INTERTWINERS FOR UNRAMIFIED GROUPS

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1. Introduction

In this note we generalize the algebraic approach to intertwiners given in [HKP], from split groups to unramified groups. Actually, it is perfectly clear that the methods of the current note extend to *arbitrary* non-split groups, but we have limited our discussion to the unramified case because some things become slightly more concrete in that situation. We give some of the standard applications as in [HKP] (such as Bernstein's description of the center of the Iwahori-Hecke algebra). We also provide an ingredient (see section 9) needed for one particularly technical lemma in [H]. Section 9 was the main motivation for writing this note.

2. Preliminaries

2.1. **Basic notation.** Let F denote a p-adic field. Let $\mathcal{O}$ denote the ring of integers in F, and $\pi \in \mathcal{O}$ a uniformizer. Let $q = p^r$ denote the cardinality of the residue field of F. Fix an algebraic closure $\overline{F}$ for F, and let L denote the completion of the maximal unramified extension of F inside $\overline{F}$. Let $\sigma \in \mathrm{Aut}(L/F)$ denote the Frobenius automorphism of L over F.

We let G denote a connected reductive group which is defined and unramified over F. Sometimes we use the symbol G to denote the group $G(F)$ of F-points. Let A denote a maximal F-split subtorus, and set $T := \mathrm{Cent}_G(A)$, a maximal torus defined over F. Let $N = \mathrm{Norm}_G(A)$. We use W to denote the (relative) Weyl group $W := N(F)/T(F)$.

We consider the Bruhat-Tits building $\mathcal{B}(G)$ for $G(F)$. Fix once and for all an alcove $\mathbf{a}$, which we can assume belongs to the apartment corresponding to A. Let I denote the Iwahori subgroup of $G(F)$ corresponding to $\mathbf{a}$. Fix a hyperspecial vertex $\mathbf{a}_0$ in the closure of $\mathbf{a}$, with corresponding hyperspecial maximal compact subgroup $K \supset I$, and designate it as the origin in the aforementioned apartment; this identifies the apartment with the vector space $V := X_*(A)_{\mathbb{R}}$.

We can embed $X_*(A)$ into $A(F)$ in two natural ways. Our convention is to identify $\mu \in X_*(A)$ with $\pi^\mu := \mu(\pi) \in A(F)$.

Let $\mathfrak{B}(T)$ denote the set of Borel subgroups $B = TU$ which contain T and are defined over F. The set $\mathfrak{B}(T)$ is a torsor for the finite Weyl group W (for $w \in W$ and $B \in \mathfrak{B}(T)$, let wB or ${}^wB$ denote wBw^{-1}). For each $B = TU \in \mathfrak{B}(T)$, define the Weyl chamber $\mathcal{C}_U$ in V, and the notion of B-positive root, as follows. Let T_b denote the unique maximal compact subgroup of $T(F)$. The chamber $\mathcal{C}_U$ is the unique one with vertex $\mathbf{a}_0$ such that T_bU is the union of the fixers of all "quartiers" $x + \mathcal{C}_U$ ($x \in V$) in the direction of $\mathcal{C}_U$. Furthermore, a B-positive root is one that appears in $\mathrm{Lie}(B)$. Equivalently, a root α is B-positive if and only if it takes positive values on the chamber $\mathcal{C}_{\overline{U}}$, where $\overline{B} = T\overline{U}$ is the unique element of $\mathfrak{B}(T)$ which is *opposite* to B.

The alcove $\mathbf{a}$ belongs to a unique Weyl chamber having vertex $\mathbf{a}_0$, which we may write in the form $\mathcal{C}_{\overline{U}_0}$ for a unique Borel $B_0 = TU_0 \in \mathfrak{B}(T)$. Thus, the roots $\alpha \in \mathrm{Lie}(B_0)$ are positive on $\mathcal{C}_{\overline{U}_0}$, and a coweight λ belonging to the closure of $\mathcal{C}_{\overline{U}_0}$ is B_0-*dominant*.

Except in a few instances where B denotes an arbitrary element of $\mathfrak{B}(T)$, the notation $B = TU$ *will always mean the opposite Borel* $\overline{B}_0$. Also, unless otherwise noted $\mathcal{C}$ denotes $\mathcal{C}_U = \mathcal{C}_{\overline{U}_0}$, the "dominant" Weyl chamber.

Note that by our conventions, the "reduction modulo π" of I is B. More precisely, we have $B \cap I = B \cap K$.

2.2. **Example of SL_2.** Our conventions amount to the following for the group SL_2. Let $A = T$ denote the diagonal torus. The base alcove $\mathbf{a}$ is the unit interval $[0,1]$ in the real line (which is identified with the apartment for A). The Iwahori subgroup I fixing $\mathbf{a}$ is the one fixing the homothety classes of the lattices $\mathcal{O} \oplus \mathcal{O}$ and $\pi\mathcal{O} \oplus \mathcal{O}$, that is,

$$I = \begin{bmatrix} \mathcal{O}^\times & \pi\mathcal{O} \\ \mathcal{O} & \mathcal{O}^\times \end{bmatrix} \cap \mathrm{SL}_2(F).$$

Furthermore, K is the group $\mathrm{SL}_2(\mathcal{O})$. Also, B_0 is the group of "upper triangular" matrices in SL_2, and B is the group of "lower triangular" matrices.

2.3. **Extended affine Weyl group.** In Bruhat-Tits theory is defined a homomorphism $\nu : N(F) \to V$, which is normalized such that $\nu(\pi^\mu) = -\mu$ (see [Tits]). Its kernel is T_b. Via ν the extended affine Weyl group $\widetilde{W} := N(F)/T_b$ can be viewed as a group of affine-linear transformations of V. It splits as a semi-direct product

$$\widetilde{W} \cong \Lambda \rtimes W,$$

where Λ is the group of translations isomorphic via ν to $T(F)/T_b$. There is a natural inclusion of lattices $X_*(A) \hookrightarrow \Lambda$. In fact Lemma 2.3.1 below shows that $X_*(A) = \Lambda$ (for another proof of this, see [Bo], 9.5).

In [Ko97] Kottwitz defined a surjective homomorphism

$$\omega_G : G(L) \twoheadrightarrow X^*(Z(\widehat{G})^{\Gamma_0})$$

where $\Gamma_0 := \mathrm{Gal}(\overline{L}/L)$ denotes the inertia group. The homomorphisms ω_G vary with G in a functorial manner. Let $G(L)_1$ denote the kernel of ω_G.

