ALCOVES ASSOCIATED TO SPECIAL FIBERS OF LOCAL MODELS

THOMAS J. HAINES AND NGÔ BAO CHÂU

Abstract. The special fiber of the local model of a PEL Shimura variety with Iwahori-type level structure admits a cellular decomposition. The set of strata is in a natural way a finite subset of the affine Weyl group determined by the Shimura data. In this paper, we generalize and give a new proof of a theorem of Kottwitz and Rapoport concerning the description of this set for the case of linear and symplectic groups. For higher rank groups not of type A_n, we produce counterexamples showing the situation is not as nice in general. We also recover a theorem of Deodhar characterizing the Bruhat order on S_n.

1. Introduction

Let G be a classical group over the p-adic field $\mathbb{Q}_p$, and let μ be a dominant minuscule coweight of G. We assume that the pair (G, μ) comes from a PEL-type Shimura datum $(\mathbf{G}, X, K)$ where the p-component K_p of the compact open subgroup K is an Iwahori subgroup of $\mathbf{G}(\mathbb{Q}_p) = G(\mathbb{Q}_p)$. We assume that the corresponding Shimura variety $\mathrm{Sh}(\mathbf{G}, X, K)$ has a model over a p-adic number ring $\mathcal{O}_\mathfrak{p}$. In this case $\mathrm{Sh}(\mathbf{G}, X, K)$ is said to have *Iwahori-type bad reduction* at the prime $\mathfrak{p}$. A fundamental problem is to understand the geometry of the special fiber. In the prototypical example of the modular curve $Y_0(p)$, the special fiber is a union of two smooth curves which intersect transversally. In the general case the global geometry cannot be easily described and the singularities can be quite complicated (see [11]).

The local geometry can be approached using the Rapoport-Zink local model M_μ [12]. This projective $\mathcal{O}_\mathfrak{p}$-scheme is a local model for the singularities in the special fiber of $\mathrm{Sh}(\mathbf{G}, X, K)$, but it is defined in terms of the pair (G, μ) using linear algebra and is somewhat easier to deal with than the Shimura variety itself.

In this paper we address certain combinatorial questions which arise in the study of the special fiber $M_{\mu,\overline{\mathbb{F}}_p}$ of M_μ. To fix ideas, consider the case where G is *split*. It turns out that $M_{\mu,\overline{\mathbb{F}}_p}$ can be regarded as a finite-dimensional union of Iwahori-orbits in the affine flag variety for $G(\overline{\mathbb{F}}_p((t)))$, see [5] and [7]. It follows that $M_{\mu,\overline{\mathbb{F}}_p}$ has a stratification indexed by a finite set $\mathrm{Perm}(\mu)$ of the extended affine Weyl group $\widetilde{W}(G)$ for G. Equivalently, $\mathrm{Perm}(\mu)$ can be regarded as a finite set of alcoves in the affine Coxeter complex determined by a splitting for G. Let W_0 denote the finite Weyl group of G. For each translation λ in the W_0-orbit of μ there is an Iwahori-orbit in $M_{\mu,\overline{\mathbb{F}}_p}$ corresponding to λ; in other words, the element t_λ in $\widetilde{W}(G)$ is contained in $\mathrm{Perm}(\mu)$. We let $\mathrm{Adm}(\mu)$ denote the subset of $\mathrm{Perm}(\mu)$ indexing those strata which lie in the closure of the stratum indexed by t_λ, for some $\lambda \in W_0(\mu)$.

We are primarily concerned with *the equality* $\mathrm{Adm}(\mu) = \mathrm{Perm}(\mu)$, which has important geometric content. For example, if the $\mathcal{O}_{\mathfrak{p}}$-scheme M_μ is *flat*, this equality automatically holds. On the other hand, the equality has been established combinatorially when $G = \mathrm{GL}_n$ or GSp_{2n} by Kottwitz and Rapoport [10], and this was exploited by U. Görtz in his proof of the flatness of the local models associated to $\mathrm{Res}_{E/F}\mathrm{GL}_n$, where E/F is an unramified extension of p-adic fields.

In [10], Kottwitz and Rapoport propose purely combinatorial definitions of the sets $\mathrm{Adm}(\mu)$ and $\mathrm{Perm}(\mu)$, for every split group G and every dominant coweight μ. An element $x \in \widetilde{W}(G)$ belongs to $\mathrm{Adm}(\mu)$ if $x \leq t_\lambda$ in the Bruhat order on $\widetilde{W}(G)$, for some $\lambda \in W_0(\mu)$. The element x belongs to $\mathrm{Perm}(\mu)$ if the vertices of the corresponding alcove satisfy certain inequalities similar to those arising from the definition of local models; see section 3 for details. (These definitions agree with the geometric interpretations above when $G = \mathrm{GL}_n$ or GSp_{2n} and μ is minuscule, see loc. cit..) Kottwitz and Rapoport show that *the inclusion* $\mathrm{Adm}(\mu) \subset \mathrm{Perm}(\mu)$ *is always valid.* We adopt their point of view and assume from now on that G is split. We address the question of whether the equality $\mathrm{Adm}(\mu) = \mathrm{Perm}(\mu)$ holds in full generality, in particular for *non-minuscule* coweights μ.

The main result of this paper is that the situation is as nice as possible for groups of type A_{n-1}.

Theorem 1. *For any root system of type A_{n-1}, the equality* $\mathrm{Adm}(\mu) = \mathrm{Perm}(\mu)$ *holds for every dominant coweight μ.*

Actually, we derive this theorem from a more general result available for any root system. We introduce the new notion of the set of *μ-strongly permissible* alcoves $\mathrm{Perm}^{st}(\mu)$. Like the Kottwitz-Rapoport notion of $\mathrm{Perm}(\mu)$, this is a set of alcoves determined by imposing conditions vertex-by-vertex. Although the condition we require on vertices might look more technical than that of Kottwitz-Rapoport, a similar notion for finite Weyl groups already occurred in a classical theorem of Bernstein-Gelfand-Gelfand [1]. We prove the following statement.

Proposition 2. *For any root system R, the inclusion* $\mathrm{Adm}(\mu) \supset \mathrm{Perm}^{st}(\mu)$ *holds for every dominant coweight μ.*

When the root system R is of type A_{n-1}, we prove that μ-permissibility and μ-strong permissibility are equivalent and derive Theorem 1 from Proposition 2 and the inverse inclusion proved by Kottwitz-Rapoport.

In this paper we introduce some notions of cones, acute or obtuse, in the set of alcoves. These notions play an important role in the proof of Proposition 2 and may also be useful for other combinatorial questions. Another ingredient is a lemma due to Deodhar [3] which is also used in [4].

The equality $\mathrm{Adm}(\mu) = \mathrm{Perm}(\mu)$ turns out to be false in general.

Theorem 3. *If R is an irreducible root system of rank ≥ 4 and not of type A_{n-1}, then* $\mathrm{Adm}(\mu) \neq \mathrm{Perm}(\mu)$ *for every sufficiently regular dominant coweight μ.*

Theorem 3 is based on counter-examples due to Deodhar [4]. Deodhar determines which finite Weyl groups W have the property that $w \leq w'$ in the Bruhat order if and only if $w(\lambda) - w'(\lambda)$ is a sum of positive coroots for every dominant coweight λ. He proves that W has this property if and only if the irreducible components of the associated root system R are of type A_{n-1} or of rank ≤ 3. Our proof of Theorem 3 uses the counter-examples he gives explicitly in the other cases. However Theorem 1 is proved independently and in fact in section 8 we give an alternate, perhaps more conceptual, proof of Deodhar's theorem for root systems of type A_{n-1}.

In light of Theorem 3, and motivated by the study of local models attached to certain *nonsplit* groups, Rapoport proposed that the equality $\mathrm{Adm}(\mu) = \mathrm{Perm}(\mu)$ might be valid for all root systems as long as μ is a *sum of minuscule coweights*. In section 10 we show this is indeed the case for the symplectic group.

Theorem 4. *Let μ be a sum of minuscule coweights for the group GSp_{2n}. Then the equality $\mathrm{Adm}(\mu) = \mathrm{Perm}(\mu)$ holds.*

Theorem 4 relies on a description of the root system of GSp_{2n} as the "fixed-point" root system $R^{[\Theta]}$ with respect to the nontrivial automorphism Θ of the root system R for GL_{2n}, à la Steinberg [14]. This idea was also exploited in [10]. In fact in section 9 we prove the following statement is valid for every dominant coweight, at least in the case of the symplectic group.

Proposition 5. *The equality $\mathrm{Adm}^{\Theta}(\mu) = \mathrm{Perm}(\mu) \cap \widetilde{W}(\mathrm{GSp}_{2n})$ holds for any dominant coweight μ of the root system $R^{[\Theta]}$ for GSp_{2n}.*

Here the sets $\mathrm{Adm}^{\Theta}(\mu)$ and $\mathrm{Perm}^{\Theta}(\mu)$ are the subsets of $\widetilde{W}(\mathrm{GSp}_{2n})$ analogous to the subsets $\mathrm{Adm}(\mu)$ and $\mathrm{Perm}(\mu)$ of $\widetilde{W}(\mathrm{GL}_{2n})$. Moreover, the equality $\mathrm{Perm}^{\Theta}(\mu) = \mathrm{Perm}(\mu) \cap \widetilde{W}(\mathrm{GSp}_{2n})$ can be proved in the case where μ is a sum of minuscule coweights by a method of Kottwitz-Rapoport [10], yielding Theorem 4. It is also worthwhile to remark that the set $\mathrm{Perm}(\mu) \cap \widetilde{W}(\mathrm{GSp}_{2n})$ has itself some geometric meaning for every μ. This set parametrizes the strata in the special fiber of the local model considered in [7], since the definition thereof uses the standard representation of the symplectic group.

Theorem 1 is expected to play a role in proving the flatness of the Pappas-Rapoport local models attached to the group $\mathrm{Res}_{E/F}\mathrm{GL}_n$, where E/F is a totally ramified extension of p-adic fields. For details we refer the reader to forthcoming work of U. Görtz [6]. Likewise, Theorem 4 (or Proposition 5) is expected to play a role in proving the flatness of analogous local models attached to $\mathrm{Res}_{E/F}\mathrm{GSp}_{2n}$. Thus, our study of $\mathrm{Adm}(\mu)$ and $\mathrm{Perm}(\mu)$ for *non-minuscule* coweights μ of split groups has ramifications for Shimura varieties attached to certain *non-split* groups.

For applications to the bad reduction of PEL-type Shimura varieties attached to orthogonal groups one would like to determine whether the analogs of Theorem 4 and Proposition 5 hold for the split orthogonal groups $\mathrm{O}(2n)$ (PEL Shimura varieties arise for groups of type A, C, and D). The methods of this paper do not seem to give much information in that situation. However, in section 11 we show that the analog of Proposition 5 for the odd orthogonal groups does not hold. This can be understood in terms of a *non-inheritance* property of the Bruhat order: the group $\mathrm{SO}(2n+1)$ can be realized as the fixed point group $\mathrm{SL}(2n+1)^{\Theta}$ for a certain involution Θ, and this gives a corresponding embedding of affine Weyl groups $W_{\mathrm{aff}}(B_n) \hookrightarrow W_{\mathrm{aff}}(A_{2n})$. However, the Bruhat order on the former is not inherited from the Bruhat order on the latter (in contrast to the symplectic case, cf. Proposition 9.6).

2. Affine Weyl group attached to a root system

Let us fix notation and recall basic facts about affine Weyl group and alcoves. For proofs, we refer to Humphreys' book [8], especially chapter 4.

