INTERTWINERS FOR UNRAMIFIED GROUPS

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1. Introduction

In this note we generalize the algebraic approach to intertwiners given in [HKP], from split groups to unramified groups. Actually, it is perfectly clear that the methods of the current note extend to arbitrary non-split groups, but we have limited our discussion to the unramified case because some things become slightly more concrete in that situation. We give some of the standard applications as in [HKP] (such as Bernstein’s description of the center of the Iwahori-Hecke algebra). We also provide an ingredient (see section 9) needed for one particularly technical lemma in [H]. Section 9 was the main motivation for writing this note.

2. Preliminaries

2.1. Basic notation. Let $F$ denote a p-adic field. Let $\mathcal{O}$ denote the ring of integers in $F$, and $\pi \in \mathcal{O}$ a uniformizer. Let $q = p^r$ denote the cardinality of the residue field of $F$. Fix an algebraic closure $\overline{F}$ for $F$, and let $L$ denote the completion of the maximal unramified extension of $F$ inside $\overline{F}$. Let $\sigma \in \text{Aut}(L/F)$ denote the Frobenius automorphism of $L$ over $F$.

We let $G$ denote a connected reductive group which is defined and unramified over $F$. Sometimes we use the symbol $G$ to denote the group $G(F)$ of $F$-points.

Let $A$ denote a maximal $F$-split subtorus, and set $T := \text{Cent}_G(A)$, a maximal torus defined over $F$. Let $N = \text{Norm}_G(A)$. We use $W$ to denote the (relative) Weyl group $W := N(F)/T(F)$.

We consider the Bruhat-Tits building $\mathcal{B}(G)$ for $G(F)$. Fix once and for all an alcove $a$, which we can assume belongs to the apartment corresponding to $A$. Let $I$ denote the Iwahori subgroup of $G(F)$ corresponding to $a$. Fix a hyperspecial vertex $a_0$ in the closure of $a$, with corresponding hyperspecial maximal compact subgroup $K \supset I$, and designate it as the origin in the aforementioned apartment; this identifies the apartment with the vector space $V := X_*(A)\mathbb{R}$.

We can embed $X_*(A)$ into $A(F)$ in two natural ways. Our convention is to identify $\mu \in X_*(A)$ with $\pi^\mu := \mu(\pi) \in A(F)$.

2. Let $\mathfrak{B}(T)$ denote the set of Borel subgroups $B = TU$ which contain $T$ and are defined over $F$. The set $\mathfrak{B}(T)$ is a torsor for the finite Weyl group $W$ (for $w \in W$ and $B \in \mathfrak{B}(T)$, let $wB$ or $wBw^{-1}$). For each $B = TU \in \mathfrak{B}(T)$, define the Weyl chamber $C_U$ in $V$, and the notion of $B$-positive root, as follows. Let $T_b$ denote the unique maximal compact subgroup of $T(F)$. The chamber $C_U$ is the unique one with vertex $a_0$ such that $T_bU$ is the union of the fixers of all “quartiers” $x + C_U$ ($x \in V$) in the direction of $C_U$. Furthermore, a $B$-positive root is one that appears in $\text{Lie}(B)$. Equivalently, a root $\alpha$ is $B$-positive if and only if it takes positive values on the chamber $C_{\overline{\mathfrak{B}}}$, where $\overline{\mathfrak{B}} = TU$ is the unique element of $\mathfrak{B}(T)$ which is opposite to $B$. 

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The alcove $a$ belongs to a unique Weyl chamber having vertex $a_0$, which we may write in the form $C_{\mathcal{T}_0}$ for a unique Borel $B_0 = TU_0 \in \mathfrak{B}(T)$. Thus, the roots $\alpha \in \text{Lie}(B_0)$ are positive on $C_{\mathcal{T}_0}$, and a coweight $\lambda$ belonging to the closure of $C_{\mathcal{T}_0}$ is $B_0$-dominant.

Except in a few instances where $B$ denotes an arbitrary element of $\mathfrak{B}(T)$, the notation $B = TU$ will always mean the opposite Borel $\overline{B}_0$. Also, unless otherwise noted $C$ denotes $C_\mathcal{T} = C_{\mathcal{T}_0}$, the “dominant” Weyl chamber.

Note that by our conventions, the “reduction modulo $\pi$” of $I$ is $B$. More precisely, we have $B \cap I = B \cap K$.

2.2. Example of $\text{SL}_2$. Our conventions amount to the following for the group $\text{SL}_2$.

Let $A = T$ denote the diagonal torus. The base alcove $a$ is the unit interval $[0, 1]$ in the real line (which is identified with the apartment for $A$). The Iwahori subgroup $I$ fixing $a$ is the one fixing the homothety classes of the lattices $O \oplus O$ and $\pi O \oplus O$, that is,

$$I = \left[ \begin{array}{cc} O^\times & \pi O \\ O & O^\times \end{array} \right] \cap \text{SL}_2(F).$$

Furthermore, $K$ is the group $\text{SL}_2(O)$. Also, $B_0$ is the group of “upper triangular” matrices in $\text{SL}_2$, and $B$ is the group of “lower triangular” matrices.

2.3. Extended affine Weyl group. In Bruhat-Tits theory is defined a homomorphism $\nu : N(F) \to V$, which is normalized such that $\nu(\pi \mu) = -\mu$ (see [Tits]). Its kernel is $T_b$. Via $\nu$ the extended affine Weyl group $\widetilde{W} := N(F)/T_b$ can be viewed as a group of affine-linear transformations of $V$. It splits as a semi-direct product $\widetilde{W} \cong \Lambda \rtimes W$, where $\Lambda$ is the group of translations isomorphic via $\nu$ to $T(F)/T_b$. There is a natural inclusion of lattices $X_\ast(A) \hookrightarrow \Lambda$. In fact Lemma 2.3.1 below shows that $X_\ast(A) = \Lambda$ (for another proof of this, see [Bo], 9.5).

In [Ko97] Kottwitz defined a surjective homomorphism $\omega_G : G(L) \to X^\ast(Z(G)^{\Gamma_0})$ where $\Gamma_0 := \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ denotes the inertia group. The homomorphisms $\omega_G$ vary with $G$ in a functorial manner. Let $G(L)_1$ denote the kernel of $\omega_G$.

Since $G$ is unramified, $T$ necessarily splits over $L$, and thus the Kottwitz homomorphism takes the simpler form $\omega_T : T(L) \to X_\ast(T)$.