Since G is unramified, T necessarily splits over L, and thus the Kottwitz homomorphism takes the simpler form

$$\omega_T : T(L) \twoheadrightarrow X_*(T).$$

[To see that T splits over L, let $S \supseteq A$ be the L-split component of T. Note that S is a maximal L-split torus in G, since any larger L-split torus $S' \supseteq S$ contains A and hence belongs to T, hence to S. Since G splits over L, there exists a maximal torus $T' \subset G$ which is defined and split over L. By a standard result, S and T' are conjugate by an element of $G(L)$, hence $\dim(S) = \dim(T') = \dim(T)$, and thus $S = T$.]

Lemma 2.3.1. *For G unramified over F, we have*

(i) $T_b = T(F) \cap T(L)_1$;
(ii) $T(F)/(T(F) \cap T(L)_1) \cong X_*(A)$ *via* ω_T.

In particular, the inclusion $X_*(A) \hookrightarrow \Lambda$ is an isomorphism, and $\widetilde{W} \cong X_*(A) \rtimes W$.

Proof. There is an inclusion $i : X_*(T) \hookrightarrow X_*(T)_{\mathbb{R}}$. By [Ko97], (7.4.5), there is a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & T(L)_1 & \longrightarrow & T(L) & \xrightarrow{i\circ\omega_T} & X_*(T)_{\mathbb{R}} \\ & & & & \uparrow & & \uparrow \\ 0 & \longrightarrow & T_b & \longrightarrow & T(F) & \xrightarrow{-\nu} & X_*(A)_{\mathbb{R}} \end{array}$$

where the vertical arrows are injective. This shows that $i \circ \omega_T$ and $-\nu$ agree as maps $T(F) \to V$, proving part (i) and the equality of the images in V of $-\nu$ and $\omega_T|_{T(F)}$. Furthermore, consider the exact sequence

$$0 \longrightarrow T(L)_1 \cap T(F) \longrightarrow T(F) \xrightarrow{\omega_T} X_*(A) \longrightarrow 0$$

resulting from taking σ-invariants of

$$0 \longrightarrow T(L)_1 \longrightarrow T(L) \xrightarrow{\omega_T} X_*(T) \longrightarrow 0,$$

and using the fact that $H^1(\langle\sigma\rangle, T(L)_1) = 0$ (cf. [Ko97], (7.6.1)). We see that $\omega_T|_{T(F)}$ has image $X_*(A)$, i.e. $X_*(A) = \Lambda$. □

2.4. **Bruhat orders and length function.** When we speak of the Bruhat order on W or on $\widetilde{W}$, we will always mean the Bruhat order defined relative to the reflections through the walls of $\mathcal{C}$ resp. $\mathbf{a}$. Also, the length function ℓ on $\widetilde{W}$ is defined in terms of the reflections through the walls of $\mathbf{a}$.

2.5. **On Bruhat-Tits and Iwasawa decompositions.** Over L, our group is split and we have the usual Bruhat-Tits and Iwasawa decompositions

$$G(L) = \coprod_{w\in\widetilde{W}(L)} I(L)wI(L) = \coprod_{w\in\widetilde{W}(L)} U(L)wI(L).$$

Taking fixed points under σ, these yield the corresponding decompositions over F:

$$G(F) = \coprod_{w\in\widetilde{W}} IwI = \coprod_{w\in\widetilde{W}} UwI. \tag{2.5.1}$$

3. Affine roots and root subgroups

Let $\Phi = \Phi(G, A)$ denote the set of relative roots for G and the F-torus A. Let Φ_{aff} denote the Bruhat-Tits affine roots (as defined in [Tits], §1.4-1.6). Let Σ denote the canonical finite reduced root system associated to Φ_{aff} and the vertex $\mathbf{a}_0$ (the pair (Φ, Σ) forms an "echelonnage" in the sense of [BT1], §1.4). Each root $a \in \Sigma$ is a positive scalar multiple of a unique non-divisible root in Φ. Let Σ_{aff} denote the affine root system

$$\Sigma_{\text{aff}} = \{a + k \mid a \in \Sigma,\ k \in \mathbb{Z}\}.$$

The basic property defining Σ is that the hyperplanes in $X_*(A)_{\mathbb{R}}$ defined by $\alpha = 0$ for $\alpha \in \Phi_{\text{aff}}$ are precisely the hyperplanes $\alpha = 0$, for $\alpha \in \Sigma_{\text{aff}}$. In fact, for each affine root $a' + k' \in \Phi_{\text{aff}}$ ($a' \in \Phi$, $k' \in \mathbb{R}$), there is a unique $a + k \in \Sigma_{\text{aff}}$ ($a \in \Sigma$, $k \in \mathbb{Z}$), such that $a + k$ is a positive real multiple of $a' + k'$.

Letting Φ_0^+ denote the B_0-positive roots in Φ, we get a corresponding set of positive roots $\Sigma_0^+ \subset \Sigma$. Let Δ_0 denote the set of simple roots in Σ_0^+. Similarly, let Δ denote the set of simple roots in the set of B-positive roots Σ^+.

To each non-divisible root $a' \in \Phi$ (corresponding to $a \in \Sigma$) there is a unipotent subgroup $U_{a'}(F)$ equipped with a filtration which is used to define the set $\Phi_{\rm aff}$ (see [Tits], §1.4). For $\alpha := a + k \in \Sigma_{\rm aff}$, there is a subgroup $U_\alpha \subset U_{a'}(F)$. The family $\{U_{a+k}\}$ possesses the following properties:

(1) $U_{a'}(F)$ is the union of the subgroups U_{a+k}, for $k \in \mathbb{Z}$;
(2) $U_{\alpha+1} \subsetneq U_\alpha$;
(3) $U_{-\alpha} - U_{-\alpha+1} \subset U_\alpha \nu^{-1}(s_\alpha) U_\alpha = U_\alpha t_a^k s_a T_b U_\alpha$, if $\alpha = a + k \in \Sigma_{\rm aff}$.

Here $\nu : N(F) \twoheadrightarrow \widetilde{W} \cong \Lambda \rtimes W$ is the homomorphism of Bruhat-Tits (cf. [Tits], §1). (In particular, $\nu(\pi^\lambda) = -\lambda \in X_*(A) \cong \Lambda$.) Moreover, for $\alpha = a + k \in \Sigma_{\rm aff}$, the element $t_\alpha \in A(F)$ is defined in the following subsection (for $\alpha = a$ it is $\pi^{a^\vee}$; see below and compare with the element a_α of [Cas80]).

For the proof of the key property (3), we refer to [Mac], Lemma (2.6.6) and [BT1], Lemme (6.3.3). See also the examples in subsection 3.2 below.