Let $(X^*, X_*, R, \check{R})$ be a root system. We assume throughout this paper that the root system is *reduced and irreducible*. When there is no chance of confusion we will denote the root system $(X^*, X_*, R, \check{R})$ simply by R. Let Π be a base of R. Let R^+ (resp. R^-) denote the set of positive (resp. negative) roots. The cardinality of Π will be called the *rank* of R. Denote by $\langle\ ,\ \rangle : X^* \times X_* \to \mathbb{Z}$ the perfect pairing making X^* and X_* dual free abelian groups.

Corresponding to $\alpha \in R$ we have a reflection s_α, acting on $V = X_* \otimes \mathbb{R}$ by $s_\alpha(x) = x - \langle \alpha, x\rangle \check{\alpha}$. The Weyl group W_0 is the subgroup of $\mathrm{GL}(V)$ generated by these reflections. It is known that W_0 is generated by $S = \{s_\alpha \mid \alpha \in \Pi\}$ as a finite Coxeter group.

Let H_α denote the hyperplane (*wall*) fixed by the reflection s_α, for every $\alpha \in R$. The connected components of the set $V - \bigcup_{\alpha\in R} H_\alpha$ will be called *chambers*. The finite Weyl group W_0 acts simply transitively on the set $\mathcal{C}$ of chambers. There is a distinguished chamber

$$C_0 = \{x \in V \mid \langle \alpha, x\rangle > 0,\ \text{ for every }\ \alpha \in \Pi\}$$

which will be called the *dominant chamber*.

Corresponding to $\alpha \in R$ and $k \in \mathbb{Z}$ we have an *affine* reflection $s_{\alpha,k}$, acting on $V = X_* \otimes \mathbb{R}$ by $s_{\alpha,k}(x) = x - (\langle \alpha, x\rangle - k)\check{\alpha}$. The affine Weyl group W_{aff} is the subgroup of $\mathrm{Aut}(V)$ generated by these reflections. It is known that W_{aff} is generated, as a Coxeter group, by $S_{\mathrm{aff}} = S \cup \{s_{\tilde{\alpha},1}\}$ where $\tilde{\alpha}$ is the unique highest root of R. Moreover, W_{aff} is the semi-direct product $W_{\mathrm{aff}} = W_0 \ltimes \check{Q}$ where $\check{Q}$ is the lattice generated by $\check{R}$ acting on V by translation.

Let us denote $\tilde{R} = R \times \mathbb{Z}$. Let $H_{\alpha,k}$ denote the hyperplane in V fixed by the reflection $s_{\alpha,k}$ for every $(\alpha, k) \in \tilde{R}$. The connected components of the set $V - \bigcup_{(\alpha,k)\in\tilde{R}} H_{\alpha,k}$ will be called *alcoves*. The affine Weyl group W_{aff} acts simply transitively on the set $\mathcal{A}$ of the alcoves. There is a distinguished alcove

$$A_0 = \{x \in C_0 \mid \langle \tilde{\alpha}, x\rangle < 1\}$$

that will be called the *base alcove*.

The extended Weyl group $\widetilde{W} = W_0 \ltimes X_*$ also acts on the set $\mathcal{A}$. Indeed for any coweight $\lambda \in X_*$ the *translation* t_λ by the vector λ sends a wall on another wall since $\langle \alpha, \lambda\rangle \in \mathbb{Z}$ for every $\alpha \in R$, therefore sends an alcove to another alcove. Let Ω be the isotropy group in $\widetilde{W}$ of the base alcove. It is known that $\widetilde{W} = W_{\mathrm{aff}} \rtimes \Omega$. With the help of this decomposition, the Bruhat order and the length function on the Coxeter group W_{aff} can be extended to $\widetilde{W}$, which is not a Coxeter group in general. For $w, w' \in W_{\mathrm{aff}}$, $\tau, \tau' \in \Omega$, we say $w\tau \leq w'\tau'$ if and only if $w \leq w'$ and $\tau = \tau'$. We put $l(w\tau) = l(w)$.

Let us recall some basic facts on minimal galleries. A gallery of length l is a sequence of alcoves $A' = A'_0, \ldots, A'_l = A''$ such that A'_{i-1} and A'_i share a wall H_i, for $i = 1, \ldots, l$.

The gallery is minimal if there does not exist a gallery going from A' to A'' with length strictly less than l. In general there exist more than one *minimal gallery* going from A' from A'', but the set of walls H_i depends only on A' and A'': it is the set of walls $H = H_{\alpha,k}$ separating A' and A'', i.e., those such that A' and A'' lie in different connected components of $V - H$.

Minimal galleries are very closely related to reduced expressions in the Coxeter system $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$. Let $x \in W_{\mathrm{aff}}$ and $x = s_1 s_2 \dots s_l$ be a *reduced expression* with $s_1, \dots, s_l \in S_{\mathrm{aff}}$, i.e., such an expression with minimal length $l = l(x)$. Let $A_0, A_1, \dots, A_l$ be the alcoves defined by $A_i = s_1 s_2 \dots s_i A_0$. Obviously, A_{i-1} and A_i share a wall H_i. Hence, this sequence of alcoves forms a gallery going from the base alcove A_0 to $A_l = xA_0$. In this way, the reduced expressions of x correspond bijectively to the minimal galleries going from A_0 to xA_0. The cardinality of the set of hyperplanes $H_{\alpha,k}$ with $(\alpha, k) \in \tilde{R}$ separating A_0 and xA_0 is equal to the length $l(x)$.

Another basic fact that will be used in the sequel is the following. If an expression $x = s_1 s_2 \dots s_l$ is not reduced, there are integers $i < j$ such that we can delete s_i and s_j without changing x:

$$x = s_1 \dots \hat{s}_i \dots \hat{s}_j \dots s_l.$$

Equivalently, a gallery $A'_0, \dots, A'_l$ is not minimal if there exist integers $i < j$ such that $H_i = H_j$, where for every i, H_i denotes the wall shared by the consecutive alcoves A'_{i-1} and A'_i.

3. The μ-admissible and μ-permissible sets

In the sequel, a dominant coweight μ will be fixed and we will denote by τ the unique element of Ω such that $t_\mu \in W_{\mathrm{aff}}\tau$. Let $\mathrm{Conv}(\mu)$ denote the convex hull of the set $\{\lambda \mid \lambda \in W_0(\mu)\}$. In [10], R. Kottwitz and M. Rapoport introduce the notions of the μ-admissible and μ-permissible subsets of $\widetilde{W}$.

Definition 3.1. (**Kottwitz-Rapoport**)

(1) Let $\mathrm{Adm}(\mu) = \{x \in \widetilde{W} \mid x \leq t_\lambda \text{ for some } \lambda \in W_0(\mu)\}$. An element $x \in \mathrm{Adm}(\mu)$ is called *μ-admissible*.
(2) Let $\mathrm{Perm}(\mu)$ be the set of elements $x \in W_{\mathrm{aff}}\tau$ such that $x(a) - a \in \mathrm{Conv}(\mu)$ for every vertex $a \in A_0$. An element $x \in \mathrm{Perm}(\mu)$ is called *μ-permissible*.

We remark that in general the facets of A_0 having minimal dimension are not points, so our use of the word "vertex" in the definition above may be considered inaccurate. Throughout this paper we shall ignore this subtlety and continue to imagine that the facets of A_0 of minimal dimension are in fact points. It is not difficult to translate our resulting arguments and statements into ones which do not abuse terminology.

The following theorem is proved in [10].

Theorem 3.2. (**Kottwitz-Rapoport**)

(1) *For any root system R and dominant coweight μ,* $\mathrm{Adm}(\mu) \subset \mathrm{Perm}(\mu)$.
(2) *Let R be the root system attached to* GL_n *or* GSp_{2n}*. Let μ be a minuscule coweight of R. Then* $\mathrm{Adm}(\mu) = \mathrm{Perm}(\mu)$.

Recall that a coweight μ is *minuscule* if $\langle\alpha,\mu\rangle \in \{-1,0,1\}$, for every root α. For root systems of type A_{n-1}, the second statement can be generalized to every dominant coweight μ.

Theorem 3.3. *For any root system of type A_{n-1}, the equality* $\mathrm{Adm}(\mu) = \mathrm{Perm}(\mu)$ *holds for every dominant coweight μ.*

According to Kottwitz-Rapoport, it is sufficient to prove the inclusion $\mathrm{Adm}(\mu) \supset \mathrm{Perm}(\mu)$.

4. Obtuse cones and μ-strongly permissible sets

Let B_0 denote the negative obtuse cone in V generated by the coroots $-\check{\alpha}$ with $\alpha \in R^+$. The convex hull $\mathrm{Conv}(\mu)$ occurring in the definition of the μ-permissible set may be usefully described as the intersection of $|W_0|$ obtuse cones

$$\mathrm{Conv}(\mu) = \bigcap_{w\in W_0} w\mu + w(B_0).$$

We can rephrase the definition of μ-permissibility by saying that $x \in W_{\mathrm{aff}}\tau$ is μ-permissible if and only if for every vertex v of the base alcove and for any $w \in W_0$, we have

$$x(v) \in t_{w\mu}(v) + w(B_0).$$

The notion of μ-strong permissibility consists in requiring a little bit more than the last inclusion.

Let v be an element of V, $W_{\mathrm{aff}}(v)$ the orbit of v under the action of the affine Weyl group W_{aff}. In the spirit of a paper of Bernstein-Gelfand-Gelfand [1], it seems reasonable to consider the subset $B(v,w)$ of $W_{\mathrm{aff}}(v)$ defined as follows.

Definition 4.1. For every $v \in V$ and $w \in W_0$, let $B(v,w)$ be the set of elements v' of the form $s_r \ldots s_1(v)$ with $s_i = s_{\alpha_i,k_i}$ with $(\alpha_i,k_i) \in \tilde{R}$ such that for all $i = 1,\ldots,r$,

$$s_i s_{i-1} \ldots s_1(v) \in s_{i-1} \ldots s_1(v) + w(B_0).$$

We often call the set $B(v,w)$ the *obtuse cone* with apex v in the w-opposite direction; notice however that $B(v,w)$ is just a discrete set of points.

We obviously have the inclusion

$$B(v,w) \subset W_{\mathrm{aff}}(v) \cap (v + w(B_0)).$$

In contrast to the notion of μ-permissibility, we use these smaller sets $B(v,w)$ to define the notion of μ-strong permissibility.

Definition 4.2. For every dominant coweight $\mu \in X_*$, let $\mathrm{Perm}^{st}(\mu)$ be the set of $x \in W_{\mathrm{aff}}\tau$ such that for every vertex a of A_0 and every $w \in W_0$ the element $x(a)$ lies in $B(t_{w\mu}(a), w)$. Such an element x is said to be *μ-strongly permissible.*

We obviously have the inclusion $\mathrm{Perm}^{st}(\mu) \subset \mathrm{Perm}(\mu)$. We will prove the following stronger statement.

Proposition 4.3. *For any root system R and dominant coweight $\mu \in X_*$, the inclusion* $\mathrm{Adm}(\mu) \supset \mathrm{Perm}^{st}(\mu)$ *holds.*

To prove this theorem we need the notion of acute cone to be discussed in section 5. The proof itself is postponed to section 6.

The following lemma shows that in the case of a root system of type A_{n-1} the notions of μ-permissibility and μ-strong permissibility coincide. Thus Theorem 3.3 follows from Proposition 4.3.