To see that $T$ splits over $L$, let $S \supseteq A$ be the $L$-split component of $T$. Note that $S$ is a maximal $L$-split torus in $G$, since any larger $L$-split torus $S' \supseteq S$ contains $A$ and hence belongs to $T$, hence to $S$. Since $G$ splits over $L$, there exists a maximal torus $T' \subset G$ which is defined and split over $L$. By a standard result, $S$ and $T'$ are conjugate by an element of $G(L)$, hence $\dim(S) = \dim(T') = \dim(T)$, and thus $S = T$.

Lemma 2.3.1. For $G$ unramified over $F$, we have

(i) $T_b = T(F) \cap T(L)_1$;

(ii) $T(F)/(T(F) \cap T(L)_1) \cong X_\ast(A)$ via $\omega_T$.

In particular, the inclusion $X_\ast(A) \hookrightarrow \Lambda$ is an isomorphism, and $\widetilde{W} \cong X_\ast(A) \rtimes W$. 
Proof. There is an inclusion $i : X_*(T) \hookrightarrow X_*(T)_\mathbb{R}$. By [Ko97], (7.4.5), there is a commutative diagram with exact rows

$$
\begin{array}{cccc}
0 & \longrightarrow & T(L)_1 & \longrightarrow & T(L) & \xrightarrow{i \omega_F} & X_*(T)_\mathbb{R} \\
0 & \longrightarrow & T_b & \longrightarrow & T(F) & \xrightarrow{\omega_F} & X_*(A)_\mathbb{R} \\
\end{array}
$$

where the vertical arrows are injective. This shows that $i \circ \omega_T$ and $-\nu$ agree as maps $T(F) \to V$, proving part (i) and the equality of the images in $V$ of $-\nu$ and $\omega_T|_{T(F)}$. Furthermore, consider the exact sequence

$$
0 \longrightarrow T(L)_1 \cap T(F) \longrightarrow T(F) \xrightarrow{\omega_T} X_*(A) \longrightarrow 0
$$

resulting from taking $\sigma$-invariants of

$$
0 \longrightarrow T(L)_1 \longrightarrow T(L) \xrightarrow{\omega_T} X_*(T) \longrightarrow 0,
$$

and using the fact that $H^1((\sigma), T(L)_1) = 0$ (cf. [Ko97], (7.6.1)). We see that $\omega_T|_{T(F)}$ has image $X_*(A)$, i.e. $X_*(A) = \Lambda$. \hfill \Box

2.4. Bruhat orders and length function. When we speak of the Bruhat order on $W$ or on $\tilde{W}$, we will always mean the Bruhat order defined relative to the reflections through the walls of $C$ resp. $a$. Also, the length function $\ell$ on $\tilde{W}$ is defined in terms of the reflections through the walls of $a$.

2.5. On Bruhat-Tits and Iwasawa decompositions. Over $L$, our group is split and we have the usual Bruhat-Tits and Iwasawa decompositions

$$
G(L) = \bigsqcup_{w \in \tilde{W}(L)} I(L)wI(L) = \bigsqcup_{w \in \tilde{W}(L)} U(L)wI(L).
$$

Taking fixed points under $\sigma$, these yield the corresponding decompositions over $F$:

$$(2.5.1) \quad G(F) = \bigsqcup_{w \in \tilde{W}} IwI = \bigsqcup_{w \in \tilde{W}} UwI.
$$

3. Affine roots and root subgroups

Let $\Phi = \Phi(G, A)$ denote the set of relative roots for $G$ and the $F$-torus $A$. Let $\Phi_{aff}$ denote the Bruhat-Tits affine roots (as defined in [Tits], §1.4-1.6). Let $\Sigma$ denote the canonical finite reduced root system associated to $\Phi_{aff}$ and the vertex $a_0$ (the pair $(\Phi, \Sigma)$ forms an “echelonnage” in the sense of [BT1], §1.4). Each root $a \in \Sigma$ is a positive scalar multiple of a unique non-divisible root in $\Phi$. Let $\Sigma_{aff}$ denote the affine root system

$$
\Sigma_{aff} = \{a + k \mid a \in \Sigma, \ k \in \mathbb{Z}\}.
$$

The basic property defining $\Sigma$ is that the hyperplanes in $X_*(A)_\mathbb{R}$ defined by $\alpha = 0$ for $\alpha \in \Phi_{aff}$ are precisely the hyperplanes $\alpha = 0$, for $\alpha \in \Sigma_{aff}$. In fact, for each affine root $a' + k' \in \Phi_{aff}$ ($a' \in \Phi$, $k' \in \mathbb{R}$), there is a unique $a + k \in \Sigma_{aff}$ ($a \in \Sigma$, $k \in \mathbb{Z}$), such that $a + k$ is a positive real multiple of $a' + k'$.

Letting $\Phi_{0}^{+}$ denote the $B_0$-positive roots in $\Phi$, we get a corresponding set of positive roots $\Sigma_{0}^{+} \subset \Sigma$. Let $\Delta_0$ denote the set of simple roots in $\Sigma_{0}^{+}$. Similarly, let $\Delta$ denote the set of simple roots in the set of $B$-positive roots $\Sigma^{+}$. 

To each non-divisible root \( a' \in \Phi \) (corresponding to \( a \in \Sigma \)) there is a unipotent subgroup \( U_a'(F) \) equipped with a filtration which is used to define the set \( \Phi_{\text{aff}} \) (see [Tits], §1.4). For \( \alpha := a + k \in \Sigma_{\text{aff}} \), there is a subgroup \( U_\alpha \subset U_a'(F) \). The family \( \{U_{a+k}\} \) possesses the following properties:

1. \( U_a'(F) \) is the union of the subgroups \( U_{a+k} \), for \( k \in \mathbb{Z} \);
2. \( U_{\alpha+1} \subset U_{\alpha} \);
3. \( U_{-\alpha} - U_{-\alpha+1} \subset U_\alpha \nu^{-1}(s_\alpha)U_\alpha = U_\alpha t_\alpha s_\alpha T_\alpha U_\alpha \), if \( \alpha = a + k \in \Sigma_{\text{aff}} \).

Here \( \nu : N(F) \to \tilde{W} \cong \Lambda \rtimes W \) is the homomorphism of Bruhat-Tits (cf. [Tits], §1). (In particular, \( \nu(\pi^\lambda) = -\lambda \in X_*(A) \cong \Lambda \).) Moreover, for \( \alpha = a + k \in \Sigma_{\text{aff}} \), the element \( t_\alpha \in A(F) \) is defined in the following subsection (for \( \alpha = a \) it is \( \pi^{a'} \); see below and compare with the element \( a_\alpha \) of [Cas80]).