3.1. **Definition of t_α and in particular of an element $a^\vee \in X_*(A)$.** Via the Bruhat-Tits homomorphism ν, the group $T(F)/T_b$ is identified with a group of translations Λ acting on the vector space $X_*(A)_\mathbb{R}$. In fact the elements of Λ belong to the lattice of coweights $P^\vee(\Sigma)$ associated to the root system Σ, and as we saw in Lemma 2.3.1 Λ also coincides in the unramified case with the group of translations by elements of $X_*(A)$. For $\alpha \in \Sigma_{\rm aff}$, the element

$$s_{\alpha-1} \circ s_\alpha$$

is a translation by a *coroot* for Σ, hence it belongs to the coroot lattice $Q^\vee(\Sigma) \subset P^\vee(\Sigma)$. The image of the Bruhat-Tits homomorphism contains every affine reflection associated to Σ (and consequently all translations by coroots). Therefore $s_{\alpha-1} \circ s_\alpha$ can be lifted to a unique coset $t_\alpha \in T(F)/T_b$ such that $-\nu(t_\alpha) = s_{\alpha-1} \circ s_\alpha$. Thus t_α can be viewed as an element in $T(F)$ uniquely determined up to T_b. By Lemma 2.3.1, in the unramified case we may also view t_α as an element in $A/A_\mathcal{O}$. Therefore, for $a \in \Sigma$ we may write $t_a = \pi^{a^\vee}$ for some unique cocharacter $a^\vee \in X_*(A)$. In fact the notation is not accidental: the element $a^\vee$ is precisely the coroot $a^\vee \in Q^\vee(\Sigma)$ corresponding to the root $a \in \Sigma$.

In the sequel, we will *sometimes* denote the element t_a by $\pi^{a^\vee}$.

For every $\alpha := a + k \in \Sigma_{\rm aff}$ and $l \in \mathbb{Z}$, we have

$$t_\alpha U_{\alpha+l} t_\alpha^{-1} = U_{\alpha+l+2}.$$

Note also that for any $w \in W$, we have

$$w t_\alpha w^{-1} = t_{w\alpha},$$

where $w\alpha$ is defined by $w(a + k) := a \circ w^{-1} + k$.

For each $\alpha \in \Sigma_{\rm aff}$, let

$$q_\alpha = [U_{\alpha-1} : U_\alpha].$$

It is clear that q_α is always the same as $q_{\alpha+2}$, but it is not necessarily the same as $q_{\alpha+1}$. Macdonald [Mac] defines a root system Σ_1 with $\Sigma \subseteq \Sigma_1 \subseteq \Sigma \cup \frac{1}{2}\Sigma$, where for $a \in \Sigma$, the element $a/2$ lies in Σ_1 if and only if $q_{a+1} \neq q_a$. For every $a \in \Sigma$, he defines $q_{a/2} = q_{a+1}/q_a$. Then:

(a) For $a \in \Sigma$, we have $[U_{a+1} : U_{a+m+1}] = q_{a/2}^{[m/2]} q_a^m$;

(b) For $a \in \Delta$, we have $[Is_aI : I] = q_{a/2}q_a$ ([Mac] 2.7.4, 3.1.6);
(c) If G has semi-simple rank 1, and $a \in \Delta$, then

$$\delta_B(t_a) = [U_a : t_a U_a t_a^{-1}]^{-1} = q_{a/2}^{-1} q_a^{-2}.$$

3.2. **Examples of the constructions above.**

Example: SL_2. If a is the unique B_0-positive root $a = e_1 - e_2$, and $\alpha = -a + 1$ (the simple affine root), then $s_{\alpha-1} \circ s_\alpha$ is the translation by $-a^\vee = (-a)^\vee$, where $a^\vee$ is the coroot associated to a, and so $t_{-a+1} = \pi^{(-a)^\vee}$, according to our conventions. More generally, for any $a \in \Sigma$, we have $t_{a+k} = \pi^{a^\vee}$, for any $k \in \mathbb{Z}$. The identity (3) above for $\alpha = a + k$ ($a \in \Delta_0$) corresponds to the following matrix identity: given $u \in \mathcal{O}^\times$ and any $k \in \mathbb{Z}$, we have

$$\begin{bmatrix} 1 & 0 \\ \pi^{-k}u & 1 \end{bmatrix} = \begin{bmatrix} 1 & \pi^k a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \pi^k \alpha & 0 \\ 0 & \pi^{-k}\alpha^{-1} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & \pi^k b \\ 0 & 1 \end{bmatrix},$$

by taking $\alpha = a = b = u^{-1}$.

Example: Quasi-split SU_3 for an unramified quadratic extension E/F. Let $a_1 \in \Phi_0^+$ be the unique non-divisible B_0-positive root, so that $2a_1$ is also a B_0-positive root. Let $a \in \Delta_0$ be the corresponding positive root of Σ. Then since the simple affine roots are $\{a_1, -2a_1 + 1\}$, we see that in fact $a = 2a_1$ and that $s_{a-1} \circ s_a = t_{(2a_1)^\vee}$. So, in this case $a^\vee = (2a_1)^\vee = a_1^\vee/2$. Concretely, $a^\vee$ is the cocharacter in $X_*(A)$ given by the formula for $x \in F^\times$

$$a^\vee(x) = \begin{bmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & x^{-1} \end{bmatrix},$$

and $a_1^\vee(x) = a^\vee(x^2)$. Furthermore, if the cardinality of $\mathcal{O}_F/\pi\mathcal{O}_F$ is q, then $q_a = q$, $q_{a+1} = q^3$, and $q_{a/2} = q^2$. For all this, see [Tits], §1.15.

Let $x \mapsto \bar{x}$ denote the generator of $\mathrm{Gal}(E/F)$. Then given an integer k, a unit $u \in \mathcal{O}_E^\times$ and an element $c \in E$ such that $-c\bar{c} = \pi^{-k}(u + \bar{u})$, the relation (3) for $\alpha = a + k$ ($a \in \Delta_0$) corresponds to the matrix identity:

$$\begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ \pi^{-k}u & -\bar{c} & 1 \end{bmatrix} = \begin{bmatrix} 1 & a & \pi^k v \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \pi^k \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \pi^{-k}\bar{\alpha}^{-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & b & \pi^k w \\ 0 & 1 & -\bar{b} \\ 0 & 0 & 1 \end{bmatrix},$$

where $\alpha = \bar{u}^{-1}$, $\beta = \bar{u}u^{-1}$, $v = w = u^{-1}$, $a = -\pi^k \bar{c}\bar{u}^{-1}$, and $b = -\pi^k \bar{c} u^{-1}$.

