{E_0} = L_{D_{\mathcal{O}_{E_0}}}\underline{\mathcal{G}_{\mathcal{O}_{K'}}}$ *(and similarly for* L^+_D*);*

(iii) $\mathrm{Gr}_{(X,\underline{\mathcal{G}},D)} \otimes_{\mathcal{O}_F} \mathcal{O}_{E_0} = \mathrm{Gr}_{(X_{\mathcal{O}_{E_0}},\underline{\mathcal{G}_{\mathcal{O}_{K'}}},D_{\mathcal{O}_{E_0}})}$.

Proof. Part (i) follows because the formation of $\underline{G}$ and $\underline{\mathcal{G}}$ as in [Lev, Prop. 3.1.2; Thm. 3.3.3] is compatible with change of base $\mathcal{O}_F[u^{\pm}] \to \mathcal{O}_{E_0}[u^{\pm}]$ (resp., $\mathcal{O}_F[u] \to \mathcal{O}_{E_0}[u]$); see also Example 4.14. Part (ii) follows formally from part (i) and the identities $R[\![D_{\mathcal{O}_{E_0}}]\!] = \varprojlim_n R[u]/Q^n = R[\![D]\!]$ (resp., $R(\!(D_{\mathcal{O}_{E_0}})\!) = (\varprojlim_n R[u]/Q^n)[1/Q] = R(\!(D)\!)$) for $\mathcal{O}_{E_0}$-algebras R. Part (iii) follows from part (ii) and Lemma 3.4 ii). □

6.2. **Statement of theorem.** Given an irreducible representation V of ${}^L\tilde{G}$, we define the parity $d_V \in \mathbb{Z}/2\mathbb{Z}$ as in [HaRi, (7.11)]. Then we define the function $\tau^{\mathrm{ss}}_{\tilde{\mathcal{G}},V}$ on $\mathrm{Gr}_{\tilde{\mathcal{G}}}(k_F)$ by the identify

$$\tau^{\mathrm{ss}}_{\tilde{\mathcal{G}},V} = (-1)^{d_V}\, \mathrm{tr}^{\mathrm{ss}}(\Phi \mid \Psi_{\mathrm{Gr}_{\tilde{\mathcal{G}}}}(\mathrm{Sat}(V))). \tag{6.1}$$

We extend this definition to general representations V of ${}^L\tilde{G}$ (not necessarily irreducible) by linearity. By Theorem 5.15, Lemma 4.12, and Corollary 4.9, we may view $\tau^{\mathrm{ss}}_{\tilde{\mathcal{G}},V}$ as an element in the Hecke algebra $\mathcal{Z}(\tilde{G}(F), \tilde{\mathcal{G}}(\mathcal{O}_F))$. Given any such V, we also define $z^{\mathrm{ss}}_{\tilde{\mathcal{G}},V} \in \mathcal{Z}(\tilde{G}(F), \tilde{\mathcal{G}}(\mathcal{O}_F))$ to be the unique function such that, if π is an irreducible smooth representation of $\tilde{G}(F)$ on a $\bar{\mathbb{Q}}_\ell$-vector space such that $\pi^{\tilde{\mathcal{G}}(\mathcal{O}_F)} \neq 0$, then $z^{\mathrm{ss}}_{\tilde{\mathcal{G}},V}$ acts on $\pi^{\tilde{\mathcal{G}}(\mathcal{O}_F)}$ by the scalar $\mathrm{tr}(s(\pi) \mid V^{1 \rtimes I_F})$, where $s(\pi)$ is the Satake parameter of π as defined in [Hai15].

Theorem 6.3. *For* $(\tilde{G}, \tilde{\mathcal{G}}, V)$ *as above, we have the equality* $\tau^{\mathrm{ss}}_{\tilde{\mathcal{G}},V} = z^{\mathrm{ss}}_{\tilde{\mathcal{G}},V}$.

