Survey of Affine Deligne-Lusztig Varieties

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Outline

A question in σ-linear algebra

- Let $k = \mathbb{F}_q$. Gal$(\overline{k}/k)$ has a canonical generator $\sigma : x \mapsto x^q$.
- Let $\mathcal{O} := \overline{k}[[\epsilon]]$ and Frac$(\mathcal{O}) =: L = \overline{k}((\epsilon))$. The **Frobenius automorphism** σ of L is defined by

$$\sigma(\sum_i a_i \epsilon^i) = \sum_i a_i^q \epsilon^i.$$

- We have $L^\sigma = F := k((\epsilon))$ and $\mathcal{O}^\sigma = \mathcal{O}_F := k[[\epsilon]]$.
- **σ-Linear Algebra Question:** Given $b \in \mathrm{GL}_n(L)$ and $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$, **does there exist** an $\mathcal{O}$-lattice $\Lambda \subset L^n$ such that $b\sigma(\Lambda) \subseteq \Lambda$, and

$$\Lambda / b\sigma(\Lambda) \cong \mathcal{O}/\epsilon^{\mu_1} \oplus \cdots \oplus \mathcal{O}/\epsilon^{\mu_n},$$

in other words, such that inv$(\Lambda, b\sigma(\Lambda)) = \mu$? If yes, what is the **dimension** of the "space of such Λ's"?
- **Goal:** Explain why this question is interesting and how it is answered.

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Examples

- Define $X_\mu^{\mathrm{GL}_n}(b) = \{\Lambda \subset L^n \mid \mathrm{inv}(\Lambda, b\sigma(\Lambda)) = \mu\}$. Call it the **Affine Deligne-Lusztig Variety (ADLV)** associated to GL_n, b, and μ.
- (I) $n = 2$, $b = 1$, and $\mu = (0, 0)$. Then
- $X_\mu^{\mathrm{GL}_2}(b) = \{\Lambda \mid \sigma(\Lambda) = \Lambda\}$
- $= \{\mathcal{O}_F\text{-lattices}\, \Lambda_F \subset F^2\}$
- $=$ the vertices in the building (a tree) for $\mathrm{GL}_2(F)$. This is an infinite discrete set ($\dim = 0$).
- (II) $n = 2$, $b = 1$ and $\mu = (\mu_1, \mu_2)$, where $\mu_1 \geq \mu_2$. Then
- $$\dim\ X_\mu^{\mathrm{GL}_2}(b) = \begin{cases} -\infty, & \text{if } \mu_1 + \mu_2 \neq 0 \\ \mu_1, & \text{if } \mu_1 + \mu_2 = 0. \end{cases}$$

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- It is instructive to prove that **non-emptiness implies** $\mu_1 + \mu_2 = 0$.
- Let $\Lambda_0 = \mathcal{O}e_1 \oplus \mathcal{O}e_2$. Let $K = \mathrm{GL}_2(\mathcal{O}) = \mathrm{Stab}_{\mathrm{GL}_2(L)}(\Lambda_0)$.
- Write $\Lambda = g\Lambda_0$ for $g \in \mathrm{GL}_2(L)$.
- Theory of elementary divisors implies

$$\Lambda \in X_\mu^{\mathrm{GL}_2}(1) \Leftrightarrow g^{-1}\sigma(g) \in K \begin{bmatrix} \epsilon^{\mu_1} & 0 \\ 0 & \epsilon^{\mu_2} \end{bmatrix} K.$$

- Taking determinants, the above implies

$$\epsilon^{\mu_1+\mu_2} \in \det(g^{-1}\sigma(g))\mathcal{O}^\times = \mathcal{O}^\times,$$