4. Hecke algebras and the universal model for unramified principal series

4.1. **Hecke algebras.** Consider the Iwahori-Hecke algebra $H := C_c(I\backslash G/I)$, where convolution is defined using the Haar measure dg which gives I measure 1. Consider also the spherical Hecke algebra $H_K := C_c(K\backslash G/K)$, and the finite Hecke algebra $H_f := C_c(I\backslash K/I)$. As a $\mathbb{C}$-vector space H has a basis given by the characteristic functions $T_w = 1_{IwI}$, where $w \in \widetilde{W}$.

The Iwahori-Hecke algebra of the torus A is the ring $R := C_c(A(F)/A_{\mathcal{O}})$, where convolution is defined using the Haar measure da on $A(F)$ which gives $A_{\mathcal{O}}$ measure 1. By using the valuation map ν the ring R is identified with the group algebra $\mathbb{C}[X_*(A)]$.

Now for any $B = TU \in \mathfrak{B}(T)$, and any $t \in T(F)$, we denote by $\delta_B(t)$ the normalized absolute value of the determinant of the adjoint action of t on $\mathrm{Lie}(U)$.

4.2. **The universal model M.** Write $B = TU$ as before. Let $\mathbf{M} = \mathbf{M}_B$ be defined as the (R, H)-bimodule

$$\mathbf{M} = C_c(A_{\mathcal{O}}U\backslash G/I) = C_c(T_bU\backslash G/I).$$

The subscript "c" means that we consider functions supported on only finitely many double cosets. As a complex vector space, $\mathbf{M}$ has a basis consisting of the functions $v_x := 1_{A_{\mathcal{O}}UxI}$ $(x \in \widetilde{W})$.

It is clear that H acts on the right on $\mathbf{M}$ by right convolutions. One proves as in [HKP] Lemma 1.6.1 that $\mathbf{M}$ is free of rank 1 as an H-module, with canonical generator v_1 (use (2.5.1)). Furthermore, R acts on the left on M by *normalized* left convolutions. More precisely, letting $f \in R$, we define the left action of f on v_x by the integral

$$f \cdot v_x(g) = \int_A \delta_B^{1/2}(a) f(a) v_x(a^{-1}g)\, da.$$

In other words, if $\lambda \in X_*(A)$ and if π^λ is regarded as both an element in $A/A_{\mathcal{O}}$ *and* as the characteristic function on $A/A_{\mathcal{O}}$ for the subset π^λ, then

$$\pi^\lambda \cdot v_x = \delta_B^{1/2}(\pi^\lambda) v_{t_\lambda x}, \tag{4.2.1}$$

where t_λ is the translation element of $\widetilde{W}$ corresponding to $\lambda \in X_*(A)$. Clearly this applies in particular to the element $\lambda = a^\vee$ we attached to $a \in \Sigma$.

5. Intertwiners for nonstandard models

5.1. **Definition and basic properties.** In this section we follow closely the construction of intertwiners given in [HKP], §1.10 (the difference being that [HKP] only treated the intertwiners for standard models attached to split groups). As in loc. cit., we let J be a set of coroots $a^\vee \in Q^\vee(\Sigma) \subset X_*(A)$ which belong to some positive subsystem of coroots. Denote by $\mathbb{C}[J]$ the $\mathbb{C}$-subalgebra of R generated by J and by $\widehat{\mathbb{C}[J]}$ the completion of $\mathbb{C}[J]$ with respect to the (maximal) ideal generated by J. Let R_J denote the R-algebra $\widehat{\mathbb{C}[J]} \otimes_{\mathbb{C}[J]} R$, a completion of R that can be viewed as the convolution algebra of complex valued functions on $X_*(A)$ supported on a finite union of sets of the form $\lambda + C_J$, where C_J is the submonoid of $X_*(A)$ consisting of all non-negative integral linear combinations of elements in J.

Given $B = TU \in \mathfrak{B}(T)$ and J as above, we denote by $\mathbf{M}_{B,J}$ the module $R_J \otimes_R C_c(A_{\mathcal{O}}U\backslash G/I)$, which can be viewed as the set of functions f on $A_{\mathcal{O}}U\backslash G/I$ satisfying the following support condition: there exists a finite union S of sets of the form $\lambda + C_J$ such that the support of f is contained in the union of the sets $A_{\mathcal{O}}U\pi^\nu K$ for $\nu \in S$. Clearly $\mathbf{M}_{B,J}$ is a left R_J-module and a right H-module. We will often write $\mathbf{M}_J$ in place of $\mathbf{M}_{B,J}$ when B is understood.

Now let w denote a fixed element of W as well as its lift to an element of $N(F) \cap K$. Let J denote the set of coroots $a^\vee \in Q^\vee(\Sigma)$ which are B-positive and wB-negative. Then as in loc. cit., we define an intertwiner

$$I_w : \mathbf{M}_{B,w^{-1}J} \to \mathbf{M}_{B,J}$$

by the integral

$$I_w(\varphi)(g) = \int_{U_w} \varphi(w^{-1}ug)\,du, \tag{5.1.1}$$

where U_w denotes $U \cap w\overline{U}w^{-1}$. The Haar measure du on U_w is normalized such that $U_w \cap K$ has measure 1. Here we view the elements φ as functions (as above). As in loc. cit. Lemma 1.10.1, one proves the convergence of the integral defining $I_w(\varphi)(g)$, and the fact that $I_w(\varphi) \in \mathbf{M}_J$. (The proof of the latter is very similar to the argument in section 5.2 below.)

We remark that in defining I_w, we may replace the set J with any larger set of coroots that is still contained in a positive subsystem, for example the set of all B-positive coroots.

Moreover, if $w = w_1w_2$ where $\ell(w) = \ell(w_1) + \ell(w_2)$ and if J is chosen as above, then the composition $I_{w_1} \circ I_{w_2}$ is defined. The following properties are immediate.