6.3. **Reducing the Main Theorem to Theorem 6.3.** As in [HaRi, 7.3], we show that the main theorem is a consequence of Theorem 6.3 as follows. Recall that $V_{\{\mu\}}$ is a representation of ${}^L\tilde{G}_E = \tilde{G}^\vee \rtimes \Gamma_E$, the L-group of $\mathrm{Res}_{K/F}G \otimes_F E$, and $I(V_{\{\mu\}})$ is a representation of ${}^L\tilde{G}_{E_0} = \tilde{G}^\vee \rtimes \Gamma_{E_0}$, the L-group of $\mathrm{Res}_{K/F}G \otimes_F E_0 = \mathrm{Res}_{KE_0/E_0}G_{KE_0}$. Arguing as in [HaRi, §7.3], up to the sign $(-1)^{d_\mu}$ the function $\tau^{\mathrm{ss}}_{\{\mu\}}$ is identified with the function

$$\mathrm{tr}^{\mathrm{ss}}(\Phi_E \mid \Psi_{\mathrm{Gr}_{\tilde{\mathcal{G}},\mathcal{O}_E}}(\mathrm{IC}_{\{\mu\}})) = \mathrm{tr}^{\mathrm{ss}}(\Phi_{E_0} \mid \Psi_{\mathrm{Gr}_{\tilde{\mathcal{G}},\mathcal{O}_{E_0}}}(\mathrm{Sat}(I(V_{\{\mu\}})))). \tag{6.2}$$

Taking into account Lemmas 6.1, 6.2 and the results cited in §6.2, the right hand side belongs to $\mathcal{Z}(\tilde{G}(E_0), \tilde{\mathcal{G}}(\mathcal{O}_{E_0}))$. Also, $z^{\mathrm{ss}}_{\{\mu\}}$ acts on $\pi^{\tilde{\mathcal{G}}(\mathcal{O}_{E_0})} \neq 0$ by

$$\mathrm{tr}(s(\pi) \mid V^{I_E}_{\{\mu\}}) = \mathrm{tr}(s(\pi) \mid I(V_{\{\mu\}})^{I_{E_0}}).$$

Furthermore, by Lemma 6.2 (iii), the Main Theorem holds for $V_{\{\mu\}}$ provided Theorem 6.3 holds for the representation $I(V_{\{\mu\}})$ of ${}^L\mathrm{Res}_{KE_0/E_0}G_{KE_0}$ and nearby cycles along $\mathrm{Gr}_{(X_{\mathcal{O}_{E_0}},\underline{\mathcal{G}_{\mathcal{O}_{K'}}},D_{\mathcal{O}_{E_0}})}$. Therefore it suffices to assume $F = E_0$ henceforth. Since all the irreducible factors of the representation $I(V_{\{\mu\}})$ of ${}^L\tilde{G}$ have the same parity, we are reduced to proving Theorem 6.3 for irreducible representations V of ${}^L\tilde{G}$. By [HaRi, Lem. 7.7], we may assume that $V|_{\tilde{G}^\vee \rtimes I_F}$ is irreducible, whenever convenient.

6.4. **Proof of Theorem 6.3.** As in the proof of [HaRi, Thm. 7.9], there are three main steps:

(1) Step 1: Reduction to minimal F-Levi subgroups of $\tilde{G}$.
(2) Step 2: Reduction from anisotropic mod center groups to quasi-split groups.
(3) Step 3: Proof for quasi-split groups.

The proofs work exactly the same way as in [HaRi], with only a few additional remarks. For Step 1, we use Lemma 4.2 to ensure that a minimal F-Levi subgroup of $\tilde{G}$ is of the form $\tilde{M} = \mathrm{Res}_{K/F}(M)$, for M a minimal K-Levi subgroup of G; in light of Theorem 5.15 and Theorem 5.8 the argument of [HaRi] goes through to reduce us to proving the Theorem 6.3 for $\tilde{M}$, i.e., for $\mathrm{Gr}_{(X,\underline{\mathcal{M}},D)}$. For Step 2, we observe that if G^* is a K-quasi-split inner form of G, then $\tilde{G}^* = \mathrm{Res}_{K/F}(G^*)$ is an F-quasi-split inner form of $\tilde{G}$. More to the point, $\mathrm{Gr}_{(X,\underline{\mathcal{G}},D)}$ and $\mathrm{Gr}_{(X,\underline{\mathcal{G}}^*,D)}$ become isomorphic over $\breve{\mathcal{O}}_F$ and hence we may think of them as the same geometrically, with differing Galois actions Φ and Φ^* of the geometric Frobenius element; applying the argument of [HaRi], we reduce to proving Theorem 6.3 for $\tilde{G}^*$, i.e., for $\mathrm{Gr}_{(X,\underline{\mathcal{G}}^*,D)}$.