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ADLVs for general G

- Let G denote a (split) connected reductive group, and put $K = G(\mathcal{O})$.
- Examples: GL_n, SL_n, $\mathrm{SO}(n)$, $\mathrm{Sp}(2n)$, G_2, E_8, etc.
- The analog of $\mu = (\mu_1, \dots, \mu_2) \in \mathbb{Z}^n$, with $\mu_1 \geq \cdots \geq \mu_n$ is a **dominant cocharacter** $\mu : \mathbb{G}_m \to A$, for A a (split) maximal torus in G. Denote these by $X_*(A)_{\mathrm{dom}}$.
- **Cartan Decomposition:** $G(L) = \coprod_{\mu \in X_*(A)_{\mathrm{dom}}} K\mu(\epsilon)K$.
- Define $X_\mu^G(b) = \{gK \in G(L)/K \mid g^{-1}b\sigma(g) \in K\mu(\epsilon)K\}$.
- This is a locally closed, finite-dimensional subvariety of the **affine Grassmannian** $G(L)/K$.

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Classical Deligne-Lusztig varieties

- Let $B \subset G$ be a Borel subgroup containing A, and let $W = N_G(A)/A$ be the Weyl group.
- **Bruhat Decomposition** $G = \coprod_{w \in W} BwB$, where $G = G(\bar{k})$ and $B = B(\bar{k})$ here.
- Define $X_w = \{gB \in G/B \mid g^{-1}\sigma(g) \in BwB\}$.
- This is a locally closed subvariety of the **flag variety** G/B which is **non-empty**, **smooth**, and has **dimension** equal to $\ell(w)$.
- Deligne and Lusztig introduced these and they are a crucial tool in the representation theory of the finite groups of Lie type, i.e., the finite groups $G(\mathbb{F}_q)$.

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Basic Questions about ADLVs

- (I) For which (μ, b) is $X_\mu^G(b) \neq \emptyset$?
- (II) If non-empty, is $X_\mu^G(b)$ equidimensional, and is there a formula for its dimension?
- (III) What is the geometric structure of $X_\mu^G(b)$ (irreducible components, singularities, etc.)?
- The fact that $X_\mu^G(b)$ can be empty should be contrasted with the classical case.
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- ADLVs arise from Shimura varieties over finite fields and isocrystals with additional structure.

Basic Questions about ADLVs

- (I) For which (μ, b) is $X_\mu^G(b) \neq \emptyset$?
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Isocrystals

- Usual context is p-adic: $F = \mathbb{Q}_p$, $\mathcal{O}_F = \mathbb{Z}_p$, $L = \widehat{\mathbb{Q}}_p^{\text{un}}$, $\mathcal{O}$ = ring of integers in L, $k = \mathbb{F}_p = \mathcal{O}_F/p\mathcal{O}_F$.
- σ is the Frobenius automorphism: either $x \mapsto x^p \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$, or as the element of $\mathrm{Gal}(L/F)$, defined by

$$\sigma\Big(\sum_{i>>-\infty} a_i p^i\Big) = \sum_{i>>-\infty} a_i^p p^i.$$

- An **isocrystal** is a pair (V, Φ), where V is a finite-dimensional L-vector space, and $\Phi : V \to V$ is a σ-linear bijection:

$$\Phi(\alpha v) = \sigma(\alpha)\Phi(v), \quad \forall v \in V, \ \alpha \in L.$$

- If V_0 is an F-vector space and $V = V_0 \otimes_F L$, then all (V, Φ) are of form $(V, b(1 \otimes \sigma))$, for $b \in \mathrm{GL}(V) = \mathrm{GL}_n(L)$.

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Dieudonne's classification of isocrystals

- Dieudonne proved that the category of isocrystals is abelian and semi-simple. The simple objects, parametrized by $\lambda = r/s \in \mathbb{Q}$, are of form

$$V_\lambda := (L^s, b_{r,s}\sigma)$$

where

$$b_{r,s} = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ p^r & & & 0 \end{bmatrix} \in \mathsf{GL}_s(L).$$

- The s-tuple $(r/s, \cdots, r/s)$ is called the Newton vector of V_λ.
- Any (V, Φ) has a Newton vector $\overline{\nu}(V, \Phi) = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{Q}^n_{\mathrm{dom}}$ by decomposing (V, Φ) as a sum of simple objects and stringing together all the Newton vectors of the simple objects, in non-increasing order.
- Given $b \in GL(V)(L)$, define its **Newton point** $\overline{\nu}_b \in \mathbb{Q}^n_{\mathrm{dom}}$ to be the Newton vector of the isocrytal $(L^n, b\sigma)$.