Lemma 4.4. *Suppose R is of type A_{n-1}. Let v be a vertex of some alcove A. For every $w \in W_0$, the equality*

$$B(v,w) = W_{\mathrm{aff}}(v) \cap (v + w(B_0))$$

holds. In particular $\mathrm{Perm}(\mu) = \mathrm{Perm}^{st}(\mu)$ *for every dominant coweight μ.*

Proof. For a root system R of type A_{n-1}, the vertices of alcoves belong to the coweight lattice $\check{P} = \{v \in V \mid \langle \alpha, v\rangle \in \mathbb{Z}$ for every root $\alpha \in R\}$. Moreover, if $v \in \check{P}$, the orbit $W_{\mathrm{aff}}(v)$ is exactly the set of elements $v' \in \check{P}$ such that $v' - v \in \check{Q}$, as one sees using the relation $t_{\check{\alpha}} = s_{\alpha,1}s_\alpha$, for any $\alpha \in R$. Since $\check{Q}$ admits $\{w(\check{\alpha}) \mid \alpha \in \Pi\}$ as $\mathbb{Z}$-basis, any element

$$v' \in W_{\mathrm{aff}}(v) \cap (v + w(B_0))$$

can be written uniquely in the form $v' = v - \sum_{\alpha\in\Pi} n_\alpha w(\check{\alpha})$ with $n_\alpha \in \mathbb{N}$. Thus, it is sufficient to prove the inclusion $v - w(\check{\alpha}) \in B(v,w)$ for any simple root $\alpha \in \Pi$. Let k be the integer $\langle w(\alpha), v\rangle$. Obviously, $v - w(\check{\alpha}) = s_{w(\alpha),k-1}(v)$ and we are done. □

Remark 4.5. Let R be any root system and let v be a *special* vertex for R, i.e. an element of $\check{P}$. Then for every element $w \in W_0$ we have inclusions

$$\{v - \sum_{\alpha\in\Pi}\mathbb{N}w(\check{\alpha})\} \ \subset\ B(v,w) \ \subset\ W_{\mathrm{aff}}(v) \cap (v + w(B_0)) \ \subset\ \{v - \sum_{\alpha\in\Pi}\mathbb{N}w(\check{\alpha})\},$$

the first and last inclusions being proved as in the proof of Lemma 4.4; in particular the displayed equality in Lemma 4.4 holds provided that v is a special vertex. The importance of the first set above stems from its interpretation as the coweights appearing in the character of the Verma module attached to the vertex v and the system of positive roots $w(R^+)$. One can see from the example of nonspecial vertices for the root systems B_2 and G_2 that in general $\{v - \sum_{\alpha\in\Pi}\mathbb{N}w(\check{\alpha})\} \not\supseteq B(v,w)$. One might hope that the second inclusion is always an equality (and thus that $\mathrm{Perm}^{st}(\mu)$ always equals $\mathrm{Perm}(\mu)$). However, this also fails in general, as we shall see in §7 using the examples of Deodhar [4]. Of the three sets above, $B(v,w)$ seems to be the most related to the Bruhat order but also the most difficult to be visualized.

The equality of the sets in Lemma 4.4 for *every* vertex imposes rather strong conditions on the underlying root system R. In fact, as we shall see in section 8 (Remark 8.3), if w_0 is the longest element of W_0 and

$$B(v,w_0) = W_{\mathrm{aff}}(v) \cap (v + w_0(B_0))$$

for every vertex v, then R is necessarily of type A_{n-1} or of rank ≤ 3.

Figure 1. The obtuse cone for G2 with marked vertex as apex in the anti-dominant direction. The base alcove is shaded.

5. Acute cones

Let $\bar{C}_0$ be the closure of the dominant chamber C_0. For any $x \in V$ and $w \in W_0$, the acute cone $\bar{C}(x, w) = x + w(\bar{C}_0)$ is the translation of $w(\bar{C}_0)$ by the vector x. For $A \in \mathcal{A}$ and $w \in W$, we want to define a subset $C(A, w)$ of $\mathcal{A}$ which is as close as possible to a cone $\bar{C}(x, w)$.

The easiest way to do so consists in choosing a point a inside the base alcove A_0. Since W_{aff} acts simply transitively on $\mathcal{A}$, each alcove A contains a unique point $A(a)$ conjugate to a.

Definition 5.1. $C_a(A, w) = \{A' \in \mathcal{A} \mid A'(a) \in A(a) + w(\bar{C}_0)\}$.

The disadvantage of this definition is that it really depends on the choice of $a \in A_0$. In order to give a more intrinsic definition, we need the notion of direction of a gallery.

Let $w \in W_0$. A wall $H = H_{\alpha,k}$ can also be given by $H_{-\alpha,-k}$. But only one root from $\{\alpha, -\alpha\}$ lies in $w(R^+)$; let us assume $\alpha \in w(R^+)$. We define the w-positive half-space H^{w+} by $H^{w+} = \{v \in V \mid \langle \alpha, x \rangle > k\}$. Let H^{w-} be the other half-space.

Definition 5.2. A gallery $A'_0, \ldots, A'_l$ is said to be in the *w-direction* if for any $i = 1, \ldots, l$, letting H_i denote the common wall of A'_{i-1} and of A'_i, the alcove A'_{i-1} lies in the w-negative half-space H_i^{w-} and the alcove A'_i lies in the w-positive half-space H_i^{w+}.

Obviously, the gallery $A'_0, \ldots, A'_l$ is in the w-direction if and only if the inverse gallery $A'_l, \ldots, A'_0$ is in the ww_0-direction where w_0 is the maximal length element of W_0. We will sometimes say the gallery $A'_l, \ldots, A'_0$ is in the *w-opposite direction*.

Lemma 5.3. *Any gallery $A'_0, \ldots, A'_l$ in some w-direction is automatically minimal. If there exists a gallery going from A'_0 to A'_l in the w-direction, then any other minimal gallery going from A'_0 to A'_l is in the w-direction too.*

Proof. Suppose the gallery $A'_0, \ldots, A'_l$ is not minimal. This means there exist two integers $i < j$ such that $H_i = H_j$. Let H denote this hyperplane and suppose $H_k \neq H$ for any integer k in between $i < k < j$. By hypothesis the alcove A'_i lies in H^{w+}. By induction, we know that A'_k also lies in H^{w+} for any $k = i, \ldots, j-1$ since the only hyperplane separating A'_{k-1} and A'_k is $H_k \neq H$. Thus $A_{j-1} \in H_j^{w+}$, which contradicts the hypothesis $A_{j-1} \in H_j^{w-}$.

In an arbitrary minimal gallery $A'_0, \ldots, A'_l$, for every i, the alcoves A'_0 and A'_{i-1} lie in the same connected component of $V - H_i$, and the alcoves A'_i and A'_l lie in the other connected component. If the gallery $A'_0, \ldots, A'_l$ is supposed to be in the w-direction, $A'_{i-1} \in H_i^{w-}$ and $A'_i \in H_i^{w+}$ and consequently we have $A'_0 \in H_i^{w-}$ and $A'_l \in H_i^{w+}$. This means for any hyperplane H separating A'_0 and A'_l, $A'_0 \in H^{w-}$ and $A'_l \in H^{w+}$. The same argument now proves that any other minimal gallery going from A'_0 to A'_l is also in the w-direction. □

Definition 5.4. Let $C(A, w)$ be the set of alcoves $A' \in \mathcal{A}$ such that there exists a gallery $A'_0, A'_1, \ldots, A'_l$ going from A to A' in the w-direction.

We call $C(A, w)$ the *acute cone from A in the w-direction.*

Proposition 5.5. *We have the inclusion $C_a(A, w) \subset C(A, w)$ for any point $a \in A_0$.*

Proof. If $A'(a) \in A(a) + w(\bar{C}_0)$ then for every root $\alpha \in w(R^+)$, the inequality

$$\langle \alpha, A(a) \rangle \leq \langle \alpha, A'(a) \rangle$$

holds. Hence, for any hyperplane H separating A and A', A lies in the w-negative half-space H^{w-} and A' lies in the w-positive half-space H^{w+}. As shown in the proof of the above lemma, any minimal gallery going from A to A' is in the w-direction. □

Corollary 5.6. *For any two alcoves A, A', there exists $w \in W_0$ and a gallery going from A to A' in the w-direction. In other words*

$$\mathcal{A} = \bigcup_{w \in W_0} C(A, w).$$

Proof. The union $\bigcup_{w \in W_0} C_a(A, w)$ obviously covers the whole set $\mathcal{A}$ for any $a \in A_0$. □

Corollary 5.7. *Let μ be a dominant coweight. Then $t_{w\mu}(A_0) \in C(A_0, w)$.*

Proof. If μ is dominant and lies in $\check{Q}$, $t_{w\mu}(A_0)$ belongs to $C_a(A_0, w)$ for all $a \in A_0$. If now μ is dominant but not necessarily in $\check{Q}$, let us choose a point a in A_0 fixed by Ω, for instance we can take the barycenter of A_0. Then $t_{w\mu}(A_0)$ belongs to $C_a(A_0, w)$ a fortiori to $C(A, w)$. □

There exists another characterization of the acute cone $C(A, w)$ which will be useful later.

Lemma 5.8. *An alcove A' lies in $C(A, w)$ if and only if it is contained in every w-positive halfspace H^{w+} containing A:*

$$A' \in \bigcap_{A \in H^{w+}} H^{w+}.$$

Proof. This is only a matter of rephrasing the proof of Lemma 5.3. In fact, we have seen there that $A' \in C(A, w)$ if and only if for any wall H separating A and A', we have $A \in H^{w-}$ and $A' \in H^{w+}$. In other words, $A' \in \bigcap_{A \in H^{w+}} H^{w+}$. □

FIGURE 2. An acute cone in A2 with marked alcove as apex

6. Proof of Proposition 4.3

Suppose $\mu \in X_*$ is dominant and $\mu \in W_{\mathrm{aff}}\tau$ for $\tau \in \Omega$, the isotropy group of the base alcove. Given an element $x \in W_{\mathrm{aff}}\tau$ which is μ-strongly permissible, we want to prove there exists $w \in W_0$ such that $x \leq t_{w\mu}$. According to the Corollary 5.6, there exists $w \in W_0$ such that $x(A_0) \in C(A_0, w)$. According to Corollary 5.7, $t_{w\mu}(A_0)$ lies in the same acute cone $C(A_0, w)$. We will prove $x \leq t_{w\mu}$ for this w.

Let us write $x = x_1\tau$ and $t_{w\mu} = x_2\tau$ with $x_1, x_2 \in W_{\mathrm{aff}}$ and $\tau \in \Omega$. To prove Proposition 2 we thus have to prove the following statement.

Proposition 6.1. *Let x_1, x_2 be two elements of W_{aff} and $w \in W_0$ such that*

- *for every vertex v of A_0, $x_1(v)$ lies in $B(x_2(v), w)$;*
- *the two alcoves $x_1(A_0)$ and $x_2(A_0)$ lie in $C(A_0, w)$.*

Then the inequality $x_1 \leq x_2$ holds.

First, we prove that the statement does not change when we move the base alcove in the w-opposite direction. By this operation we can then move the alcoves $x(A_0)$ and $t_{w\mu}(A_0)$ far from the walls. As we will see, the proof is much easier in that case, compared to the proof of Kottwitz-Rapoport in the minuscule case, which deals with alcoves very close to the origin.

Lemma 6.2. *Let $x_1, x_2 \in W_{\mathrm{aff}}$ be such that the alcoves $x_1(A_0)$ and $x_2(A_0)$ lie in $C(A_0, w)$. Let $y \in W_{\mathrm{aff}}$ be such that $A'_0 \in C(A_0, ww_0)$ where $y(A'_0) = A_0$. Then $x_1 \leq x_2$ if and only if $yx_1 \leq yx_2$.*

Note that the condition $A'_0 \in C(A_0, ww_0)$ means there exists a gallery going from A'_0 to A_0 in the w-direction.