For Step 3, we apply Step 1 to $\tilde{G}^*$, and we are reduced to proving the theorem for a torus of the form $\tilde{T}^* = \mathrm{Res}_{K/F}(T^*)$, i.e., for $\mathrm{Gr}_{(X,\underline{\mathcal{T}},D)}$. The theorem for any torus is easy. Let us explain following the method of [HaRi, §7.6]). Let V be a representation of $\tilde{T}^\vee \rtimes \Gamma_F$ such that $V|_{\tilde{T}^\vee \rtimes I_F}$ is irreducible. As in [HaRi, Def. 7.11], let $\omega_V \in \pi_1(\tilde{G})^{\Phi}_{I_F}$ be the common image of the $\tilde{B}^\vee$-highest $\tilde{T}^\vee$-weights in $V|_{\tilde{T}^\vee}$. Then ω_V can be viewed as the unique k-rational point in the support of $\Psi(\mathrm{Sat}(V|_{\tilde{T}^\vee \rtimes I_F}))$, and also as the element indexing the unique coset in the support of $z^{\mathrm{ss}}_{\tilde{\mathcal{T}},V}$. By the Grothendieck-Lefschetz fixed point theorem, it suffices to prove

$$z^{\mathrm{ss}}_{\tilde{\mathcal{T}},V}(\omega_V) = \mathrm{tr}(\Phi_F \,|\, V^{1 \rtimes I_F}) = \mathrm{tr}^{\mathrm{ss}}(\Phi_F \,|\, \mathbb{H}^*(\mathrm{Gr}_{\tilde{\mathcal{T}},\bar{F}}, \mathrm{Sat}(V))).$$

The second equality comes from the identifications $\mathbb{H}^*(\mathrm{Gr}_{\tilde{\mathcal{T}},\bar{F}}, \mathrm{Sat}(V)) = \mathbb{H}^0(\mathrm{Gr}_{\tilde{\mathcal{T}},\bar{F}}, \mathrm{Sat}(V)) = V$ as ${}^L\tilde{T}$-representations under the Satake correspondence. Therefore we prove the first equality. Note that $V^{1\rtimes I_F}$ has a single $(\tilde{T}^\vee)^{I_F}$-weight which identifies with ω_V, and $z^{\mathrm{ss}}_{\tilde{\mathcal{T}},V}$ acts on a weakly unramified character $\chi : \tilde{T}(F)/\tilde{T}(F)_1 \to \bar{\mathbb{Q}}_\ell^\times$ by the scalar

$$\mathrm{tr}(s(\chi) \,|\, V^{1\rtimes I_F}) = \chi(\omega_V)\, \mathrm{tr}(\Phi_F \,|\, V^{1\rtimes I_F}),$$

so that $z^{\mathrm{ss}}_{\tilde{\mathcal{T}},V} = \mathrm{tr}(\Phi_F \,|\, V^{1\rtimes I_F})\, \mathbb{1}_{\omega_V}$, as desired. This completes the proof of Step 3 and therefore of Theorem 6.3. □

6.5. **Values of Test Functions.** As in the Main Theorem of [HaRi], the function $q_{E_0}^{d_\mu/2} z^{\mathrm{ss}}_{\tilde{\mathcal{G}},\{\mu\}}$ takes values in $\mathbb{Z}$ and is independent of $\ell \neq p$. The proof given in [HaRi, §7.7] uses only general facts about the Bernstein functions and related combinatorics, and applies equally well to all groups, including those which are Weil-restricted groups such as $\tilde{G}$.

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