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Elementary computation of Newton points

- $\overline{\nu}_b$ is unchanged if b is replaced with $g^{-1}b\sigma(g)$ (since isomorphism class of $(V, b\sigma)$ is unchanged).
- Therefore we can replace b with an element of form $\epsilon^\lambda w$, i.e., a monomial matrix in $\mathrm{GL}_n(L)$.
- Let N be the order of the permutation matrix w. Then $\overline{\nu}_b$ is the unique dominant element in $\mathbb{Q}^n_{\mathrm{dom}}$ which is some permutation of $\frac{1}{N}\sum_{i=0}^{N-1} w^i(\lambda)$.
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Mazur's inequality and its converse

- For an $\mathcal{O}$-lattice $\Lambda \subset V$, define its **Hodge point** $\mu = \mu(\Lambda) \in \mathbb{Z}^n_{\text{dom}}$ by $\text{inv}(\Lambda, \Phi(\Lambda)) = \mu$. This makes sense even when $\Phi(\Lambda) \not\subseteq \Lambda$.
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- That is, **The Hodge polygon lies above the Newton polygon (with same endpoints)**.
- Gives a necessary condition for non-emptiness of $X_\mu^{\text{GL}_n}(b)$.
- Question: does the converse of Mazur's $\leq$ hold? That is, given $\mu \geq \overline{\nu}(V, \Phi)$, does there exist a lattice $\Lambda \in V$ whose Hodge point is μ?
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Newton and Hodge Polygons

Example

$(1, 1, 0, 0, 0) \geq (\frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$

Mazur inequality and non-emptiness in general

- For general G, Kottwitz defined notions of G-isocrystal, and also the **Newton point** $\overline{\nu}_b \in X_*(A)_{\mathbb{Q},\text{dom}}$ for $b \in G(L)$.
- The inequality $\mu \geq \overline{\nu}_b$ now is for usual dominance order on $X_*(A)_{\mathbb{R},\text{dom}}$.

Theorem

$X_\mu^G(b) \neq \emptyset \Leftrightarrow \mu \geq \overline{\nu}_b$.

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Application of G-isocrystals: moduli of abelian varieties over $\overline{k}$

- Dieudonne: every polarized n-dim'l abelian variety $\mathcal{A}$ over $\overline{k}$ gives rise to a GSp_{2n}-isocrystal $(L^{2n}, b\sigma)$. The Newton point $\overline{\nu}_b$ is therefore an invariant of $\mathcal{A}$.
- Define the **Newton stratum** $\mathcal{S}_b$ in the moduli space of all $\mathcal{A}$ to consist of those $\mathcal{A}$ with fixed Newton point $\overline{\nu}_b$.
- Examples: ordinary abelian varieties form a single Newton stratum (which is open and dense in the moduli space). Supersingular AVs form another Newton stratum.

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- Dieudonne: every polarized n-dim'l abelian variety $\mathcal{A}$ over $\overline{k}$ gives rise to a GSp_{2n}-isocrystal $(L^{2n}, b\sigma)$. The Newton point $\overline{\nu}_b$ is therefore an invariant of $\mathcal{A}$.
- Define the **Newton stratum** $\mathcal{S}_b$ in the moduli space of all $\mathcal{A}$ to consist of those $\mathcal{A}$ with fixed Newton point $\overline{\nu}_b$.
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What about dim$X_\mu^G(b)$?

Theorem (GHKR + Viehmann)

If $X_\mu^G(b) \neq \emptyset$, then

$$\dim\, X_\mu^G(b) = \langle \rho, \mu - \overline{\nu}_b \rangle - \frac{1}{2}(\mathrm{rk}_F G - \mathrm{rk}_F J_b).$$

We write $\mathrm{def}_G(b) := \mathrm{rk}_F G - \mathrm{rk}_F J_b$.