For the proof of the key property (3), we refer to [Mac], Lemma (2.6.6) and [BT1], Lemme (6.3.3). See also the examples in subsection 3.2 below.

### 3.1. Definition of \( t_\alpha \) and in particular of an element \( a^{\vee} \in X_*(A) \).

Via the Bruhat-Tits homomorphism \( \nu \), the group \( T(F)/T_b \) is identified with a group of translations \( \Lambda \) acting on the vector space \( X_*(A)_{\mathbb{R}} \). In fact the elements of \( \Lambda \) belong to the lattice of coweights \( P^\vee(\Sigma) \) associated to the root system \( \Sigma \), and as we saw in Lemma 2.3.1 \( \Lambda \) also coincides in the unramified case with the group of translations by elements of \( X_*(A) \). For \( \alpha \in \Sigma_{\text{aff}} \), the element

\[
\begin{align*}
\ s_{\alpha-1} \circ s_\alpha & \\
\end{align*}
\]

is a translation by a coroot for \( \Sigma \), hence it belongs to the coroot lattice \( Q^\vee(\Sigma) \subset P^\vee(\Sigma) \). The image of the Bruhat-Tits homomorphism contains every affine reflection associated to \( \Sigma \) (and consequently all translations by coroots). Therefore \( s_{\alpha-1} \circ s_\alpha \) can be lifted to a unique coset \( t_\alpha \in T(F)/T_b \) such that \( -\nu(t_\alpha) = s_{\alpha-1} \circ s_\alpha \).

Thus \( t_\alpha \) can be viewed as an element in \( T(F) \) uniquely determined up to \( T_b \).

By Lemma 2.3.1, in the unramified case we may also view \( t_\alpha \) as an element in \( A/A_\Sigma \). Therefore, for \( a \in \Sigma \) we may write \( t_\alpha = \pi^{a^{\vee}} \) for some unique cocharacter \( a^{\vee} \in X_*(A) \). In fact the notation is not accidental: the element \( a^{\vee} \) is precisely the coroot \( a^{\vee} \in Q^\vee(\Sigma) \) corresponding to the root \( a \in \Sigma \).

In the sequel, we will sometimes denote the element \( t_\alpha \) by \( \pi^{a^{\vee}} \).

For every \( \alpha := a + k \in \Sigma_{\text{aff}} \) and \( l \in \mathbb{Z} \), we have

\[
\begin{align*}
\ t_\alpha U_{\alpha+l} & = U_{\alpha+l+2}.
\end{align*}
\]

Note also that for any \( w \in W \), we have

\[
\begin{align*}
\ w t_\alpha w^{-1} & = t_{w(a)}.
\end{align*}
\]

where \( w(a) \) is defined by \( w(a + k) := a \circ w^{-1} + k \).

For each \( \alpha \in \Sigma_{\text{aff}} \), let

\[
\begin{align*}
\ q_\alpha & = [U_{\alpha-1} : U_\alpha].
\end{align*}
\]

It is clear that \( q_\alpha \) is always the same as \( q_{\alpha+2} \), but it is not necessarily the same as \( q_{\alpha+1} \). Macdonald [Mac] defines a root system \( \Sigma_1 \) with \( \Sigma = \Sigma_1 \subseteq \Sigma \cup \frac{1}{2} \Sigma \), where for \( a \in \Sigma \), the element \( a/2 \) lies in \( \Sigma_1 \) if and only if \( q_{\alpha+1} \neq q_\alpha \). For every \( a \in \Sigma \), he defines \( q_{a/2} = q_{\alpha+1}/q_\alpha \). Then:

1. For \( a \in \Sigma \), we have \( [U_{\alpha+1} : U_{\alpha+m+1}] = \left[ q_{a/2} \right]^{m} q_\alpha \).
(b) For $a \in \Delta$, we have $[Is_a I : I] = q_{a/2}a$ ([Mac] 2.7.4, 3.1.6);
(c) If $G$ has semi-simple rank 1, and $a \in \Delta$, then
$$
\delta_B(t_a) = [U_a : t_a U_a t_a]^{-1} = q_{a/2}^{-2}.
$$

3.2. Examples of the constructions above.

Example: SL$_2$. If $a$ is the unique $B_0$-positive root $a = e_1 - e_2$, and $\alpha = -a + 1$ (the simple affine root), then $s_{\alpha - 1} \circ s_a$ is the translation by $-a\check{\gamma} = (-a)\check{\gamma}$, where $a\check{\gamma}$ is the coroot associated to $a$, and so $t_{-a+1} = \pi(-a)\check{\gamma}$, according to our conventions. More generally, for any $a \in \Sigma$, we have $t_{a+k} = \pi^a\gamma$, for any $k \in \mathbb{Z}$. The identity (3) above for $\alpha = a + k$ ($a \in \Delta_0$) corresponds to the following matrix identity: given $u \in \mathcal{O}_x^\times$ and any $k \in \mathbb{Z}$, we have

$$
\begin{bmatrix}
1 & 0 \\
\pi^{-k}u & 1
\end{bmatrix} =
\begin{bmatrix}
1 & \pi^k a \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
\pi^k a & 0 \\
0 & \pi^{-k}a^{-1}
\end{bmatrix}
\begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
\pi^k b & 0 \\
0 & 1
\end{bmatrix},
$$

by taking $a = a = b = u^{-1}$.

Example: Quasi-split SU$_3$ for an unramified quadratic extension $E/F$. Let $a_1 \in \Phi_0^+$ be the unique non-divisible $B_0$-positive root, so that $2a_1$ is also a $B_0$-positive root. Let $a \in \Delta_0$ be the corresponding positive root of $\Sigma$. Then since the simple affine roots are $\{a_1, -2a_1 + 1\}$, we see that in fact $a = 2a_1$ and that $s_{a_1} \circ s_a = t_{(2a_1)^\vee}$. So, in this case $a\check{\gamma} = (2a_1)^\vee = a_1^\vee/2$. Concretely, $a\check{\gamma}$ is the cocharacter in $X_*(A)$ given by the formula for $x \in F^\times$

$$
a\check{\gamma}(x) =
\begin{bmatrix}
x & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & x^{-1}
\end{bmatrix},
$$

and $a_1^\vee(x) = a^\vee(x^2)$. Furthermore, if the cardinality of $\mathcal{O}_F/\pi \mathcal{O}_F$ is $q$, then $q_a = q$, $q_{a+1} = q^3$, and $q_{a/2} = q^2$. For all this, see [Tits], §1.15.