Lemma 5.1.1. *We have*

(i) $I_w \circ \pi^\mu = \pi^{w\mu} \circ I_w$ *for each* $\mu \in X_*(A)$*;*
(ii) $I_{w_1w_2} = I_{w_1} \circ I_{w_2}$ *if* $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$*;*
(iii) I_w *is a right* H*-module homomorphism.*

Now suppose that $B_1 = TU_1$ and $B_2 = TU_2$ are two elements of the set $\mathfrak{B}(T)$, and that $B_2 = vB_1$ for $v \in W$. Suppose we have chosen the sets J_1 and J_2 such that $J_2 = vJ_1$. Given a function φ on G, define the function $L(v)\varphi$ by $L(v)\varphi(g) := \varphi(v^{-1}g)$. The following diagram commutes:

$$\begin{array}{ccc} \mathbf{M}_{B_2,w^{-1}j_2} & \xrightarrow{L(v^{-1})} & \mathbf{M}_{B_1,v^{-1}w^{-1}J_2} \\ {\scriptstyle I_w}\downarrow & & \downarrow{\scriptstyle I_{v^{-1}wv}} \\ \mathbf{M}_{B_2,J_2} & \xleftarrow{L(v)} & \mathbf{M}_{B_1,J_1}. \end{array} \tag{5.1.2}$$

We also have the following relation for any $v, w \in W$ and any $B \in \mathfrak{B}(T)$:

$$L(v)v_w^B = v_{vw}^{vB}. \tag{5.1.3}$$

Because of (5.1.2), there is no loss in generality in studying only the *standard* intertwiners, i.e. those for which $U = \overline{U}_0$. That is what we will do, and so from now on *all statements about intertwiners refer to the standard ones.* We leave it to the reader to derive the analogous statements about the nonstandard intertwiners using (5.1.2).

For this *standard* situation $B = \overline{B}_0$, we have the following three useful equalities. They are proved the same way as in [HKP] (using the Iwahori factorization and along the way $I \cap B = K \cap B$ and $I \cap \overline{B} \subset wIw^{-1}$).

$$v_1T_w = v_w, \text{ for every } w \in W, \tag{5.1.4}$$

$$v_{\pi^\mu}T_w = v_{\pi^\mu w}, \text{ for every } w \in W \text{ and } \mu \in X_*(A), \tag{5.1.5}$$

$$v_1T_{\pi^\mu} = v_{\pi^\mu}, \text{ for } \mu \in X_*(A) \text{ } B\text{-dominant.} \tag{5.1.6}$$

5.2. **On the support of $I_w(v_1)$.** In this subsection we give some necessary conditions for an element $\pi^\lambda v \in \widetilde{W}$ to be in the support of $I_w(v_1)$. First we define a relation $x \rightharpoonup y$ on W which is reflexive and transitive (but not anti-symmetric), and which depends on our choice of Weyl chamber $\mathcal{C}$. Given $x \in W$, let $S(x)$ denote the set of simple reflections in the set $\{s_a \mid a \in \Delta\}$ which appear in some (equivalently, all) reduced expressions for x. Then we write $x \rightharpoonup y$ if and only if $S(x) \subseteq S(y)$.

Secondly, we recall the definition of the *retraction* functions r_B (for any $B \in \mathfrak{B}(T)$): if $g = u\pi^\mu k$ in the Iwasawa decomposition $G = UAK$, then we set $r_B(g) := \mu$. Note that the Iwasawa decomposition which a priori takes the form $G = UTK$ agrees with the one we used, since $A(F)/A_{\mathcal{O}} = T(F)/T_b$ (Lemma 2.3.1) and $T_b \subset K$.

The basic property of the family of retractions $\{r_B\}_{B\in\mathfrak{B}(T)}$ is that for any $g \in G$ and $B_1, B_2 \in \mathfrak{B}(T)$, the difference $r_{B_1}(g) - r_{B_2}(g)$ is a sum of coroots $a^\vee \in Q^\vee(\Sigma)$ which are B_1-positive and B_2-negative.

One needs to check this only for adjacent Borels in $\mathfrak{B}(T)$. In the split case, a simple computation in SL_2 does the job[1].

Recall that we are currently assuming $B = \overline{B}_0$, so that J can be taken as the set of coroots $a^\vee$ such that $a^\vee$ is B-positive (i.e. B_0-negative) and wB-negative (i.e. wB_0-positive).

Lemma 5.2.1. *If $I_w(v_1)(\pi^\lambda v) \neq 0$, then λ is a sum of elements of J, and $v \rightharpoonup w$. In particular, if $a \in \Delta$ and $I_{s_a}(v_1)(\pi^\lambda v) \neq 0$, then $\lambda = ka^\vee$ for some $k \in \mathbb{Z}_{\geq 0}$, and $v \in \{1, s_a\}$.*

Proof. The nonvanishing of $I_w(v_1)(\pi^\mu v)$ implies that we may write $g := u\pi^\mu v = u'wi$ for some $u \in U_w$, $u' \in wU$, and $i \in I$. It follows that $\mu = r_B(g) - r_{wB}(g)$ is a sum of coroots which are B-positive and wB-negative.

Next, let $M \supset T$ denote the Levi subgroup corresponding to the set of simple positive roots $a \in \Delta$ such that $s_a \in S(w)$. Note that this is a Levi subgroup of a parabolic subgroup which contains B, and that $U_w \subset M$.

We see that $gv^{-1} = u\pi^\mu \in Bv^{-1}\,{}^vI$. Since $u\pi^\mu \in M$, then using the Iwasawa decomposition for that group we see that $u\pi^\mu \in (B\cap M)w_M({}^vI \cap M)$, for some w_M in the Weyl group for M. Comparing the two Iwasawa decompositions we see $v^{-1} = w_M \in M$. But since the Weyl group of M is generated by the simple reflections $s_a \in S(w)$, we see finally that $v \rightharpoonup w$. $\square$

6. Formula for $I_{s_a}(v_1)$

Here we again fix $B = TU \in \mathfrak{B}(T)$ to be the opposite Borel $\overline{B}_0$. Let J denote the set of B-positive coroots $a^\vee \in Q^\vee(\Sigma)$.

For $a \in \Sigma^+$, we define an element $c_a \in R_J$ by

$$c_a = \frac{(1 - q_{a/2}^{-1/2}q_a^{-1}\pi^{a^\vee})(1 + q_{a/2}^{-1/2}\pi^{a^\vee})}{1 - \pi^{2a^\vee}}.$$

(In a similar way we may define c_a for any $a \in \Sigma$, but we have to choose J appropriately to ensure that $c_a \in R_J$.) Also, for $w \in W$, define

$$c_w := \prod_{a\in\Sigma_w} c_a,$$

where $\Sigma_w = \{a \in \Sigma \mid a \text{ is } B\text{-positive and } w^{-1}a \text{ is } B\text{-negative}\}$.

Theorem 6.0.2. *Let $a \in \Delta$. We have the following equality in $\mathbf{M}_{B,J}$:*

$$I_{s_a}(v_1) = (q_{a/2}q_a)^{-1}v_{s_a} + (c_a - 1)\cdot v_1$$

[1]The same proof in the unramified case works, and in the end requires also a simple computation in the group SU_3.