Remarks

- $J_b(F) = \{g \in G(L) \mid g^{-1}b\sigma(g) = b\}$.
- Conjectured by Rapoport, who pointed out the similarity with Chai's conjecture.
- In particular, if $b = 1$, get dim $X_\mu^G(1) = \langle \rho, \mu \rangle$ (cf. GL_2 example).
- After some work, **Chai's conjecture** takes the form surprising form

$$\dim(\mathcal{S}_b) = \langle \rho, \mu + \overline{\nu}_b \rangle - \frac{1}{2}(\mathrm{rk}_F G - \mathrm{rk}_F J_b),$$

  where $\mu = (1^n, 0^n)$, a cocharacter for GSp_{2n}. There is a geometric reason for this similarity.
- $X_\mu^G(b)$ is conjectured to be equidimensional. This is proved when $b \in A(L)$ [GHKR] and when b is "basic" [Hartl-Viehmann].

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ADLVs in the affine flag variety

- Let $I \subset G(L)$ be an Iwahori subgroup, and call $G(L)/I$ the **affine flag variety**.
- $I\backslash G(L)/I = \widetilde{W} = X_*(A) \rtimes W$.
- For $x \in \widetilde{W}$ and $b \in G(L)$, define

$$X_x^G(b) = \{gI \in G(L)/I \mid g^{-1}b\sigma(g) \in IxI\}.$$

- **Questions:** When are $X_x^G(b) \neq \emptyset$? Are they equidimensional? Is there a formula for the dimensions?
- Much less is known, but progress has been made.
- The following picture shows the dimensions of ADLVs for $G = G_2$, $b = 1$.

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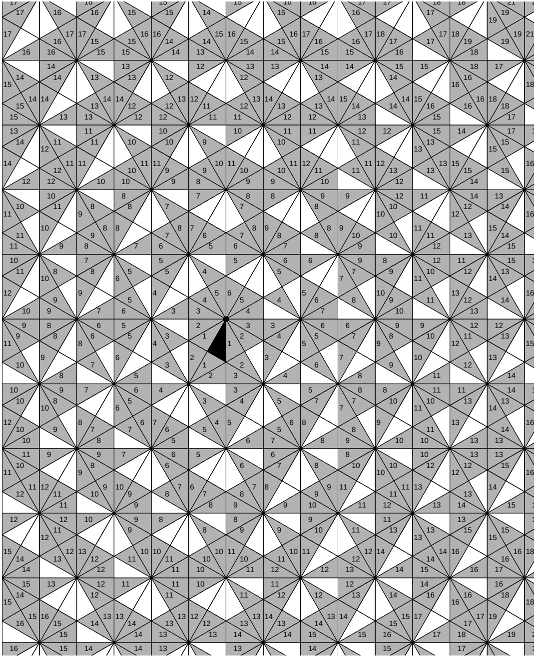

Some results

Theorem (GHKR)

(i) *There is an algorithm, in terms of foldings in Bruhat-Tits building of* $G(L)$*, to compute* dim $X_x^G(b)$ *for all* G,x*, and* b.

(ii) *There is a conjectural (non-algorithmic) description of when* $X_x^G(b)$ *is empty, for* b *"basic", and we can prove emptiness occurs when predicted.*

(iii) *There is a conjectural formula for* x *"generic" and* b *"basic" which is supported by computer evidence: write* $x = w_2\epsilon^\lambda w_1 w_2^{-1}$*, for* $w_i \in W$ *and* $\lambda \in X_*(A)_{\mathrm{dom}}$*. Conjecture:* $X_x(b) \neq \emptyset \Leftrightarrow w_1 \notin \bigcup_{T \subsetneq S} W_T$*, in which case*

$$\dim X_x^G(b) = \frac{1}{2}(\ell(x) + \ell(w_1) - \mathrm{def}_G(b)).$$