Let $x \mapsto \tilde{x}$ denote the generator of $\text{Gal}(E/F)$. Then given an integer $k$, a unit $u \in \mathcal{O}_E^\times$, and an element $c \in E$ such that $-c\tilde{c} = \pi^{-k}(u + \tilde{u})$, the relation (3) for $\alpha = a + k$ ($a \in \Delta_0$) corresponds to the matrix identity:

$$
\begin{bmatrix}
1 & 0 & 0 \\
\pi^{-k}u & 1 & 0 \\
0 & 1 & -\tilde{a}
\end{bmatrix} =
\begin{bmatrix}
1 & a & \pi^k w \\
0 & 1 & -\tilde{u} \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\pi^k \alpha & 0 & 0 \\
0 & \beta & 0 \\
0 & 0 & \pi^k \alpha^{-1}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
0 & -1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & b & \pi^k w \\
0 & 1 & -\tilde{b} \\
0 & 0 & 1
\end{bmatrix},
$$

where $\alpha = \tilde{u}^{-1}$, $\beta = \tilde{u}^{-1}$, $v = w = u^{-1}$, $a = -\pi^k \check{c} \tilde{u}^{-1}$, and $b = -\pi^k \check{c} u^{-1}$.

4. Hecke algebras and the universal model for unramified principal series

4.1. Hecke algebras. Consider the Iwahori-Hecke algebra $H := C_c(G/I)$, where convolution is defined using the Haar measure $dg$ which gives $I$ measure 1. Consider also the spherical Hecke algebra $H_K := C_c(K\backslash G/K)$, and the finite Hecke algebra $H_f := C_c(I \backslash K/I)$. As a $\mathbb{C}$-vector space $H$ has a basis given by the characteristic functions $T_w = 1_{twI}$, where $w \in \tilde{W}$.

The Iwahori-Hecke algebra of the torus $A$ is the ring $R := C_c(A(F)/A_{\mathbb{Q}})$, where convolution is defined using the Haar measure $da$ on $A(F)$ which gives $A_{\mathbb{Q}}$ measure 1. By using the valuation map $\nu$ the ring $R$ is identified with the group algebra $\mathbb{C}[X_*(A)]$. 
Now for any $B = TU \in \mathcal{B}(T)$, and any $t \in T(F)$, we denote by $\delta_B(t)$ the normalized absolute value of the determinant of the adjoint action of $t$ on $\operatorname{Lie}(U)$.

4.2. The universal model $M$. Write $B = TU$ as before. Let $M = M_B$ be defined as the $(R, H)$-bimodule

$$M = C_c(A_0 U \backslash G/I) = C_c(T_B U \backslash G/I).$$

The subscript “c” means that we consider functions supported on only finitely many double cosets. As a complex vector space, $M$ has a basis consisting of the functions $v_x := 1_{A_0 U x I} (x \in \widetilde{W})$.

It is clear that $H$ acts on the right on $M$ by right convolutions. One proves as in [HKP] Lemma 1.6.1 that $M$ is free of rank 1 as an $H$-module, with canonical generator $v_1$ (use (2.5.1)). Furthermore, $R$ acts on the left on $M$ by normalized left convolutions. More precisely, letting $f \in R$, we define the left action of $f$ on $v_x$ by the integral

$$f \cdot v_x(g) = \int_A \delta_B^{1/2}(a)f(a)v_x(a^{-1}g) \, da.$$ 

In other words, if $\lambda \in X_*(A)$ and if $\pi^\lambda$ is regarded as both an element in $A/A_0$ and as the characteristic function on $A/A_0$ for the subset $\pi^\lambda$, then

$$(4.2.1) \quad \pi^\lambda \cdot v_x = \delta_B^{1/2}(\pi^\lambda)v_{t_\lambda x},$$

where $t_\lambda$ is the translation element of $\widetilde{W}$ corresponding to $\lambda \in X_*(A)$. Clearly this applies in particular to the element $\lambda = a^\vee$ we attached to $a \in \Sigma$.

5. Intertwiners for nonstandard models

5.1. Definition and basic properties. In this section we follow closely the construction of intertwiners given in [HKP], §1.10 (the difference being that [HKP] only treated the intertwiners for standard models attached to split groups). As in loc. cit., we let $J$ be a set of coroots $a^\vee \in Q^\vee(\Sigma) \subset X_*(A)$ which belong to some positive subsystem of coroots. Denote by $\mathbb{C}[J]$ the $\mathbb{C}$-subalgebra of $R$ generated by $J$ and by $\mathbb{C} \widetilde{[J]}$ the completion of $\mathbb{C}[J]$ with respect to the (maximal) ideal generated by $J$. Let $R_J$ denote the $R$-algebra $\mathbb{C} \widetilde{[J]} \otimes_{\mathbb{C}[J]} R$, a completion of $R$ that can be viewed as the convolution algebra of complex valued functions on $X_*(A)$ supported on a finite union of sets of the form $\lambda + C_J$, where $C_J$ is the submonoid of $X_*(A)$ consisting of all non-negative integral linear combinations of elements in $J$.

Given $B = TU \in \mathcal{B}(T)$ and $J$ as above, we denote by $M_{B,J}$ the module $R_J \otimes_R C_c(A_0 U \backslash G/I)$, which can be viewed as the set of functions $f$ on $A_0 U \backslash G/I$ satisfying the following support condition: there exists a finite union $S$ of sets of the form $\lambda + C_J$ such that the support of $f$ is contained in the union of the sets $A_0 U \pi^\nu K$ for $\nu \in S$. Clearly $M_{B,J}$ is a left $R_J$-module and a right $H$-module. We will often write $M_J$ in place of $M_{B,J}$ when $B$ is understood.

Now let $w$ denote a fixed element of $W$ as well as its lift to an element of $N(F) \cap K$. Let $J$ denote the set of coroots $a^\vee \in Q^\vee(\Sigma)$ which are $B$-positive and $wB$-negative. Then as in loc. cit., we define an intertwiner

$$I_w : M_{B,w^{-1}J} \rightarrow M_{B,J}.$$
by the integral

\[ I_w(\varphi)(g) = \int_{U_w} \varphi(w^{-1}ug) \, du, \]

where \( U_w \) denotes \( U \cap wUw^{-1} \). The Haar measure \( du \) on \( U_w \) is normalized such that \( U_w \cap K \) has measure 1. Here we view the elements \( \varphi \) as functions (as above). As in loc. cit. Lemma 1.10.1, one proves the convergence of the integral defining \( I_w(\varphi)(g) \), and the fact that \( I_w(\varphi) \in M_J \). (The proof of the latter is very similar to the argument in section 5.2 below.)