Proof. According to Bernstein-Gelfand-Gelfand [2] (see also [3]) every Coxeter group satisfies the following property : let $x_1, x_2 \in W_{\mathrm{aff}}$ and $s \in S_{\mathrm{aff}}$ be a simple reflection, then

$$x_1 \leq x_2 \quad \Leftrightarrow \quad sx_1 \leq sx_2$$

whenever $l(sx_1) > l(x_1)$ and $l(sx_2) > l(x_2)$. It follows easily by induction on $l(y)$ that $x_1 \leq x_2$ is equivalent to $yx_1 \leq yx_2$ whenever $l(yx_i) = l(y) + l(x_i)$ for $i = 1, 2$.

Therefore what we really have to prove is that the lengths add

$$l(yx_1) = l(y) + l(x_1) \qquad l(yx_2) = l(y) + l(x_2).$$

Of course, we only need to prove the first of these equalities.

Let us choose a minimal gallery $A'_0, \ldots, A'_l$ going from A'_0 to A_0, with $l = l(y)$, and $A_0, \ldots, A_r$ a minimal gallery going from A_0 to $x_1(A_0)$, with $r = l(x_1)$. According to Lemma 5.3, any minimal galleries going from A'_0 to A_0 or going from A_0 to $x_1(A_0)$ are automatically in the w-direction. Obviously, the concatenation of two galleries in the w-direction is still in the w-direction. By Lemma 5.3 again, the concatenated gallery $A'_0, \ldots, A'_l = A_0, \ldots, A_r$ is minimal.

Let us prove that the concatenated gallery $A'_0, \ldots, A_0, \ldots, A_r = x_1yA'_0$ is minimal only if the equality $l(yx_1) = l + r$ holds.

Let $S'_{\mathrm{aff}} = \{y^{-1}sy \mid s \in S_{\mathrm{aff}}\}$ be the set of reflections by walls of A'_0; obviously $(W_{\mathrm{aff}}, S'_{\mathrm{aff}})$ is also a Coxeter system. Now, by viewing A'_0 as base alcove, the gallery $A'_0, \ldots, A_0, \ldots, A_r = x_1yA'_0$ gives rise to a reduced expression

$$x_1y = s'_1 \ldots s'_{l+r}$$

with $s'_1, \ldots, s'_{l+r}$ in S'_{aff}. Thus the length of x_1y in the Coxeter system $(W_{\mathrm{aff}}, S'_{\mathrm{aff}})$ is equal to $l + r$. Consequently, the length of

$$yx_1 = (ys'_1y^{-1}) \ldots (ys'_{l+r}y^{-1})$$

in the Coxeter system $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ is equal to $l + r$. □

Another important ingredient in the proof is a lemma due to Deodhar [3], which is available for any Coxeter system, in particular for the Coxeter system $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$. Let J be a subset of S_{aff} ; let W_J be the subgroup of W_{aff} generated by J. It is known that there exists a subset W^J of W_{aff} such that any element $x \in W_{\mathrm{aff}}$ admits a *unique decomposition* $x = x^Jx_J$ with $x^J \in W^J$ and $x_J \in W_J$ such that $l(x) = l(x^J) + l(x_J)$.

Lemma 6.3 (Deodhar [3]). *Let $\mathcal{J}$ be a set whose elements are subsets of S_{aff} such that $\bigcap_{J\in\mathcal{J}} J = \emptyset$. Let $x, y \in W_{\mathrm{aff}}$ and $x = x^Jx_J$ and $y = y^Jy_J$ be the unique decompositions of x and y for $J \in \mathcal{J}$. Then $x \le y$ if and only if $x^J \le y^J$ for every $J \in \mathcal{J}$.*

Elements of S_{aff} correspond to walls of the base alcove A_0. For any vertex a of A_0, let J_a denote the subset of elements of S_{aff} corresponding to walls containing a. Obviously, the intersection of such subsets J_a for all vertices a of the base alcove is empty. Thus, the lemma of Deodhar applies.

Let us fix a vertex a of the base alcove A_0 and let J denote J_a. It is useful to visualize the decomposition $x = x^Jx_J$ in terms of alcoves. Let x be an element of W_{aff} and $A = x(A_0)$ the corresponding alcove. The coset xW_J is in bijection with the set of alcoves sharing with A the vertex $x(a)$. The element x^J corresponds to the minimal element among those alcoves.

Lemma 6.4. *Let a be a vertex of the base alcove A_0 and let J denote J_a. Let x_1, x_2 be elements of W_{aff} such that $x_1(a) \in B(x_2(a), w)$. There exists an alcove A such that for every $y \in W_{\mathrm{aff}}$ satisfying $y^{-1}A_0 \in C(A, ww_0)$ we have $(yx_1)^J \le (yx_2)^J$.*

In more intuitive terms, the inequality $(yx_1)^J \le (yx_2)^J$ holds for the alcove $y^{-1}A_0$ far enough in w-opposite direction.

Proof. Let us denote $v_1 = x_1(a)$ and $v_2 = x_2(a)$. We can easily reduce to the case $v_1 = sv_2$, where s is some reflection $s_{\alpha,k}$ and $v_1 \in v_2 + w(B_0)$. In other words v_1 and v_2 are symmetric with respect to the hyperplane $H = H_{\alpha,k}$; v_1 lies in the w-negative half-space H^{w-} ; v_2 lies in the w-positive half-space H^{w+}. We can also suppose $v_1 \neq v_2$, thus v_1 and v_2 lie in the interiors of the corresponding half-spaces determined by H.

Let A be an alcove in the w-negative half-space H^{w-}. The whole acute cone $C(A, ww_0)$ belongs then to H^{w-}.

Let $A'_0 = y^{-1}(A_0)$ be an alcove lying in $C(A, ww_0)$, a fortiori in the w-negative half-space H^{w-}. Just as in the proof of the last lemma, it will be convenient to consider A'_0 as base alcove. Let S'_{aff} denote the set of reflections by walls of A'_0. Let $a' = y^{-1}(a)$

and let J' denote the subset of reflections by the walls of A'_0 containing the vertex a'. Since the conjugation by y^{-1} makes the Coxeter $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ isomorphic to the Coxeter system $(W_{\mathrm{aff}}, S'_{\mathrm{aff}})$, the inequality we want to prove in the former system

$$(yx_1)^J \leq (yx_2)^J$$

is equivalent to the inequality we are going to prove

$$(x_1y)^{J'} \leq (x_2y)^{J'}$$

in the latter system. In the remainder of the proof, all inequalities are understood to be relative to the Coxeter system $(W_{\mathrm{aff}}, S'_{\mathrm{aff}})$.

Obviously, $(x_1y)^{J'}(a') = x_1y(a') = v_1$ and $(x_2y)^{J'}(a') = x_2y(a') = v_2$. The alcove $(x_1y)^{J'}(A'_0)$, resp. $(x_2y)^{J'}(A'_0)$, is minimal among the alcoves sharing the vertex v_1, resp. v_2. Like v_1, the alcove $(x_1y)^{J'}(A'_0)$ lies in the w-negative half-space H^{w-}. Like v_2, the alcove $(x_2y)^{J'}(A'_0)$ lies in the w-positive half-space H^{w+}.

Let $A'_0, A'_1, \ldots, A'_l$ be a minimal gallery going from A'_0 to $A'_l = (x_2y)^{J'}A'_0$. Since $A'_0 \in H^{w-}$ and $A'_l \in H^{w+}$, there exists an integer j such that A'_{j-1} and A'_j share the wall H. The minimal gallery $A'_0, A'_1, \ldots, A'_l$ corresponds to a reduced expression

$$(x_2y)^{J'} = s'_1 s'_2 \ldots s'_l$$

with $s'_1, \ldots, s'_l \in S'_{\mathrm{aff}}$. By removing the reflection s'_j from the reduced expression, we get

$$s'_1 \ldots \hat{s}'_j \ldots s'_l = s(x_2y)^{J'}.$$

Therefore $s(x_2y)^{J'} \leq (x_2y)^{J'}$.

But the alcove $s(x_2y)^{J'}A'_0$ contains the vertex $s(x_2y)^{J'}(a') = v_1$. Thus by minimality, we know $(x_1y)^{J'} \leq s(x_2y)^{J'}$, therefore $(x_1y)^{J'} \leq (x_2y)^{J'}$. □

Proof of Proposition 6.1. The proposition follows from Lemmas 6.2, 6.3 and 6.4 as follows. By hypothesis we know that $x_1(A_0)$, $x_2(A_0) \in C(A_0, w)$ and that, for every vertex a of the base alcove, $x_1(a) \in B(x_2(a), w)$. By 6.4, for every vertex $a \in A_0$ there exists an alcove A_a such that, if $y \in W_{\mathrm{aff}}$ satisfies $y^{-1}(A_0) \in C(A_a, ww_0)$, then $(yx_1)^{J_a} \leq (yx_2)^{J_a}$. Now choose y such that

$$y^{-1}(A_0) \in C(A_0, ww_0) \cap \bigcap_a C(A_a, ww_0).$$

(Note that the intersection of finitely many acute cones in the same direction is nonempty.) Then $(yx_1)^{J_a} \leq (yx_2)^{J_a}$ for every a and thus by Lemma 6.3, $yx_1 \leq yx_2$. But then Lemma 6.2 implies $x_1 \leq x_2$.

□

7. Deodhar's counter-examples

In [4] Deodhar considers the following question: which finite Weyl groups W_0 have the property that $w \leq w'$ in the Bruhat order if and only if $w(\lambda) - w'(\lambda)$ is a sum of positive coroots for every dominant coweight λ? Surprisingly enough, at least for us, he proves that W_0 has this property if and only if the irreducible components of the associated root system R are of type A_{n-1} or of rank ≤ 3. For part of his proof, for an irreducible root system of rank ≥ 4 and not of type A_{n-1}, he proves the following statement by giving explicit examples.

Proposition 7.1 (**Deodhar [4]**). *Let R be an irreducible root system of rank ≥ 4 and not of type A_{n-1}. Then there exist elements $w, w' \in W_0$, $w \neq w'$ such that*

- *for every dominant coweight λ, $w(\lambda) - w'(\lambda)$ is a sum of positive coroots;*
- *$l(w) = l(w')$.*

We derive the following statement about μ-admissible and μ-permissible sets, thus proving Theorem 3.

Proposition 7.2. *For R an irreducible root system of rank ≥ 4 and not of type A_{n-1}, we have $\mathrm{Adm}(\mu) \neq \mathrm{Perm}(\mu)$ for every sufficiently regular dominant coweight μ.*

Proof. The idea of the proof is the remark, due to Kottwitz and Rapoport, that nearby the extreme elements $t_{w\mu}$, the picture looks like the Bruhat order in the finite Weyl group W_0. (This is proved in a precise form in Lemma 7.5 below.)

Let μ be a *regular* dominant coweight. For simplicity, suppose $\mu \in \check{Q}$. Let $w, w' \in W_0$ be chosen as in Proposition 7.1. Let us consider the element

$$x = t_{w^{-1}\mu} w^{-1} w'.$$

We prove x is μ-permissible but not μ-admissible, if μ is sufficiently regular.

Since $w \neq w'$, we know $x \neq t_{\mu'}$ for any $\mu' \in W_0(\mu)$. Therefore to prove x is not μ-admissible, it is enough to show $l(x) = l(t_\mu)$. Since μ is regular dominant, by a theorem of Iwahori-Masumoto [9] the maximal length element in the coset $t_{w^{-1}\mu} W_0$ is $t_{w^{-1}\mu} w^{-1}$. Moreover we have

$$\begin{aligned} l(x) &= l(t_{w^{-1}\mu} w^{-1}) - l(w') \\ &= l(t_{w^{-1}\mu} w^{-1}) - l(w) \\ &= l(t_{w^{-1}\mu}), \end{aligned}$$

so $l(x)$ is equal to $l(t_\mu)$.