Note that this implies that $(1-\pi^{2a^\vee})I_{s_a}$ takes $\mathbf{M}$ into itself.

7. Proof of Theorem 6.0.2

The proof below is the result of combining the approach of [HKP] with some ideas of Casselman [Cas80].

7.1. **Reduction to semi-simple rank 1.** The calculation of $I_{s_a}(v_1)$ immediately reduces from the group G to the Levi subgroup corresponding to $a \in \Delta$, by Lemma 5.2.1.

7.2. **Proof in the case of semi-simple rank 1.** Now we assume G has semi-simple rank 1. Let a' denote the unique non-divisible B-positive (thus B_0-negative) root in Φ, and let $a \in \Delta$ be the corresponding simple B-positive root in Σ.

By Lemma 5.2.1, we can write $I_{s_a}(v_1)$ as a sum

$$I_{s_a}(v_1) = \sum_{k,w} J(k,w)\, v_{t_{ka^\vee}w},$$

where $w \in \{1, s_a\}$ and $k \geq 0$. We need to find all the coefficients $J(k,w)$.

First consider the case where $k=0$. Then $J(0,1) \neq 0$ implies that we can solve the equation $\overline{u}s_a = ui$, where $\overline{u} \in \overline{U}$, $u \in U$, and $i \in I$. Remembering that $U = U_{a'}(F)$ and $\overline{U} = U_{-a'}(F)$, we may find a very B_0-dominant (regular) element $\lambda \in X_*(A)$ such that $\pi^\lambda \overline{u}\pi^{-\lambda} \in I$. Then we see that $I\pi^\lambda s_a I$ meets $U\pi^\lambda I$, and thus that $t_\lambda \leq t_\lambda s_a$ in the Bruhat order determined by I. But this is impossible, since $\ell(t_\lambda s_a) = \ell(t_\lambda) - 1$ for regular B_0-dominant λ. Thus $J(0,1) = 0$.

Next we examine $J(0, s_a)$. We need to determine for which $u \in U_{s_a} = U$ we have $s_a u s_a \in UI$. Using the Iwahori decomposition $I = (U\cap I)\cdot(T\cap I)\cdot(\overline{U}\cap I)$, we see that $s_a u s_a \in UI$ iff $s_a u s_a \in \overline{U} \cap I = U_{-a+1}$. This happens iff $u \in U_{a+1}$. So $J(0,s_a) = \mathrm{meas}_{du}(U_{a+1}) = [U_a : U_{a+1}]^{-1}$ since $\mathrm{meas}_{du}(U_a) = 1$ by definition (recall $U\cap I = U \cap K$). So $J(0,s_a) = q_{a+1}^{-1} = (q_{a/2}q_a)^{-1}$.

Next we consider the case $k>0$. Suppose $w = s_a$. Then $J(k,s_a) \neq 0$ iff there exists $u \in U$ with $s_a u \pi^{ka^\vee} s_a \in UI$. We can then write $\overline{u}\pi^{-ka^\vee} = ui$, for some $\overline{u} \in \overline{U}$, $u \in U$, and $i \in I$. This is also $\pi^{-ka^\vee}\overline{u}' = ui$ for some $\overline{u}' \in \overline{U}$. We write ui in the form $u_+ \cdot t_0 \cdot u_-$ for unique $u_+ \in U$, $t_0 \in T\cap I$ and $u_- \in \overline{U} \cap I$. Since the factors in $U \cdot T \cdot \overline{U}$ are uniquely determined, we must have $\pi^{-ka^\vee} \in T \cap I$. But this contradicts $k > 0$. Hence $J(k,s_a) = 0$.

Finally consider $w=1$. For $u \in U$, we need to determine when $s_a u \pi^{ka^\vee} \in UI$. Suppose $u \neq 1$. Then there is a unique integer l such that $u \in U_{a+l} - U_{a+l+1} \subset U_{-a-l}\pi^{la^\vee}T_b s_a U_{-a-l}$. We may write $u = u'_{-a-l}\pi^{la^\vee}t_0 s_a u_{-a-l}$, where $t_0 \in T_b$. Then we have for each $u \in U_{a+l} - U_{a+l+1}$:

$$\begin{aligned} s_a u \pi^{ka^\vee} \in UI &\Leftrightarrow u_{a-l}\pi^{-la^\vee}t'_0 u_{-a-l}\pi^{ka^\vee} \in UI \\ &\Leftrightarrow u_{a-l}\pi^{-(l-k)a^\vee}t'_0 u'_{-a+2k-l} \in UI, \end{aligned}$$

where $u_{a-l} := s_a u'_{-a-l} s_a$, $t'_0 := s_a t_0 s_a$, and $u'_{-a+2k-l} := \pi^{-ka^\vee}u_{-a-l}\pi^{ka^\vee}$, the latter being in $U_{-a+2k-l}$. Using the decomposition of UI as above, we see that this holds if and only if $l = k$. (The "only if" if clear. To check the "if", note that we just need to see that $u'_{-a+2k-l} \in I$. But $l=k$ implies that this element belongs to $U_{-a+k} \subset I$, as desired.)

In particular, we see that
$$J(k,1) = \mathrm{meas}_{du}(U_{a+k} - U_{a+k+1}).$$
Now for each $k \geq 0$, we have
$$\mathrm{meas}_{du}(U_{a+k}) = q_{a/2}^{-[\frac{k+1}{2}]} q_a^{-k}.$$
Therefore,
$$\begin{aligned} J(k, s_a) &= q_{a/2}^{-[\frac{k+1}{2}]} q_a^{-k} - q_{a/2}^{-[\frac{k+2}{2}]} q_a^{-(k+1)} \\ &= q_{a/2}^{-k/2} q_a^{-k} \Big(q_{a/2}^{-[\frac{k+1}{2}]+\frac{k}{2}} - q_{a/2}^{-[\frac{k+2}{2}]+\frac{k}{2}} q_a^{-1} \Big) \\ &= \delta_B(\pi^{a^\vee})^{k/2} \begin{cases} 1 - q_{a/2}^{-1} q_a^{-1}, & \text{if } k \text{ is even} \\ q_{a/2}^{-1/2} - q_{a/2}^{-1/2} q_a^{-1}, & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$
Thus we get
$$I_{s_a}(v_1) = (q_{a/2} q_a)^{-1} v_{s_a} + \sum_{k=1}^{\infty} \delta_B(\pi^{a^\vee})^{k/2} \begin{cases} 1 - q_{a/2}^{-1} q_a^{-1}, & \text{if } k \text{ is even} \\ q_{a/2}^{-1/2} - q_{a/2}^{-1/2} q_a^{-1}, & \text{if } k \text{ is odd} \end{cases} \cdot v_{t_{ka^\vee}}.$$
That is,
$$I_{s_a}(v_1) = (q_{a/2} q_a)^{-1} v_{s_a} + \Big(\frac{(1 - q_{a/2}^{-1/2} q_a^{-1} \pi^{a^\vee})(1 + q_{a/2}^{-1/2} \pi^{a^\vee})}{1 - \pi^{2a^\vee}} - 1 \Big) \cdot v_1.$$
This completes the proof of the main theorem in the case of G semi-simple rank 1, thus also in general.