We remark that in defining \( I_w \), we may replace the set \( J \) with any larger set of coroots that is still contained in a positive subsystem, for example the set of all \( B \)-positive coroots.

Moreover, if \( w = w_1w_2 \) where \( \ell(w) = \ell(w_1) + \ell(w_2) \) and if \( J \) is chosen as above, then the composition \( I_{w_1} \circ I_{w_2} \) is defined. The following properties are immediate.

**Lemma 5.1.1.** We have

1. \( I_w \circ \pi^\mu = \pi^{w\mu} \circ I_w \) for each \( \mu \in X_+(A) \);
2. \( I_{w_1w_2} = I_{w_1} \circ I_{w_2} \) if \( \ell(w_1w_2) = \ell(w_1) + \ell(w_2) \);
3. \( I_w \) is a right \( H \)-module homomorphism.

Now suppose that \( B_1 = TU_1 \) and \( B_2 = TU_2 \) are two elements of the set \( \mathfrak{B}(T) \), and that \( B_2 = vB_1 \) for \( v \in W \). Suppose we have chosen the sets \( J_1 \) and \( J_2 \) such that \( J_2 = vJ_1 \). Given a function \( \varphi \) on \( G \), define the function \( L(v)\varphi \) by \( L(v)\varphi(g) := \varphi(v^{-1}g) \). The following diagram commutes:

\[ \begin{array}{ccc}
M_{B_2,w^{-1}J_2} & \xrightarrow{L(v^{-1})} & M_{B_1,w^{-1}w^{-1}J_2} \\
I_w \downarrow & & \downarrow I_{w^{-1}w} \\
M_{B_2,J_2} & \xrightarrow{L(v)} & M_{B_1,J_1}.
\end{array} \]

We also have the following relation for any \( v, w \in W \) and any \( B \in \mathfrak{B}(T) \):

\[ L(v)v^B_w = v^B_{vw} \]

Because of (5.1.2), there is no loss in generality in studying only the standard intertwiners, i.e. those for which \( U = U_0 \). That is what we will do, and so from now on all statements about intertwiners refer to the standard ones. We leave it to the reader to derive the analogous statements about the nonstandard intertwiners using (5.1.2).

For this standard situation \( B = \overline{B}_0 \), we have the following three useful equalities. They are proved the same way as in [HKP] (using the Iwahori factorization and along the way \( I \cap B = K \cap B \) and \( I \cap \overline{B} \subset wIw^{-1} \)).

\[ v_1T_w = v_w, \text{ for every } w \in W, \]
\[ v_wT_w = v_\mu w, \text{ for every } w \in W \text{ and } \mu \in X_+(A), \]
\[ v_1T_{\mu w} = v_\mu w, \text{ for } \mu \in X_+(A) \text{ } B\text{-dominant.} \]

### 5.2. On the support of \( I_w(v_1) \)

In this subsection we give some necessary conditions for an element \( \pi^\lambda v \in \overline{W} \) to be in the support of \( I_w(v_1) \). First we define a relation \( x \to y \) on \( W \) which is reflexive and transitive (but not anti-symmetric), and which depends on our choice of Weyl chamber \( \mathcal{C} \). Given \( x \in W \), let \( S(x) \) denote the
set of simple reflections in the set \( \{ s_a \mid a \in \Delta \} \) which appear in some (equivalently, all) reduced expressions for \( x \). Then we write \( x \to y \) if and only if \( S(x) \subseteq S(y) \).

Secondly, we recall the definition of the retraction functions \( r_B \) (for any \( B \in \mathfrak{B}(T) \)): if \( g = u \pi^\mu v \) in the Iwasawa decomposition \( G = UAK \), then we set \( r_B(g) := \mu \). Note that the Iwasawa decomposition which a priori takes the form \( G = UTK \) agrees with the one we used, since \( A(F)/A_O = T(F)/T_b \) (Lemma 2.3.1) and \( T_b \subset K \).

The basic property of the family of retractions \( \{ r_B \}_{B \in \mathfrak{B}(T)} \) is that for any \( g \in G \) and \( B_1, B_2 \in \mathfrak{B}(T) \), the difference \( r_{B_1}(g) - r_{B_2}(g) \) is a sum of coroots \( \alpha^\vee \in Q^\vee(\Sigma) \) which are \( B_1 \)-positive and \( B_2 \)-negative.

One needs to check this only for adjacent Borels in \( \mathfrak{B}(T) \). In the split case, a simple computation in \( \text{SL}_2 \) does the job\(^1\).

Recall that we are currently assuming \( B = \overline{B}_0 \), so that \( J \) can be taken as the set of coroots \( \alpha^\vee \) such that \( \alpha^\vee \) is \( B \)-positive (i.e. \( B_0 \)-negative) and \( wB \)-negative (i.e. \( wB_0 \)-positive).

**Lemma 5.2.1.** If \( I_{w}(v_1)(\pi^\lambda v) \neq 0 \), then \( \lambda \) is a sum of elements of \( J \), and \( v \to w \).

In particular, if \( a \in \Delta \) and \( I_{s_a}(v_1)(\pi^\lambda v) \neq 0 \), then \( \lambda = ka^\vee \) for some \( k \in \mathbb{Z}_{\geq 0} \), and \( v \in \{ 1, s_a \} \).

**Proof.** The nonvanishing of \( I_{w}(v_1)(\pi^\mu v) \) implies that we may write \( g := u \pi^\mu v = u'w_i \) for some \( u \in U_w, u' \in wU \), and \( i \in I \). It follows that \( \mu = r_B(g) - r_wB(g) \) is a sum of coroots which are \( B \)-positive and \( wB \)-negative.

Next, let \( M \supset T \) denote the Levi subgroup corresponding to the set of simple positive roots \( a \in \Delta \) such that \( s_a \in S(w) \). Note that this is a Levi subgroup of a parabolic subgroup which contains \( B \), and that \( U_w \subset M \).