Let us prove that x is μ-permissible. Let a be a vertex of A_0. By construction of the elements w and w', the difference

$$t_{w^{-1}\mu}(a) - t_{w^{-1}\mu} w^{-1} w'(a) = a - w^{-1} w'(a) = w^{-1}(w(a) - w'(a))$$

lies in $-w^{-1}(B_0)$, since a is dominant.

For μ sufficiently regular dominant, we have

$$t_{w^{-1}\mu} w^{-1} w'(a) - a \in w^{-1}(\bar{C}_0).$$

(This relation can be used as the *definition* of "sufficiently regular".) According to the following well-known statement, x is μ-permissible. □

Lemma 7.3. *The equality*

$$w\bar{C}_0 \cap (w\mu + wB_0) = w\bar{C}_0 \cap \mathrm{Conv}(\mu)$$

holds for any μ dominant and $w \in W_0$.

In some sense, this lemma was the starting point of this work.

Remark 7.4. The following lemma, which is not used in the sequel, makes precise the remark of Kottwitz-Rapoport which inspired Proposition 7.2 above.

Lemma 7.5. *Let R be any irreducible root system. Suppose $\lambda \in X_*$, and let $t_\lambda w_\lambda$ be the unique element of minimal length in the coset $t_\lambda W_0$. Suppose w_1, $w_2 \in W_0$. Then $t_\lambda w_\lambda w_1 \leq t_\lambda w_\lambda w_2$ if and only $w_1 \leq w_2$.*

Proof. By a formula of Iwahori-Matsumoto (loc. cit.), $l(t_\lambda w_\lambda w_i) = l(t_\lambda w_\lambda) + l(w_i)$ for $i = 1, 2$. The lemma is a consequence of these equalities, as explained already in the proof of Lemma 6.2. □

The geometric meaning of the lemma can be stated as follows.

Corollary 7.6. *Suppose x, $y \in W_{\text{aff}}\tau$ are such that the alcoves $x(A_0)$ and $y(A_0)$ share a vertex $v = t_\lambda(0)$, where $t_\lambda \in W_{\text{aff}}\tau$. Then $x < y$ if and only if there exists a sequence of reflections $s_{H_1}, \dots, s_{H_n}$ such that*

$$x < s_{H_1}x < s_{H_2}s_{H_1}x < \cdots < s_{H_n} \dots s_{H_1}x = y,$$

where every hyperplane H_i, $i = 1, \dots, n$, contains the vertex v.

8. Proof of Deodhar's theorem for A_{n-1}

The goal of this section is to present a short proof of the following theorem of Deodhar [4] characterizing the Bruhat order on $W_0 = S_{n-1}$. As in the proof of Theorem 1 we rely on Proposition 6.1 as the key ingredient.

Proposition 8.1 (Deodhar [4]). *Let R be a root system of type A_{n-1}. Let $w, w' \in W_0$. Then $w' \leq w$ in the Bruhat order on W_0 if and only if for every dominant coweight λ, $w'(\lambda) - w(\lambda)$ is a sum of positive coroots.*

Proof. We need to show that if $w'(\lambda) - w(\lambda)$ is a sum of positive coroots for all dominant coweights λ, then $w' \leq w$, the reverse implication being a general fact for all root systems which is easily proved. Recall that for root systems of type A_{n-1} every vertex v of the base alcove is either a dominant fundamental coweight or zero. Let w_0 denote the longest element of W_0.

The assumption on w and w' implies that, for every vertex $v \in A_0$,

$$w'(v) \in W_{\text{aff}}(w(v)) \cap (w(v) + w_0(B_0))$$

and thus

$$w'(v) \in B(w(v), w_0),$$

by Lemma 4.4. By Lemma 8.2 below, $w(A_0)$ and $w'(A_0)$ both belong to $C(A_0, w_0)$. Therefore by Proposition 6.1, $w' \leq w$, as desired.

□

We have used a special case ($\mu = 0$) of the following general lemma. Intuitively, it says that all the alcoves "on the boundary" of the antidominant Weyl chamber are contained in the antidominant acute cone of alcoves.

Lemma 8.2. *Let R be an irreducible root system with Weyl group W_0. Let w_0 denote the longest element of W_0. Then for any dominant coweight μ and any $w \in W_0$ we have*

$$t_{w_0\mu}w(A_0) \in C(A_0, w_0).$$

Proof. Since the concatenation of two galleries in the w_0-direction is still in the w_0-direction, it suffices to prove that $t_{w_0\mu}(A_0) \in C(A_0, w_0)$ and $t_{w_0\mu}w(A_0) \in C(t_{w_0\mu}(A_0), w_0)$. The first statement results from Corollary 5.7, and since the translation of a gallery in the w_0-direction is still in the w_0-direction, the second is equivalent to $w(A_0) \in C(A_0, w_0)$.

Choose a reduced expression $w = s_1 s_2 \cdots s_r$, where for each $i = 1, \dots r$, $s_i = s_{\alpha_i}$ is the reflection corresponding to a simple root α_i. We claim that the gallery $A_0, s_1(A_0), \dots, s_1 \cdots s_r(A_0)$ is in the w_0-direction. Fix i and let $H_i = H_{s_1 \cdots s_{i-1}(\alpha_i)}$ denote the hyperplane separating $s_1 \cdots s_{i-1}(A_0)$ and $s_1 \cdots s_i(A_0)$. We need to show that

$$s_1 \cdots s_{i-1}(A_0) \subset H_i^{w_0-}$$

and

$$s_1 \cdots s_i(A_0) \subset H_i^{w_0+}.$$

But since $s_1 \cdots s_{i-1}(\alpha_i) > 0$, we have

$$H_i^{w_0-} = \{x \in V \mid \langle s_1 \cdots s_{i-1}(\alpha_i), x\rangle > 0\}$$

and

$$H_i^{w_0+} = \{x \in V \mid \langle s_1 \cdots s_{i-1}(\alpha_i), x\rangle < 0\},$$

yielding the desired inclusions.

□

Remark 8.3. If R is any root system such that

$$B(v, w_0) = W_{\text{aff}}(v) \cap (v + w_0(B_0))$$

for every vertex v, then as in the proof of Proposition 8.1. we see that W_0 enjoys the property that $w' \leq w$ in the Bruhat order if and only if $w'(\lambda) - w(\lambda)$ is a sum of positive coroots for every dominant coweight λ. (Indeed, letting λ range over fundamental coweights and recalling that every vertex of A_0 is a positive scalar multiple of a fundamental coweight, we see that the condition on the differences $w'(\lambda) - w(\lambda)$ ensures that $w'(v) \in W_{\text{aff}}(w(v)) \cap (w(v) + w_0(B_0))$ for every vertex $v \in A_0$, and the rest of the proof goes over word-for-word.) Consequently, Deodhar's theorem (see Proposition 7.1) implies that R is of type A_{n-1} or of rank ≤ 3 (cf. Remark 4.5).

9. Admissible sets and automorphisms of root systems

In this section we study the behavior of admissible sets relative to an automorphism Θ of the underlying root system. This will yield information about the admissible and permissible sets in the extended affine Weyl group for GSp_{2n}, to be discussed at the end of this section and in section 10. We begin by recalling some facts about automorphisms of root systems; the main reference is Steinberg's article [14], especially §1.30 − 1.33.

Let $R = (X^*, X_*, R, \check{R}, \Pi)$ be an irreducible and reduced (based) root system, as in section 2. We may equip $V = X_* \otimes \mathbb{R}$ with a W_0-invariant inner product $(\ ,\ )$. By assumption $X^* = \mathrm{Hom}(X_*, \mathbb{Z})$, and we let $\langle\ ,\ \rangle : X^* \times X_* \to \mathbb{Z}$ denote the canonical pairing. A root $\beta \in R$ is a linear functional on V. Using the isomorphism of the vector space V with its dual defined with the help of $(\ ,\ )$, we have the identification $\beta = 2\check{\beta}/(\check{\beta}, \check{\beta})$; in this way β may be regarded as an element of V.

Let Θ be an automorphism of R, in the sense of [14]. This means that Θ is a linear automorphism of $(V, (\ ,\ ))$ leaving stable the sets X^*, R, and Π, when these are regarded as functionals on V. Hence, Θ leaves stable the subsets X_* and $\check{R}$ of V as well. Clearly Θ induces automorphisms of the groups $\widetilde{W}$, W_{aff}, W_0, and $\check{Q}$; in particular we may consider the fixed-point subgroups $W_{\mathrm{aff}}^{\Theta}$, W_0^{Θ}, and $\check{Q}^{\Theta}$.

The group W_0^{Θ} is the Weyl group of a root system $R^{[\Theta]}$, which is defined as follows. Define $\mathcal{Z} = \{x \in X_* \mid \langle \alpha, x\rangle = 0, \text{ for all } \alpha \in R\}$. It is known that $\mathcal{Z} \cap \check{Q} = \{0\}$ and $\mathcal{Z} + \check{Q}$ has finite index in X_* (see Lemma 1.2 of [13]). Let $X_*^{[\Theta]} = \{x \in X_* \mid x - \Theta(x) \in \mathcal{Z}\}$, and let $V^{[\Theta]} = X_*^{[\Theta]} \otimes \mathbb{R}$. For any $\beta \in R$, let $\bar{\Theta}(\beta)$ denote the average of the Θ-orbit of β. Then according to [14] §1.33, $R^{[\Theta]}$ is the subset of $V^{[\Theta]}$ consisting of all elements $\bar{\Theta}(\beta)$, $\beta \in R$, except those which are smaller multiples of others. Moreover, for any Θ-orbit $\pi \subset \Pi$, let V_π denote the real vector space generated by π and let $\tilde{\alpha}_\pi$ be any highest root of the (possibly reducible) root system $R \cap V_\pi$. Then the set $\Pi^{[\Theta]}$ of simple roots in $R^{[\Theta]}$ consists of the elements $\alpha_\pi := \bar{\Theta}(\tilde{\alpha}_\pi)$ as π ranges over all Θ-orbits in Π. (The highest roots of $R \cap V_\pi$ form a single Θ-orbit, so the choice of $\tilde{\alpha}_\pi$ for each orbit π is immaterial.) By loc. cit. §$1.32 - 1.33$, $(V^{[\Theta]}, R^{[\Theta]}, \Pi^{[\Theta]})$ is a root system in the sense of loc. cit., §$1.1 - 1.6$.

For any Θ-orbit $\pi \subset \Pi$, let s_π denote the longest element of the Weyl group W_π generated by $\{s_\alpha \mid \alpha \in \pi\}$. The restriction of s_π to $V^{[\Theta]}$ is the reflection s_{α_π} corresponding to α_π. In the sequel we will abuse notation and write s_{α_π} instead of s_π.

Let $X^{*[\Theta]} = \mathrm{Hom}(X_*^{[\Theta]}, \mathbb{Z})$. Regarding $R^{[\Theta]}$ as a set of linear functionals on $V^{[\Theta]}$, it follows easily that $R^{[\Theta]} \subset X^{*[\Theta]}$. Regarding $R^{[\Theta]}$ as a subset of $V^{[\Theta]}$, we define another subset $\check{R}^{[\Theta]} = \{2\beta/(\beta,\beta) \mid \beta \in R^{[\Theta]}\}$. The following statement is certainly well-known, but lacking a convenient reference, we provide some details.