Corollary 7.2.1. *We have* $I_{s_a}(v_1 + v_{s_a}) = c_a \cdot (v_1 + v_{s_a})$.

Proof. Let $q(w) := [IwI : I]$. Use the main theorem combined with $v_{s_a} = v_1 T_{s_a}$, $q(s_a) = q_{a/2} q_a$, and $T_{s_a}^2 = (q(s_a) - 1) T_{s_a} + q(s_a) T_1$. $\square$

Corollary 7.2.2. *We have* $I_w(1_{A_\mathcal{O} UK}) = c_w \, 1_{A_\mathcal{O} UK}$.

8. Sketch of proof of Bernstein isomorphism for unramified groups

We may now define an embedding $R \hookrightarrow H$ by sending $\pi^\mu \in R$ to the element $\Theta_\mu \in H$ which is characterized by the identity

$$\pi^\mu v_1 = v_1 \Theta_\mu. \tag{8.0.1}$$

(Often we abuse notation and write π^μ instead of Θ_μ in the sequel.) Our aim is to show that R^W maps onto the center $Z(H)$ of H. We follow closely the argument for split groups given in [HKP].

For each $a \in \Sigma$, let d_a denote the denominator of c_a. If $q_{a/2} = 1$, then $d_a = 1 - \pi^{a^\vee}$. If $q_{a/2} \neq 1$, then $d_a = 1 - \pi^{2a^\vee}$. For each $w \in W$, define
$$d_w := \prod_{a \in \Sigma_w} d_a.$$
We define the intertwiners without denominators
$$J_w := d_w I_w.$$

We have the relation

$$J_w \circ \pi^\mu = \pi^{w\mu} \circ J_w. \tag{8.0.2}$$

For any reduced expression $w = s_1 \cdots s_r$, where each $s_i \in \{s_a \mid a \in \Delta\}$, we see that

$$J_w = J_{s_1} \cdots J_{s_r},$$

and hence J_w maps $\mathbf{M}$ to itself (since by Theorem 6.0.2 this holds for each J_{s_i}). Thus each J_w can be represented by an element of H. In the case of $w = s_a$, Theorem 6.0.2 gives us

$$J_{s_a} = q(s_a)^{-1} d_a T_{s_a} + d_a(c_a - 1). \tag{8.0.3}$$

Using this together with (8.0.2) in the case $w = s_a =: s$, we recover Bernstein's relation

$$T_s \pi^\mu - \pi^{s\mu} T_s = q(s_a)(c_a - 1)(\pi^{s\mu} - \pi^\mu). \tag{8.0.4}$$

Here we are writing π^μ in place of its image Θ_μ under our embedding $R \hookrightarrow H$.

An elementary computation (done by writing out $J_{s_a}^2 v_1$ and simplifying), yields the relation

$$J_{s_a}^2 = d_a c_a d_{-a} c_{-a} \in R. \tag{8.0.5}$$

Note that we can already see that R^W maps *into* $Z(H)$: if $r \in R^W$, then (8.0.2) shows that r commutes with J_{s_a}, and hence with $d_a T_{s_a}$ (use (8.0.3) and note that r clearly commutes with $d_a(c_a - 1) \in R$). But then since H is torsion-free as an R-module, we see that r commutes with each T_{s_a} as well, and so $r \in Z(H)$.

Using this remark, we define the normalized intertwiners. Let $L = \mathrm{Frac}(R)$, so that $L^W = \mathrm{Frac}(R^W)$. We consider the module $\mathbf{M}_{\mathrm{gen}} := L \otimes_R \mathbf{M} = L^W \otimes_{R^W} \mathbf{M}$, which is an (L, H_{gen})-module, where $H_{\mathrm{gen}} := L^W \otimes_{R^W} H$. Let K_w be the intertwiner on $\mathbf{M}_{\mathrm{gen}}$ defined by

$$K_w := \frac{1}{c_w} I_w = \frac{1}{d_w c_w} J_w. \tag{8.0.6}$$

By (8.0.5) we see that $K_{s_a}^2 = 1$, and we see from this and Lemma 5.1.1 that

$$K_{w_1 w_2} = K_{w_1} K_{w_2}$$

for *every* $w_1, w_2 \in W$. Thus, $w \mapsto K_w$ defines a group homomorphism $W \to H_{\mathrm{gen}}^\times$ and thus an algebra homomorphism from the twisted group algebra $L[W]$ to H_{gen}. Then exactly as in [HKP], Lemma 2.3.1, we conclude that R^W maps onto the center of H.

9. Descent of intertwiners from universal to specific unramified principal series

The results of this section are used to complete a technical lemma in [H], which was the main motivation for this note. The strength of the algebraic approach to intertwiners is apparent here: it works perfectly well for *arbitrary unramified characters*. In most other approaches, one first develops the theory of intertwiners for *regular* characters, and then proves by analytic continuation that they may be defined for certain other characters (depending on w). But then it seems more difficult to analyze when the resulting intertwiners are not identically zero. Below, we give a simple criterion which applies to the intertwiners without denominators. This is sufficient for the application in [H].

9.1. **Rough description of I_w on M.** Fix $w \in W$. Let J denote the set of coroots $a^\vee$ which are B-positive and wB-negative. Set $q(w) := [IwI : I]$ and note that

$$q(w_1 w_2) = q(w_1)q(w_2) \tag{9.1.1}$$

whenever $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$.

Lemma 9.1.1. *We have the following rough description of $I_w(v_1)$ as an element in $\mathbf{M}_{B,J}$:*

$$I_w(v_1) = q(w)^{-1} v_w + \sum_{w', a^\vee, k} a_{w', a^\vee, k} \; v_{t_{ka^\vee} w'}, \tag{9.1.2}$$

where $w' \in W$ satisfies $w' < w$ in the Bruhat order on W, the coroots $a^\vee$ range over the set J, the integers k are non-negative, and the scalars $a_{w', a^\vee, k}$ are complex numbers.