We see that \( v^{-1} = w\pi^\mu \in Bv^{-1}wJ \). Since \( w\pi^\mu \in M \), then using the Iwasawa decomposition for that group we see that \( w\pi^\mu \in (B \cap M)w_M(\ast I \cap M) \), for some \( w_M \) in the Weyl group for \( M \). Comparing the two Iwasawa decompositions we see \( v^{-1} = w_M \in M \). But since the Weyl group of \( M \) is generated by the simple reflections \( s_a \in S(w) \), we see finally that \( v \to w \).

\[ \square \]

6. **FORMULA FOR \( I_{s_a}(v_1) \)**

Here we again fix \( B = TU \in \mathfrak{B}(T) \) to be the opposite Borel \( \overline{B}_0 \). Let \( J \) denote the set of \( B \)-positive coroots \( \alpha^\vee \in Q^\vee(\Sigma) \).

For \( a \in \Sigma^+ \), we define an element \( c_a \in R_J \) by

\[
 c_a = \frac{(1 - q_a^{-1/2} \pi^{-1/2} \alpha^\vee)(1 + q_a^{-1/2} \pi^a^\vee)}{1 - \pi^2 \alpha^\vee}.
\]

(In a similar way we may define \( c_a \) for any \( a \in \Sigma \), but we have to choose \( J \) appropriately to ensure that \( c_a \in R_J \).) Also, for \( w \in W \), define

\[
 c_w := \prod_{a \in \Sigma_w} c_a,
\]

where \( \Sigma_w = \{ a \in \Sigma \mid a \text{ is } B \text{-positive and } w^{-1}a \text{ is } B \text{-negative} \} \).

**Theorem 6.0.2.** Let \( a \in \Delta \). We have the following equality in \( M_{B,J} \):

\[
 I_{s_a}(v_1) = (q_{a}/2q_a)^{-1}v_{s_a} + (c_a - 1) \cdot v_1
\]

\(^1\)The same proof in the unramified case works, and in the end requires also a simple computation in the group \( \text{SU}_3 \).
Note that this implies that \((1 - \pi^{2a'})I_{s_a} \) takes \(M\) into itself.

7. Proof of Theorem 6.0.2

The proof below is the result of combining the approach of \([HKP]\) with some ideas of Casselman \([\text{Cas}80]\).

7.1. Reduction to semi-simple rank 1. The calculation of \(I_{s_a}(v_1)\) immediately reduces from the group \(G\) to the Levi subgroup corresponding to \(a \in \Delta\), by Lemma 5.2.1.

7.2. Proof in the case of semi-simple rank 1. Now we assume \(G\) has semi-simple rank 1. Let \(a'\) denote the unique non-divisible \(B\)-positive (thus \(B_0\)-negative) root in \(\Phi\), and let \(a \in \Delta\) be the corresponding simple \(B\)-positive root in \(\Sigma\).

By Lemma 5.2.1, we can write \(I_{s_a}(v_1)\) as a sum

\[
I_{s_a}(v_1) = \sum_{k,w} J(k, w) v_{k,a,w},
\]

where \(w \in \{1, s_a\}\) and \(k \geq 0\). We need to find all the coefficients \(J(k, w)\).

First consider the case where \(k = 0\). Then \(J(0, 1) \neq 0\) implies that we can solve the equation \(\overline{u}s_a = u_i\), where \(\overline{u} \in \overline{U}\), \(u \in U\), and \(i \in I\). Remembering that \(U = U_a'(F)\) and \(\overline{U} = U_{-a'}(F)\), we may find a very \(B_0\)-dominant (regular) element \(\lambda \in X_a(\mathbb{A})\) such that \(\pi^\lambda \pi^{a'} \in I\). Then we see that \(I\pi^\lambda s_a I\) meets \(U\pi^\lambda I\), and thus that \(t_\lambda \leq t_a s_a\) in the Bruhat order determined by \(I\). But this is impossible, since \(\ell(t_a s_a) = \ell(t_\lambda) - 1\) for regular \(B_0\)-dominant \(\lambda\). Thus \(J(0, 1) = 0\).

Next we examine \(J(0, s_a)\). We need to determine for which \(u \in U_{s_a} = U\) we have \(s_a u s_a \in UI\). Using the Iwahori decomposition \(I = (U \cap I) \cdot (T \cap I) \cdot (\overline{U} \cap I)\), we see that \(s_a u s_a \in UI\) iff \(s_a u s_a \in \overline{U} \cap I = U_{-a+1}\). This happens iff \(u \in U_{a+1}\). So \(J(0, s_a) = \text{meas}_{s_a}(U_{a+1}) = [U_a : U_{a+1}]^{-1}\) since \(\text{meas}_{s_a}(U_a) = 1\) by definition (recall \(U \cap I = U \cap K\)). So \(J(0, s_a) = \left(\frac{q_a}{q_a+1}\right)\).

Next we consider the case \(k > 0\). Suppose \(w = s_a\). Then \(J(k, s_a) \neq 0\) iff there exists \(u \in U\) with \(s_a u \pi^{ka'} s_a \in UI\). We can then write \(\pi^{ka'} = u_i\), for some \(\pi \in \overline{U}\), \(u \in U\), and \(i \in I\). This is also \(\pi^{ka'} = u_i\), for some \(\pi' \in \overline{U}\). We write \(u_i\) in the form \(u_+ \cdot t_0 \cdot u_-\) for unique \(u_+ \in U\), \(t_0 \in T \cap I\) and \(u_- \in \overline{U} \cap I\). Since the factors in \(U \cdot T \cdot \overline{U}\) are uniquely determined, we must have \(\pi^{ka'} \in T \cap I\). But this contradicts \(k > 0\). Hence \(J(k, s_a) = 0\).