Proposition 9.1. $R^{[\Theta]} = (X^{*[\Theta]}, X_*^{[\Theta]}, R^{[\Theta]}, \check{R}^{[\Theta]}, \Pi^{[\Theta]})$ *is a reduced and irreducible (based) root system in the vector space* $V^{[\Theta]}$*. Its affine Weyl group is* $W_{\mathrm{aff}}^{\Theta} = \check{Q}^{\Theta} \rtimes W_0^{\Theta}$*.*

Proof. In light of loc. cit. §1.32, to prove $R^{[\Theta]}$ is a based root system it remains only to show that, (i) every element of $R^{[\Theta]}$ is an *integral* linear combination of elements of $\Pi^{[\Theta]}$, and (ii) $\check{R}^{[\Theta]} \subset X_*^{[\Theta]}$. For statement (i), consider the following property of a Θ-orbit $\pi \subset \Pi$:

$(*)$ *The elements of* π *are pairwise orthogonal.*

If $(*)$ holds for every π, then each $\alpha \in \pi$ is a highest root and thus $\bar{\Theta}(\alpha) = \alpha_\pi$. Therefore (i) follows from the analogous property of the root system $R = (X^*, X_*, R, \check{R}, \Pi)$. By examining automorphisms of simple Dynkin diagrams, we see that $(*)$ holds for every π unless R is of type A_{2n} (which we assume realized in $V = \mathbb{R}^{2n+1}$), $\Theta(x_1, \ldots, x_{2n+1}) = (-x_{2n+1}, \ldots, -x_1)$, and $\pi = \{e_n - e_{n+1}, e_{n+1} - e_{n+2}\}$; therefore, $\alpha_\pi = e_n - e_{n+2}$. In this case one can verify by direct calculation that (i) holds (and in fact $R^{[\Theta]}$ is of type C_n). We note that (i) is equivalent to the statement that $2(\alpha,\beta)/(\beta,\beta) \in \mathbb{Z}$ for every $\alpha, \beta \in R^{[\Theta]}$.

Next, one can show, following loc. cit. §1.32, that $W_{\mathrm{aff}}^{\Theta}$ is the group of affine transformations of $V^{[\Theta]}$ generated by the reflections through the hyperplanes $\beta + k = 0$, for $\beta \in R^{[\Theta]}$ and $k \in \mathbb{Z}$. It follows that $W_{\mathrm{aff}}^{\Theta}$ is the affine Weyl group for the root system $R^{[\Theta]}$ in $V^{[\Theta]}$, and therefore the fixed-point group $\check{Q}^{\Theta}$ is precisely the corresponding coroot lattice. But then $\check{R}^{[\Theta]} \subset \check{Q}^{\Theta} \subset X_*^{[\Theta]}$, and (ii) is proved.

Finally, if $\tilde{\alpha}$ is the highest root of R, then $\Theta(\tilde{\alpha})$ is also highest and thus $\Theta(\tilde{\alpha}) = \tilde{\alpha}$ is the unique highest root of $R^{[\Theta]}$. Therefore $R^{[\Theta]}$ is irreducible. □

Viewing roots as linear functionals on V, we have the inclusion

$$R^{[\Theta]} \subset R|_{V^{[\Theta]}},$$

since $\alpha|_{V^{[\Theta]}}$ can be identified with $\bar{\Theta}(\alpha)$, for any $\alpha \in R$. Although the opposite inclusion does not always hold, every element of $R|_{V^{[\Theta]}}$ is of the form $c\beta$ for some $\beta \in R^{[\Theta]}$ and $c \in [0,1]$. In fact $c \in \{\frac{1}{2}, 1\}$, as is shown by the following lemma which is implicit in Steinberg's book [14]; we provide a proof since we could not locate a reference.

Lemma 9.2. *Given a root $\alpha \in R$, $\bar{\Theta}(\alpha)$ is either a root α' or a half-root $\frac{1}{2}\alpha'$, for $\alpha' \in R^{[\Theta]}$. In other words*

$$\bar{\Theta}(R) \subset W_0^{\Theta}\Pi^{[\Theta]} \cup \frac{1}{2}W_0^{\Theta}\Pi^{[\Theta]}.$$

Conversely, if α' is any root in $R^{[\Theta]}$, there is at least one root $\alpha \in R$ such that $\bar{\Theta}(\alpha) = \alpha'$. Moreover α' is a positive root in $R^{[\Theta]}$ if and only if α is a positive root in R.

Proof. First note that if $\alpha \in V_\pi \cap R$, then $\bar{\Theta}(\alpha)$ is either α_π or $\frac{1}{2}\alpha_\pi$ (the latter occurring only in the case R is of type A_{2n} and $\pi = \{e_n - e_{n+1}, e_{n+1} - e_{n+2}\}$; cf. the proof of Prop. 9.1). Thus it is enough to prove the following statement.

(†) Given $\beta \in R^+$, there exist $\pi \subset \Pi$ and $w \in W_0^{\Theta}$ such that $w\beta \in V_\pi \cap R$.

For any $\beta \in R^+$, we have $\bar{\Theta}(\beta) = \sum_\pi c_\pi \alpha_\pi$, where $c_\pi \geq 0$ for every Θ-orbit $\pi \subset \Pi$. We will prove (†) by induction on the quantity $\Sigma\bar{\Theta}(\beta) = \sum_\pi c_\pi$.

Fix a positive root β; we may assume that β is not supported in any $\pi \subset \Pi$. Since the pairing $(\ ,\ )$ is positive definite, we have $(\bar{\Theta}(\beta), \bar{\Theta}(\beta)) > 0$ and thus there is a Θ-orbit π such that $(\bar{\Theta}(\beta), \alpha_\pi) > 0$. By [14] §1.15, s_{α_π} permutes the positive roots not supported on π, and thus $s_{\alpha_\pi}\beta$ is positive. On the other hand

$$\bar{\Theta}(s_{\alpha_\pi}\beta) = s_{\alpha_\pi}\bar{\Theta}(\beta) = \bar{\Theta}(\beta) - \frac{2(\bar{\Theta}(\beta), \alpha_\pi)}{(\alpha_\pi, \alpha_\pi)}\alpha_\pi,$$

and thus $\Sigma\bar{\Theta}(s_{\alpha_\pi}\beta) < \Sigma\bar{\Theta}(\beta)$. By induction $s_{\alpha_\pi}\bar{\Theta}(\beta) \in W_0^{\Theta}\Pi^{[\Theta]} \cup \frac{1}{2}W_0^{\Theta}\Pi^{[\Theta]}$, whence the first statement follows. The converse statement, as well as the one concerning positivity, is obvious. □

Given a fixed affine root (α, k) for R, let $k' = k$ if $\bar{\Theta}(\alpha) = \alpha'$ and let $k' = 2k$ if $\bar{\Theta}(\alpha) = \frac{1}{2}\alpha'$. Given a hyperplane $H_{\alpha,k}$ for R in V we have

$$H_{\alpha,k} \cap V^{[\Theta]} = H_{\alpha',k'},$$

where α' and k' are defined as above. Moreover, every hyperplane for $R^{[\Theta]}$ in $V^{[\Theta]}$ is of this form. This has the following consequence for the set of alcoves in $V^{[\Theta]}$ that arise from the root system $R^{[\Theta]}$.

Proposition 9.3. *Given an alcove A in V, the set $A \cap V^{[\Theta]}$ is either empty or is an alcove A' in $V^{[\Theta]}$; moreover every alcove A' in $V^{[\Theta]}$ arises as $A' = A \cap V^{[\Theta]}$ for a uniquely determined alcove A in V.*

If A_0 (resp. A'_0) is the base alcove in V (resp. in $V^{[\Theta]}$), we have $A'_0 = A_0 \cap V^{[\Theta]}$.

Proof. If $x, y \in V^{[\Theta]}$ are in the same alcove A', they belong to the same connected component of $V - \bigcup H_{\alpha,k}$; therefore $x, y \in A$ for some alcove A in V. If $x, y \in V^{[\Theta]}$ belong to two different alcoves in $V^{[\Theta]}$, there exists $(\alpha', k') \in R^{[\Theta]} \times \mathbb{Z}$ such that $\langle \alpha', x\rangle < k'$ and $\langle \alpha', y\rangle > k'$. Let (α, k) be a corresponding affine root in R (i.e., $\bar{\Theta}(\alpha) = \alpha'$ and $k = k'$). We have $\langle \alpha, x\rangle < k$ and $\langle \alpha, y\rangle > k$, thus x and y belong to different alcoves in V.

An element $x \in A'_0$ satisfies the inequalities $\langle \bar{\Theta}(\alpha), x\rangle > 0$ for all positive roots α of R, and the inequality $\langle \bar{\Theta}(\tilde{\alpha}), x\rangle < 1$ for the highest root $\tilde{\alpha}$ of R. Since $x \in V^{[\Theta]}$ these inequalities are equivalent with $\langle \alpha, x\rangle > 0$ for all positive roots α of R, and $\langle \tilde{\alpha}, x\rangle < 1$. Therefore $x \in A_0$. □

Lemma 9.4. *For any hyperplane $H_{\alpha,k}$ for R in V, the equality*

$$H^{w+}_{\alpha,k} \cap V^{[\Theta]} = H^{w+}_{\alpha',k'}$$

holds for any $w \in W_0^\Theta$.

Proof. This follows easily from the definitions, noting that $\alpha \in w(R^+)$ if and only if $\alpha' \in w(R^{[\Theta]+})$ under the assumption $w \in W_0^\Theta$. □

It is now easy to prove an inheritance property for acute cones of alcoves. Let $C(A_0, w) \cap V^{[\Theta]}$ denote the set of alcoves in $V^{[\Theta]}$ of the form $A \cap V^{[\Theta]}$ such that $A \in C(A_0, w)$.

Proposition 9.5. *Let $w \in W_0^\Theta$. Then*

$$C(A_0^{[\Theta]}, w) = C(A_0, w) \cap V^{[\Theta]}.$$

Proof. This follows easily from Lemma 9.4 and the description of acute cones of alcoves as intersections of half-spaces as in Lemma 5.8. □

As in section 2, define the extended affine Weyl group $\widetilde{W}^{[\Theta]} = X_*^{[\Theta]} \rtimes W_0^\Theta$, which acts on the set $\mathcal{A}^{[\Theta]}$ of alcoves of $V^{[\Theta]}$. Let A'_0 denote the base alcove and $\Omega^{[\Theta]}$ the stabilizer of A'_0 in $\widetilde{W}^{[\Theta]}$. Because W_{aff}^Θ acts simply transitively on $\mathcal{A}^{[\Theta]}$, we have the decomposition $\widetilde{W}^{[\Theta]} = W_{\text{aff}}^\Theta \rtimes \Omega^{[\Theta]}$.

According to Proposition 9.3, we have $A_0 \cap V^{[\Theta]} = A'_0$, thus $\Omega \cap \widetilde{W}^{[\Theta]} = \Omega^{[\Theta]}$. We also have $W_{\text{aff}} \cap \widetilde{W}^{[\Theta]} = W_{\text{aff}}^\Theta$. This allows us to generalize the Kottwitz-Rapoport result on the inheritance property of the Bruhat order from affine Weyl groups to extended affine Weyl groups.