Proof. This follows easily by induction on $\ell(w)$, taking into account Theorem 6.0.2, the equation (9.1.1), and the Bernstein relation (8.0.4). □

In particular, we see that $J_w(v_1)(w) \neq 0$, so that J_w is never identically zero on $\mathbf{M}$.

9.2. **Criterion for nonvanishing of $\widetilde{J}_w$ on $i_B^G(\chi)$.** Let $i_B^G(\chi_{\text{univ}}^{-1})$ denote the induced representation whose elements consist of the locally-constant R-valued functions ϕ on G satisfying

$$\phi(aug) = \delta_B(a)^{1/2} \cdot a^{-1} \cdot \phi(g)$$

for $a \in A$, $u \in U$, and $g \in G$. The group G acts on $i_B^G(\chi_{\text{univ}}^{-1})$ by right translations.

There is a canonical H-equivariant isomorphism $\mathbf{M} = i_B^G(\chi_{\text{univ}}^{-1})^I$, which is given by associating to $\varphi \in \mathbf{M}$ the element $\phi \in i_B^G(\chi_{\text{univ}}^{-1})^I$ defined by

$$\phi(g) = \sum_{a \in A/A_{\mathcal{O}}} \delta_B^{-1/2}(a)\varphi(ag) \cdot a. \tag{9.2.1}$$

Now let χ be an unramified character of $T(F)$ (that is, trivial on $T_b = T(F) \cap T(L)_1$). It can be viewed as a homomorphism from $A/A_{\mathcal{O}} = X_*(A)$ to $\mathbb{C}^\times$; thus it determines a unique $\mathbb{C}$-algebra homomorphism $\chi : R \to \mathbb{C}$. Let $i_B^G(\chi)$ denote the (normalized) unramified principal series representation. Using χ to extend scalars, we have a canonical identification

$$\mathbb{C} \otimes_R \mathbf{M} = i_B^G(\chi^{-1})^I.$$

As in (9.2.1), we can make this explicit: we associate to $1 \otimes \varphi$ the element $\phi \in i_B^G(\chi^{-1})^I$ defined by the formula (for $g \in G$)

$$\phi(g) = \sum_{a \in A/A_{\mathcal{O}}} \delta_B^{-1/2}(a)\chi(a)\varphi(ag). \tag{9.2.2}$$

In the following proposition, χ and ξ are unramified characters of $T(F)$, and $w \in W$ is an element of the Weyl group such that ${}^w\chi = \xi$ (where ${}^w\chi(t) := \chi(w^{-1}tw)$). Note that the intertwiner without denominator $J_w : \mathbf{M} \to \mathbf{M}$ descends to an H-module homomorphism

$$1 \otimes J_w : \mathbb{C} \otimes_{R,\chi} \mathbf{M} \to \mathbb{C} \otimes_{R,\xi} \mathbf{M}. \tag{9.2.3}$$

Via (9.2.2), this determines the descended intertwiner $\widetilde{J}_w : i_B^G(\chi^{-1})^I \to i_B^G(\xi^{-1})^I$. Explicitly, we have for $\phi \in i_B^G(\chi^{-1})^I$ corresponding to $\varphi \in \mathbf{M}$

$$\widetilde{J}_w\phi(g) = \sum_{a \in A/A_{\mathcal{O}}} \delta_B^{-1/2}(a)\, {}^w\chi(a)\, J_w\varphi(ag). \tag{9.2.4}$$

Recall that d_w denotes the denominator of c_w. Thanks to the previous subsection, it is easy to understand when $1 \otimes J_w$ is identically zero: this happens if and only if $\xi(d_w) = 0$. Therefore we derive the following criterion for the nonvanishing of $\widetilde{J}_w$.

Proposition 9.2.1. *The descended intertwiner $\widetilde{J}_w : i_B^G(\chi^{-1})^I \to i_B^G(\xi^{-1})^I$ is non-zero if and only if $\xi(d_w) \neq 0$.*

The following shows that we can easily arrange for $\widetilde{J}_w$ to be non-zero.

Proposition 9.2.2. *If w is a* minimal *element (in the Bruhat order) such that ${}^w\chi = \xi$, then $\xi(d_w) \neq 0$.*

Proof. It is enough to show that for $a \in \Sigma_w$, we have $\xi(\pi^{a^\vee}) \neq 1$ in case $q_{a/2} = 1$, and $\xi(\pi^{2a^\vee}) \neq 1$ in case $q_{a/2} \neq 1$. Let us check only the second case (the first is similar, and easier). Assume that $\xi(\pi^{2a^\vee}) = 1$. Note that $q_{a/2} \neq 1$ implies that $a \in 2X^*(A)$ (in fact for the example of $G = \mathrm{SU}_3$ explained in subsection 3.2, we see that $a = 2a_1$, where a_1 is the unique nondivisible B-positive root). This implies that for each $\lambda \in X_*(A)$, the pairing $\langle a, \lambda\rangle$ is an *even integer*. It follows that ${}^{s_a}\xi = \xi$. Indeed, for each $\lambda \in X_*(A)$, we have

$$\begin{aligned} {}^{s_a}\xi(\pi^\lambda) &= \xi(\pi^{s_a\lambda}) \\ &= \xi(\pi^\lambda)\xi(\pi^{2a^\vee})^{-\langle a,\lambda\rangle/2} \\ &= \xi(\pi^\lambda). \end{aligned}$$

But now we have ${}^{s_a w}\chi = \xi$. This violates the minimality of w: since $a \in \Sigma_w$, we have $w^{-1}a$ is B-negative and so $s_a w < w$ in the Bruhat order on W. This completes the proof that $\xi(d_w) \neq 0$. □

We finish with another result which is used in [H]. For $\tau \in W$, consider the unique function $\phi_\tau \in i_B^G(\xi^{-1})^I$ which is supported on the set $B\tau I$ and satisfies $\phi_\tau(\tau) = 1$. The functions ϕ_τ ($\tau \in W$) form a $\mathbb{C}$-vector space basis for $i_B^G(\chi^{-1})^I$. Via (9.2.2), ϕ_τ corresponds to the function $1 \otimes \varphi_\tau \in \mathbb{C} \otimes_R \mathbf{M}$ where $\varphi_\tau := 1_{A_{\mathcal{O}}U\tau I} = v_\tau$.

Proposition 9.2.3. *For $\tau, \tau' \in W$, we have $\widetilde{J}_w(\phi_\tau)(\tau') \neq 0$ only if*

$$w^{-1}U_w\tau' \cap B\tau I \neq \emptyset.$$

Proof. This follows immediately from (9.2.4) and the integral formula defining $I_w(\varphi_\tau)(a\tau')$ for $a \in A$. □

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