Finally consider \(w = 1\). For \(u \in U\), we need to determine when \(s_a u \pi^{ka'} \in UI\). Suppose \(u \neq 1\). Then there is a unique integer \(l\) such that \(u \in U_{a+l} \cap U_{a+l+1} \subset U_{-a-\pi^{la'}} U_{-a-\pi^{la'}} T_b s_a U_{-a-l}\). We may write \(u = \pi^{l-a} u_1\pi^{la'} t_0 s_{a-l} u_{-a-l}\), where \(t_0 \in T_b\). Then we have for each \(u \in U_{a+l} \cap U_{a+l+1}\):

\[
s_a u \pi^{ka'} \in UI \iff u_{-a} \pi^{-la'} t_0 u_{-a-l} \pi^{ka'} \in UI \iff u_{-a-l} \pi^{-(l-k)a'} t_0 u_{-a+2k-l} \in UI,
\]

where \(u_{-a-l} := s_a u_{-a-l} s_a\), \(t_0' := s_a t_0 s_a\), and \(u_{-a+2k-l} := \pi^{ka'} u_{-a-l} \pi^{ka'}\), the latter being in \(U_{-a+2k-l}\). Using the decomposition of \(UI\) as above, we see that this holds if and only if \(l = k\). (The “only if” if clear. To check the “if”, note that we just need to see that \(u_{-a+2k-l} \in I\). But \(k = l\) implies that this element belongs to \(U_{-a+k} \subset I\), as desired.)
In particular, we see that
\[ J(k, 1) = \text{meas}_{du}(U_{a+k} - U_{a+k+1}). \]
Now for each \( k \geq 0 \), we have
\[ \text{meas}_{du}(U_{a+k}) = q_a^{-k} q_a^{-k}. \]
Therefore,
\[ J(k, s_a) = q_a^{-k} q_a^{-k} - q_a^{-k} q_a^{-k+1} \]
\[ = q_a^{-k} q_a^{-k} \left( q_a^{-k+1} + \frac{1}{2} - q_a^{-k} + \frac{1}{2} q_a^{-1} \right) \]
\[ = \delta_B(\pi^\odot)^{k/2} \left\{ \begin{array}{ll} 1 - q_a^{-1/2} q_a^{-1}, & \text{if } k \text{ is even} \\ q_a^{-1/2} - q_a^{-1/2} q_a^{-1}, & \text{if } k \text{ is odd} \end{array} \right. \cdot v_{ tk^\odot}. \]
Thus we get
\[ I_{s_a}(v_1) = (q_a / 2q_a)^{-1} v_{s_a} + \sum_{k=1}^{\infty} \delta_B(\pi^\odot)^{k/2} \left\{ \begin{array}{ll} 1 - q_a^{-1/2} q_a^{-1}, & \text{if } k \text{ is even} \\ q_a^{-1/2} - q_a^{-1/2} q_a^{-1}, & \text{if } k \text{ is odd} \end{array} \right. \cdot v_{ tk^\odot}. \]
That is,
\[ I_{s_a}(v_1) = (q_a / 2q_a)^{-1} v_{s_a} + \left( \frac{q_a^{-1/2} q_a^{-1} \pi^\odot (1 + q_a^{-1/2} \pi^\odot)}{1 - \pi^{2\odot}} - 1 \right) \cdot v_1. \]
This completes the proof of the main theorem in the case of \( G \) semi-simple rank 1, thus also in general.

**Corollary 7.2.1.** We have \( I_{s_a}(v_1 + v_{s_a}) = c_a \cdot (v_1 + v_{s_a}) \).

**Proof.** Let \( q(w) := [W : I] \). Use the main theorem combined with \( v_{s_a} = v_1 T_{s_a} \), \( q(s_a) = q_a / 2q_a \), and \( T_{s_a} = (q(s_a) - 1)T_{s_a} + q(s_a)T_1. \) \( \square \)

**Corollary 7.2.2.** We have \( I_w(1_{A_0 U K}) = c_w 1_{A_0 U K} \).

**8. Sketch of proof of Bernstein isomorphism for unramified groups**

We may now define an embedding \( R \hookrightarrow H \) by sending \( \pi^\mu \in R \) to the element \( \Theta_\mu \in H \) which is characterized by the identity
\[ \pi^\mu v_1 = v_1 \Theta_\mu. \]
(Often we abuse notation and write \( \pi^\mu \) instead of \( \Theta_\mu \) in the sequel.) Our aim is to show that \( R^W \) maps onto the center \( Z(H) \) of \( H \). We follow closely the argument for split groups given in [HKP].

For each \( a \in \Sigma \), let \( d_a \) denote the denominator of \( c_a \). If \( q_a / 2 = 1 \), then \( d_a = 1 - \pi^\odot \). If \( q_a / 2 \neq 1 \), then \( d_a = 1 - \pi^{2\odot} \). For each \( w \in W \), define
\[ d_w := \prod_{a \in \Sigma_w} d_a. \]
We define the intertwiners without denominators
\[ J_w := d_w I_w. \]

We have the relation
\[ J_w \circ \pi^\mu = \pi^\mu \circ J_w. \]
For any reduced expression $w = s_1 \cdots s_r$, where each $s_i \in \{s_a \mid a \in \Delta\}$, we see that

$$J_w = J_{s_1} \cdots J_{s_r},$$

and hence $J_w$ maps $M$ to itself (since by Theorem 6.0.2 this holds for each $J_{s_i}$). Thus each $J_w$ can be represented by an element of $H$. In the case of $w = s_a$, Theorem 6.0.2 gives us

$$(8.0.3) \quad J_{s_a} = q(s_a)^{-1} d_a T_{s_a} + d_a (c_a - 1).$$

Using this together with (8.0.2) in the case $w = s_a =: s$, we recover Bernstein's relation

$$(8.0.4) \quad T_s \pi^\mu - \pi^\mu T_s = q(s_a)(c_a - 1)(\pi^\mu - \pi^\mu).$$

Here we are writing $\pi^\mu$ in place of its image $\Theta_\mu$ under our embedding $\mathbb{R} \hookrightarrow H$.

An elementary computation (done by writing out $J_{s_a}^2 v_1$ and simplifying), yields the relation

$$(8.0.5) \quad J_{s_a}^2 = d_ac_a d_a c_a^{-1} \in \mathbb{R}.$$
9.1. Rough description of $I_w$ on $M$. Fix $w \in W$. Let $J$ denote the set of coroots $\alpha^\vee$ which are $B$-positive and $WB$-negative. Set $q(w) := [Iw : I]$ and note that
\[ q(w_1w_2) = q(w_1)q(w_2) \]
whenever $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$.

**Lemma 9.1.1.** We have the following rough description of $I_w(v_1)$ as an element in $M_{B,J}$:
\[ I_w(v_1) = q(w)^{-1}v_w + \sum_{w',\alpha^\vee} a_{w',\alpha^\vee,k} v_{k_\alpha^\vee w'}, \]
where $w' \in W$ satisfies $w' < w$ in the Bruhat order on $W$, the coroots $\alpha^\vee$ range over the set $J$, the integers $k$ are non-negative, and the scalars $a_{w',\alpha^\vee,k}$ are complex numbers.

**Proof.** This follows easily by induction on $\ell(w)$, taking into account Theorem 6.0.2, the equation (9.1.1), and the Bernstein relation (8.0.4).