Proposition 9.6. *The Bruhat order $\leq$ on $\widetilde{W}^{[\Theta]}$ is inherited from the Bruhat order $\preceq$ on $\widetilde{W}$. In other words, if $x, y \in \widetilde{W}^{[\Theta]}$, then $x \leq y$ if and only if $x \preceq y$.*

Proof. According to Proposition 2.3 of Kottwitz-Rapoport [10], the statement is already valid if $\widetilde{W}^{[\Theta]}$ is replaced by $W_{\rm aff}^{\Theta}$ and $\widetilde{W}$ is replaced by $W_{\rm aff}$. Since the Bruhat order on $W_{\rm aff}^{\Theta}$ (resp. $W_{\rm aff}$) is extended in the obvious way to $\widetilde{W}^{[\Theta]}$ (resp. $\widetilde{W}$) as explained in section 2, the proposition follows. □

Now let $\mu \in X_*^{[\Theta]}$ be a fixed coweight, which we suppose is dominant (note that it is dominant with respect to R if and only if it is dominant with respect to $R^{[\Theta]}$). We denote by ${\rm Adm}^{\Theta}(\mu)$, ${\rm Perm}^{\Theta}(\mu)$, and ${\rm Perm}^{st,\Theta}(\mu)$ the subsets of $\widetilde{W}^{[\Theta]}$ analogous to the subsets ${\rm Adm}(\mu)$, ${\rm Perm}(\mu)$, and ${\rm Perm}^{st}(\mu)$ of $\widetilde{W}$. The main goal of this section is the following proposition. If R is taken to be the root system for ${\rm GL}_{2n}$ and Θ its nontrivial automorphism (cf. section 10), then $R^{[\Theta]}$ is the root system for ${\rm GSp}_{2n}$. Therefore this result implies Proposition 5.

Proposition 9.7. *Suppose that the root system R is of type A_m. Then the equality*

$$
{\rm Adm}^{\Theta}(\mu) = {\rm Perm}(\mu) \cap \widetilde{W}^{[\Theta]}
$$

holds for every dominant coweight μ of $R^{[\Theta]}$.

Proof. Under the assumption that R is of type A_m, we have

$$
{\rm Adm}(\mu) = {\rm Perm}(\mu) = {\rm Perm}^{st}(\mu).
$$

If $x \in {\rm Adm}^{\Theta}(\mu)$, we have $x \in {\rm Adm}(\mu)$ according to Proposition 9.6, thus $x \in {\rm Perm}(\mu) \cap \widetilde{W}^{[\Theta]}$.

Suppose $x \in {\rm Perm}(\mu) \cap \widetilde{W}^{[\Theta]}$. Let $A = x(A_0)$ and $A' = x(A_0')$; the alcoves A in V and A' in $V^{[\Theta]}$ satisfy $A' = A \cap V^{[\Theta]}$. By Corollary 5.6, there is an element $w \in W_0^{\Theta}$ such that $A' \in C(A_0', w)$. By Proposition 9.5, $A \in C(A_0, w)$. Since $x \in {\rm Perm}^{st}(\mu)$, we have $x(v) \in B(t_{w\mu}(v), w)$ for every vertex $v \in A_0$. We can now apply Proposition 6.1. to x and $t_{w\mu}$ and we obtain $x \leq t_{w\mu}$ relative to the Bruhat order on $\widetilde{W}$. By Proposition 9.6. the same inequality holds for the Bruhat order on $\widetilde{W}^{[\Theta]}$. □

Note that the set ${\rm Perm}(\mu) \cap \widetilde{W}^{[\Theta]}$ has some geometric meaning, when $m = 2n - 1$, which directly relates to the local models of [7] attached to ${\rm GSp}_{2n}$ and a dominant coweight μ. More precisely, let us write Θ for the automorphism of ${\rm GL}_{2n}$ given by $X \mapsto \tilde{J}^{-1}(X^t)^{-1}\tilde{J}$, where $\tilde{J}$ is the matrix

$$
\begin{pmatrix} 0 & J \\ -J & 0 \end{pmatrix},
$$

and where J is the anti-diagonal matrix with entries equal to 1. This determines a symplectic group ${\rm Sp}_{2n} = {\rm SL}_{2n}^{\Theta}$, and a corresponding group ${\rm GSp}_{2n}$. The automorphism Θ preserves the standard "upper triangular" Borel subgroup and the diagonal torus in ${\rm GL}_{2n}$, hence induces an automorphism, also denoted Θ, of the root system R for ${\rm GL}_{2n}$.

The root system $R^{[\Theta]}$ of Steinberg is the root system of the group GSp_{2n} determined by its standard "upper triangular" Borel subgroup and diagonal torus.

Fix a dominant cocharacter $\mu = (\mu_1, \dots, \mu_n, \nu_n, \dots, \nu_1)$ of GSp_{2n}. We can interpret the set $\mathrm{Perm}(\mu) \cap \widetilde{W}(\mathrm{GSp}_{2n})$ in terms of lattice chains. Let k denote an algebraic closure of $\mathbb{F}_p$, and let $\mathcal{V}_i = t^{-1}k[[t]]^i \oplus k[[t]]^{2n-i}$, for $0 \leq i \leq 2n$; extend by periodicity to get the "standard" infinite lattice chain $\mathcal{V}_\bullet$. Consider the following set of periodic lattice chains

$$\cdots \subset \mathcal{L}_{-1} \subset \mathcal{L}_0 \subset \cdots \subset \mathcal{L}_{2n} = t^{-1}\mathcal{L}_0 \subset \dots$$

consisting of $k[[t]]$-lattices in $k((t))^{2n}$ with the following properties:

- $\mathrm{inv}(\mathcal{L}_i, \mathcal{V}_i) \preceq \mu$, for every $i \in \mathbb{Z}$,
- $\mathcal{L}_i^\perp = t^{-c(\mu)}\mathcal{L}_{-i}$, for every $i \in \mathbb{Z}$.

Here $\mathrm{inv}(L, L')$ denotes the standard notion of invariant between two $k[[t]]$-lattices, and $\preceq$ denotes the usual partial order on dominant coweights for GL_{2n}. Moreover, $\perp$ is defined using the symplectic form $(x, y) \mapsto x^t \tilde{J} y$ on $k((t))^{2n}$, and $c(\mu)$ is the common value for $\mu_i + \nu_i$, $1 \leq i \leq n$.

This is precisely the set of k-points $M_\mu(k)$ for the local model M_μ considered in [7]. It carries an action of the standard Iwahori subgroup I_k of $\mathrm{GSp}_{2n}(k[[t]])$, namely the stabilizer of the standard lattice chain $\mathcal{V}_\bullet$.

The I_k-orbits are clearly parametrized by the set $\mathrm{Perm}(\mu) \cap \widetilde{W}(\mathrm{GSp}(2n))$. The content of Proposition 9.7 is therefore that the strata in the special fiber of the model M_μ are indexed by the set of μ-admissible elements of $\widetilde{W}(\mathrm{GSp}_{2n})$. As indicated by Görtz [5],[6], this makes it very likely that the models M_μ from [7] are *flat*.

10. Admissible and permissible sets for GSp_{2n}

In light of Theorem 3, it makes sense to ask when the equality $\mathrm{Adm}(\mu) = \mathrm{Perm}(\mu)$ holds for a dominant coweight μ of the root system for GSp_{2n}. This root system arises as the "fixed-points" under an automorphism Θ of that of GL_{2n}, as in section 9. By Proposition 9.7, we obviously have the following criterion:

$$\mathrm{Adm}^\Theta(\mu) = \mathrm{Perm}^\Theta(\mu) \iff \mathrm{Perm}^\Theta(\mu) \subset \mathrm{Perm}(\mu) \cap \widetilde{W}^{[\Theta]}.$$

Using this we will deduce that the equality $\mathrm{Adm}^\Theta(\mu) = \mathrm{Perm}^\Theta(\mu)$ holds whenever μ is a *sum of minuscule coweights* of GSp_{2n}.

Define $X^* = X_* = \mathbb{Z}^{2n}$ and equip $V = \mathbb{R}^{2n}$ with the standard inner product $(\ , \ )$. Let $R = \check{R} = \{e_i - e_j \mid 1 \leq i < j \leq 2n\}$, where the e_i are the standard basis vectors. Let $\Pi = \{e_i - e_{i+1} \mid 1 \leq i < 2n\}$. Then $R = (X^*, X_*, R, \check{R}, \Pi)$ is the root system for GL_{2n}. The finite Weyl group $W_0 = S_{2n}$ acts on $X_* = \mathbb{Z}^{2n}$ by permuting the coordinates, and the extended affine Weyl group is $\widetilde{W} = \mathbb{Z}^{2n} \rtimes S_{2n}$.

Let Θ denote the automorphism of $X_* = \mathbb{Z}^{2n}$ defined by

$$\Theta(x_1, \dots, x_{2n}) = (-x_{2n}, \dots, -x_1).$$

We can then form the root system $R^{[\Theta]}$ as in section 9. It is easy to check that $R^{[\Theta]}$ is the root system of GSp_{2n}, so all the results of section 9 are in force in studying the latter.

Theorem 10.1. *Let μ be a sum of minuscule coweights for* GSp_{2n}. *Then*

$$\mathrm{Adm}(\mu) = \mathrm{Perm}(\mu).$$

Proof. According to Proposition 9.7, we only need to show

$$\mathrm{Perm}^{\Theta}(\mu) \subset \mathrm{Perm}(\mu) \cap \widetilde{W}^{[\Theta]}.$$

The arguments we use are very close to those in Lemma 12.4 of [10].

We will need some notation. For a vector $v \in \mathbb{Z}^{2n}$, let $v(m)$ denote its mth entry, so that $v = (v(1), \dots, v(2n))$. Let $r = -\Theta$, that is, r is the automorphism of $\mathbb{R}^{2n}$ which reverses the order of the coordinate entries. If $v, v' \in \mathbb{R}^{2n}$, we write $v \leq v'$ if $v(m) \leq v'(m)$ for each m. For $c \in \mathbb{Z}$, let $\mathbf{c} = (c^{2n}) = (c, \dots, c) \in \mathbb{Z}^{2n}$.

Let $\omega_i = (1^i, 0^{2n-i})$ for $1 \leq i \leq 2n$. The vectors ω_i, $1 \leq i \leq 2n-1$, together with $\mathbf{0}$ serve as "vertices" of the base alcove A_0 for GL_{2n}.

The only minuscule coweights are the vectors ν in $\mathbb{Z}^{2n}$ of the form $\nu = (1^n, 0^n) + \mathbf{c}$ or $\nu = \mathbf{c}$, for $c \in \mathbb{Z}$. Therefore we may write $\mu = (a^n, b^n)$, where $a, b \in \mathbb{Z}$ and $a \geq b$. For simplicity we discuss only the case where $b = 0$, the more general case being similar.

We choose a coordinate system $(x_1, \dots, x_n, y_n, \dots, y_1)$ on $V = \mathbb{R}^{2n}$. Then $X_*^{[\Theta]} = \{(x_1, \dots, x_n, y_n, \dots, y_1) \in \mathbb{Z}^{2n} \mid x_1 + y_1 = \cdots = x_n + y_n\}$. The finite Weyl group $W_0^{[\Theta]}$ is $W_n = (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$. The element $e_i \in (\mathbb{Z}/2\mathbb{Z})^n$ acts by switching x_i and y_i, and an element $\sigma \in S_n$ acts by simultaneously permuting the x_i's and the y_i's. Let $\mathrm{Conv}^{\Theta}(\mu)$ denote the convex hull in $\mathbb{R}^{2n}$ of the set $W_n(\mu)$. We have

$$\begin{aligned} \mathrm{Conv}(\mu) &= \{(x_1, \dots, x_n, y_n, \dots, y_1) \in \mathbb{R}^{2n} \mid 0 \leq x_i, y_i \leq a, \forall i, \text{ and } \textstyle\sum_i x_i + y_i = na\} \\ &= \{v \in \mathbb{R}^{2n} \mid \mathbf{0} \leq v \leq \mathbf{a} \text{ and } \textstyle\sum_m v(m) = na\}, \end{aligned}$$

and

$$\begin{aligned} \mathrm{Conv}^{\Theta}(\mu) &= \{(x_1, \dots, x_n, y_n, \dots, y_1) \in \mathbb{R}^{2n} \mid 0 \leq x_i, y_i \leq a \text{ and } x_i + y_i = a, \forall i\} \\ &= \{v \in \mathbb{R}^{2n} \mid \mathbf{0} \leq v \leq \mathbf{a} \text{ and } v + r(v) = \mathbf{a}\}. \end{aligned}$$

The vectors $\eta_i := (\omega_i + \omega_{2n-i})/2$, $1 \leq i \leq n$, together with $\mathbf{0}$, serve as "vertices" for the base alcove A_0'. We now fix $x \in \widetilde{W}^{[\Theta]}$, and let $v_i := x(\omega_i)$, for $1 \leq i \leq 2n-1$. One can easily verify that $x(\eta_i) - \eta_i = (v_i - \omega_i + v_{2n-i} - \omega_{2n-i})/2$, for $1 \leq i \leq n$.