In particular, we see that $J_w(v_1)(w) \neq 0$, so that $J_w$ is never identically zero on $M$.

9.2. Criterion for nonvanishing of $\tilde{J}_w$ on $i_B^G(\chi)$. Let $i_B^G(\chi_{\text{univ}})$ denote the induced representation whose elements consist of the locally-constant $R$-valued functions $\phi$ on $G$ satisfying
\[ \phi(\text{aug}) = \delta_B(a)^{1/2} \cdot a^{-1} \cdot \phi(g) \]
for $a \in A$, $u \in U$, and $g \in G$. The group $G$ acts on $i_B^G(\chi_{\text{univ}})$ by right translations.

There is a canonical $H$-equivariant isomorphism $M = i_B^G(\chi_{\text{univ}})^I$, which is given by associating to $\varphi \in M$ the element $\phi \in i_B^G(\chi_{\text{univ}})^I$ defined by
\[ \phi(g) = \sum_{a \in A/A_O} \delta_B^{-1/2}(a) \varphi(\text{aug}) \cdot a. \]

Now let $\chi$ be an unramified character of $T(F)$ (that is, trivial on $T_k = T(F) \cap T(L_1)$). It can be viewed as a homomorphism from $A/A_O = X_*(A)$ to $\mathbb{C}^\times$; thus it determines a unique $C$-algebra homomorphism $\chi : R \to \mathbb{C}$. Let $i_B^G(\chi)$ denote the (normalized) unramified principal series representation. Using $\chi$ to extend scalars, we have a canonical identification
\[ \mathbb{C} \otimes_R M = i_B^G(\chi^{-1})^I. \]
As in (9.2.1), we can make this explicit: we associate to $1 \otimes \varphi$ the element $\phi \in i_B^G(\chi^{-1})^I$ defined by the formula (for $g \in G$)
\[ \phi(g) = \sum_{a \in A/A_O} \delta_B^{-1/2}(a) \chi(a) \varphi(\text{aug}). \]

In the following proposition, $\chi$ and $\xi$ are unramified characters of $T(F)$, and $w \in W$ is an element of the Weyl group such that $u_w \chi = \xi$ (where $u_w t := \chi(w^{-1}tw)$). Note that the intertwiner without denominator $J_w : M \to M$ descends to an $H$-module homomorphism
\[ 1 \otimes J_w : \mathbb{C} \otimes_{R,\chi} M \to \mathbb{C} \otimes_{R,\xi} M. \]
Via (9.2.2), this determines the descended intertwiner \( \tilde{J}_w : i_B^G(\chi^{-1})^I \to i_B^G(\xi^{-1})^I \). Explicitly, we have for \( \phi \in i_B^G(\chi^{-1})^I \) corresponding to \( \varphi \in M \)

\[
(9.2.4) \quad \tilde{J}_w \phi(g) = \sum_{a \in A/A_\circ} \delta_B^{-1/2}(a) w\chi(a) J_w \varphi(ag).
\]

Recall that \( d_w \) denotes the denominator of \( c_w \). Thanks to the previous subsection, it is easy to understand when \( 1 \otimes J_w \) is identically zero: this happens if and only if \( \xi(d_w) = 0 \). Therefore we derive the following criterion for the nonvanishing of \( \tilde{J}_w \).

**Proposition 9.2.1.** The descended intertwiner \( \tilde{J}_w : i_B^G(\chi^{-1})^I \to i_B^G(\xi^{-1})^I \) is non-zero if and only if \( \xi(d_w) \neq 0 \).

The following shows that we can easily arrange for \( \tilde{J}_w \) to be non-zero.

**Proposition 9.2.2.** If \( w \) is a minimal element (in the Bruhat order) such that \( w\chi = \xi \), then \( \xi(d_w) \neq 0 \).

**Proof.** It is enough to show that for \( a \in \Sigma_w \), we have \( \xi(\pi^{a\gamma}) \neq 1 \) in case \( q_{a/2} = 1 \), and \( \xi(\pi^{2a\gamma}) \neq 1 \) in case \( q_{a/2} \neq 1 \). Let us check only the second case (the first is similar, and easier). Assume that \( \xi(\pi^{2a\gamma}) = 1 \). Note that \( q_{a/2} \neq 1 \) implies that \( a \in 2X^*(A) \) (in fact for the example of \( G = SU_3 \) explained in subsection 3.2, we see that \( a = 2a_1 \), where \( a_1 \) is the unique nondivisible \( B \)-positive root). This implies that for each \( \lambda \in X_*(A) \), the pairing \( \langle a, \lambda \rangle \) is an even integer. It follows that \( s_a \xi = \xi \). Indeed, for each \( \lambda \in X_*(A) \), we have

\[
\begin{align*}
\ &s_a \xi(\pi^\lambda) = \xi(\pi^{s_a \lambda}) = \\
&\quad = \xi(\pi^{a\gamma} \xi(\pi^{2a\gamma})^{-a,\lambda}/2 = \\
&\quad = \xi(\pi^\lambda).
\end{align*}
\]

But now we have \( s_a w \chi = \xi \). This violates the minimality of \( w \): since \( a \in \Sigma_w \), we have \( w^{-1}a \) is \( B \)-negative and so \( s_aw < w \) in the Bruhat order on \( W \). This completes the proof that \( \xi(d_w) \neq 0 \).

We finish with another result which is used in [H]. For \( \tau \in W \), consider the unique function \( \phi_\tau \in i_B^G(\xi^{-1})^I \) which is supported on the set \( B\tau I \) and satisfies \( \phi_\tau(\tau) = 1 \). The functions \( \phi_\tau(\tau \in W) \) form a \( \mathbb{C} \)-vector space basis for \( i_B^G(\chi^{-1})^I \). Via (9.2.2), \( \phi_\tau \) corresponds to the function \( 1 \otimes \varphi_\tau \in \mathbb{C} \otimes_R M \) where \( \varphi_\tau := 1_{A_\circ} U_{\tau I} = v_\tau \).

**Proposition 9.2.3.** For \( \tau, \tau' \in W \), we have \( J_w(\phi_\tau)(\tau') \neq 0 \) only if

\[
w^{-1}U_{w\tau' \cap B\tau I} = \emptyset.
\]

**Proof.** This follows immediately from (9.2.4) and the integral formula defining \( J_w(\varphi_\tau)(a\tau') \) for \( a \in A \).

**References**