Now suppose $x = t_\lambda w \in \mathrm{Perm}^{\Theta}(\mu)$, i.e., $\lambda = x(\mathbf{0}) - \mathbf{0}$ and $x(\eta_i) - \eta_i$ belong to $\mathrm{Conv}^{\Theta}(\mu)$ for $1 \leq i \leq n$. Therefore

$$\mathbf{0} \leq \lambda \leq \mathbf{a} \tag{10.1}$$

$$\lambda + r(\lambda) = \mathbf{a} \tag{10.2}$$

$$\mathbf{0} \leq v_i - \omega_i + v_{2n-i} - \omega_{2n-i} \leq \mathbf{2a} \tag{10.3}$$

$$v_i - \omega_i + v_{2n-i} - \omega_{2n-i} + r(v_i - \omega_i + v_{2n-i} - \omega_{2n-i}) = \mathbf{2a}, \tag{10.4}$$

the last two equations holding for all $i = 1, \dots, n$. We also have

$$v_i + r(v_{2n-i}) = \mathbf{a} + \mathbf{1} \tag{10.5}$$

$$\omega_i + r(\omega_{2n-i}) = \mathbf{1}, \tag{10.6}$$

for all $i = 1, \dots, 2n$ (for (10.5), use (10.2) and (10.6) together with the observation that $w \in W_0^{\Theta} \Rightarrow w$ commutes with r).

We need to show that $x \in \mathrm{Perm}(\mu)$, i.e., λ and $x(\omega_i) - \omega_i$ belong to $\mathrm{Conv}(\mu)$ for $1 \le i \le 2n-1$. Thus we need to show

$$\mathbf{0} \le \lambda \le \mathbf{a} \tag{10.7}$$

$$\sum_m \lambda(m) = na \tag{10.8}$$

$$\mathbf{0} \le v_i - \omega_i \le \mathbf{a} \tag{10.9}$$

$$\sum_m v_i(m) - \omega_i(m) = na, \tag{10.10}$$

the last two equations holding for all $i = 1, \dots, 2n-1$. It is easy to check that (10.7), (10.8), and (10.10) are satisfied. It is enough to verify (10.9) for i such that $1 \le i \le n$: applying r to (10.9) and using the identities in (10.5-10.6) yields (10.9) with $2n - i$ replacing i. Henceforth we fix i in this range. From (10.5-10.6) we can easily verify that $\mathbf{a} - r(v_i - \omega_i) = v_{2n-i} - \omega_{2n-i}$. Therefore (10.2) and (10.3), respectively, yield

$$\mathbf{a} \le v_i + r(v_i) \le \mathbf{a} + \mathbf{1} \tag{10.11}$$

$$-\mathbf{a} \le v_i - \omega_i - r(v_i - \omega_i) \le \mathbf{a}. \tag{10.12}$$

Let $u = v_i(m) - \omega_i(m)$ and $v = r(v_i)(m) - r(\omega_i)(m)$ for any m in the range $1 \le m \le 2n$. Then (10.11) and (10.12) yield

$$a \le u + v + \delta \le a + 1 \tag{10.13}$$

$$-a \le u - v \le a, \tag{10.14}$$

where $\delta = \omega_i(m) + r(\omega_i)(m)$. Since $1 \le i \le n$, the term δ is either 0 or 1. Adding (10.13) and (10.14) gives

$$0 \le 2u + \delta \le 2a + 1 \tag{10.15}$$

and thus $0 \le u = v_i(m) - \omega_i(m) \le a$. Since this inequality holds for every m, we have $\mathbf{0} \le v_i - \omega_i \le \mathbf{a}$, as desired.

□

11. Remarks on PSO$(2n+1)$

The results we obtained in the symplectic case depend in a crucial way on the fact that the restriction of the Bruhat order of $\widetilde{W}(\mathrm{GL}(2n))$ to $\widetilde{W}(\mathrm{GSp}(2n))$ is the Bruhat order of the latter group. This inheritance property is no longer true for the odd orthogonal group. We are grateful to Kottwitz for his help in preparing this section.

The group PSO$(2n+1)$ whose root system is B_n can be defined as the fixed point subgroup of an involution of PGL$(2n+1)$. This induces an involution Θ on the root system A_{2n}. However the fixed point root system of Θ in the sense of Steinberg, see [14] or section 9, is not B_n but rather C_n. We have the natural inclusions

$$W_{\mathrm{aff}}(B_n) \subset W_{\mathrm{aff}}(C_n) \subset W_{\mathrm{aff}}(A_{2n}).$$

According to [14] and [10], the Bruhat order of $W_{\mathrm{aff}}(C_n)$ is inherited from $W_{\mathrm{aff}}(A_{2n})$. We will show by the following examples that the Bruhat order of $W_{\mathrm{aff}}(B_n)$ is not.

Let denote $s_0, \dots, s_n$ the simple reflections of $W_{\mathrm{aff}}(B_n)$ and $s'_0, \dots, s'_n$ those of $W_{\mathrm{aff}}(C_n)$. The inclusion $W_{\mathrm{aff}}(B_n) \subset W_{\mathrm{aff}}(C_n)$ is given by $s_i \mapsto s'_i$ for $i \neq 0$ and $s_0 \mapsto s'_0 s'_1 s'_0$. The elements s_0 and s_1 are not related in the Bruhat order on $W_{\mathrm{aff}}(B_n)$ but their images $s'_0 s'_1 s'_0$ and s'_1 in $W_{\mathrm{aff}}(C_n)$ are related. See figure 3 below.

The extended affine Weyl group $\widetilde{W}(B_n) = W_{\mathrm{aff}}(B_n) \rtimes \{1, \tau\}$ is canonically identified with $W_{\mathrm{aff}}(C_n)$ by $\tau \mapsto s'_0$. One can show by similar examples that the extended Bruhat order on the coset $W_{\mathrm{aff}}(B_n)\tau$ is also not inherited from $W_{\mathrm{aff}}(C_n)$.

Let $\mu \in \mathbb{Z}^n$ be a dominant coweight of type B_n. Concerning the analog of Theorem 4 for $\mathrm{PSO}(2n+1)$, we do not know whether the equality

$$\mathrm{Adm}^{B_n}(\mu) = \mathrm{Perm}^{B_n}(\mu)$$

holds for μ a sum of minuscule coweights. However, it is not hard to see that the analog of Proposition 5 is false.

Example. Let $n = 2$ and $\mu = (1, 0)$. Then $\mathrm{Adm}^{B_n}(\mu)$ consists of 13 elements. But

$$\mathrm{Perm}^{A_{2n}}(\mu) \cap \widetilde{W}(B_n) = \mathrm{Perm}^{A_{2n}}(\mu) \cap W_{\mathrm{aff}}(C_n) = \mathrm{Adm}^{C_n}(\mu)$$

consists of 19 elements. The final equality above follows from Proposition 9.7.

Figure 3. The alcoves on the right are related in the Bruhat order while the corresponding alcoves on the left are not.

12. Acknowledgments

We thank R. Kottwitz for encouragement, helpful comments, and a careful reading of the paper. We are also grateful to M. Rapoport for communicating his guess that Theorem 4 might hold, and for his continued interest in this work. We thank W. Casselman for his comments on the paper. We are very grateful to U. Görtz for his help in drawing pictures using graphics from Casselman's home page.

This article was written during the *Semestre Hecke* workshop on automorphic forms at the Institut Henri Poincaré in Paris, held in the spring semester of 2000. The first author thanks the IHP for its hospitality. His research is partially supported by an NSF-NATO postdoctoral fellowship, an NSERC research grant, a Connaught New Staff grant, and NSF grant DMS 97-29992. The second author would like to thank the department of mathematics of the University of Toronto for a fruitful visit in February 2000 during which this research in collaboration could begin.

References

[1] I.N. Bernstein, I.M. Gelfand, and S.I. Gelfand, *Differential operators on the base affine space and a study of $\mathfrak{g}$-modules*, Prepr. 77, IPM Akad. Nauk SSSR (1972) (English translation in: Lie groups and their representations. Proc. Summer School in Group Representations. Bolyai Janos Math. Soc., Budapest 1971, pp.21-64, New York: Halsted (1975)).

[2] I.N. Bernstein, I.M. Gelfand and S.I. Gelfand, *Schubert cells and cohomology of the spaces G/P*, Uspekhi Mat. Nauk. **28** (1973), 3-26; English translation, Russ. Math. Surv. **28** (1973), 1-26.

[3] V. Deodhar, *Some characterizations of the Bruhat ordering on a Coxeter group and determination of the relative Möbius function*, Invent. Math. **39** (1977), 187-198.

[4] V. Deodhar, *On Bruhat ordering and weight-lattice ordering for a Weyl group*, Indag. Math. **40** (1978), 423-435.

[5] U. Görtz, *On the flatness of certain Shimura varieties of PEL-type*, to appear Math. Ann.

[6] U. Görtz, forthcoming paper.

[7] T. Haines, B.C. Ngô, *Nearby cycles for local models of some Shimura varieties*, preprint (2000), to appear, Compositio Math.

[8] J.E. Humphreys, *Reflection Groups and Coxeter Groups*, Cambridge Studies in Advanced Mathematics, no. 29, Cambridge Univ. Press (1990).

[9] N. Iwahori, H. Matsumoto, *On Some Bruhat Decomposition and the Structure of the Hecke Rings of p-adic Chevalley Groups*, Publ. Math. IHES, No. 25, (1965) 5-48.

[10] R. Kottwitz, M. Rapoport, *Minuscule Alcoves for Gl_n and GSp_{2n}*, manuscripta math. **102**, 403-428 (2000).

[11] M. Rapoport, *On the Bad Reduction of Shimura Varieties*, Automorphic Forms, Shimura Varieties and L-functions, part II, Perspectives in Mathematics, vol. 11, Academic Press, San Diego, CA, 1990, 253-321.

[12] M. Rapoport, Th. Zink.*Period spaces for p-divisible groups.* Annals of Math. Studies 144, Princeton Univ. Press 1996

[13] T.A. Springer, *Reductive groups*, in *Automorphic forms, Representations, and L-functions*, part I, Proc. of the Symp. in Pure Math., vol. XXXIII, Amer. Math. Soc. 1977.

[14] R. Steinberg, *Endomorphisms of linear algebraic groups*, Mem. Amer. Math. Soc. **80** (1968), 1-108.

University of Toronto
Department of Mathematics

100 St. George Street
Toronto, ON M5S 3G3, Canada
email:haines@math.toronto.edu

CNRS, Université de Paris Nord
Département de mathématiques,
93430 Villetaneuse, FRANCE
email: ngo@math.univ-paris13.fr