On connected components of Shimura varieties

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Abstract

We study the cohomology of connected components of Shimura varieties S_{K^p} coming from the group GSp_{2g}, by an approach modeled on the stabilization of the twisted trace formula, due to Kottwitz and Shelstad. More precisely, for each character $\overline{\omega}$ on the group of connected components of S_{K^p} we define an operator $L(\omega)$ on the cohomology groups with compact supports $H^i_c(S_{K^p}, \overline{\mathbb{Q}}_\ell)$, and then we prove that the virtual trace of the composition of $L(\omega)$ with a Hecke operator f away from p and a sufficiently high power of a geometric Frobenius Φ^r_p, can be expressed as a sum of ω-*weighted* (twisted) orbital integrals (where ω-*weighted* means that the orbital integrals and twisted orbital integrals occuring here each have a weighting factor coming from the character $\overline{\omega}$). As the crucial step, we define and study a new invariant $\alpha_1(\gamma_0; \gamma, \delta)$ which is a refinement of the invariant $\alpha(\gamma_0; \gamma, \delta)$ defined by Kottwitz. This is done by using a theorem of Reimann and Zink.

Introduction

A Shimura variety S_K is constructed from the data of a reductive group G over $\mathbb{Q}$, a $G(\mathbb{R})$-conjugacy class X of $\mathbb{R}$-group homomorphisms $\mathbb{C}^\times \to G_\mathbb{R}$, and a compact open subgroup K of $G(\mathbb{A}_f)$, subject to certain conditions [1]. The field of definition of a Shimura variety is a number field, called the reflex (or Shimura) field, denoted here by E. The relevance of Shimura varieties to automorphic forms stems from the following basic facts:

1) the étale cohomology groups $H^i_c(S_K \otimes \bar{\mathbb{Q}}, \bar{\mathbb{Q}}_\ell)$ admit an action of the Hecke algebra $\mathcal{H} = C_c(K\backslash G(\mathbb{A}_f)/K)$ of K-bi-invariant compactly supported $\bar{\mathbb{Q}}_\ell$-valued functions on $G(\mathbb{A}_f)$ (here ℓ is a fixed rational prime), which commutes with the natural action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/E)$; and
2) the Lefschetz number of this action contains important arithmetic information which can be related (via the trace formula) to certain automorphic representations of the group G.

A Shimura variety is (usually) disconnected, and each connected component is defined over a finite extension of the reflex field. If one forgets the Hecke algebra action and considers only the action of the Galois group, no extra information is gained by studying the individual connected components of the

Shimura variety. However, if one considers the simultaneous actions of the Galois group and the Hecke algebra, some extra information is encoded in the the story for the individual connected components. The goal of this paper is to prove a formula, in the setting of the individual connected components of the Shimura varieties associated to the group of symplectic similitudes over $\mathbb{Q}$, for the Lefschetz number of a Hecke operator times a high power of Frobenius at primes where the variety and the Hecke operator both have good reduction. The ultimate goal, to be pursued in a future work, is to use this formula to extract the extra information alluded to above, which should result in a better understanding of the way the individual connected components contribute to the hypothetical correspondence between certain autormorphic representations of the group of symplectic similitudes and certain Galois representations. Part of this understanding should include the description of the cohomology groups of the individual connected components in terms of automorphic representations of various groups. Since the individual connected components are themselves algebraic varieties defined over number fields, such a description is expected due to general conjectures of Langlands.

To be more precise, let (G, X, K) be a PEL-type Shimura datum and let S_K denote the corresponding Shimura variety over the reflex field E. Suppose p is a prime number such that $G_{\mathbb{Q}_p}$ is unramified, suppose $K_p \subset G(\mathbb{Q}_p)$ is a hyperspecial maximal compact open subgroup and $K^p \subset G(\mathbb{A}_f^p)$ is a sufficiently small compact open subgroup; further suppose $K = K^p K_p$, a compact open subgroup of $G(\mathbb{A}_f)$. Then S_K has good reduction at any prime $\mathfrak{p}$ of E dividing p. Choose a prime ℓ different from p and consider the alternating sum of the étale cohomology groups with compact support $H_c^\bullet(S_K \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell) = \sum (-1)^i H_c^i(S_K \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$ as a virtual $\overline{\mathbb{Q}}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/E)] \times \mathcal{H}$-module, where $\mathcal{H} = C_c(G(\mathbb{A}_f)//K)$ is the Hecke algebra. According to Langlands' conjectures, it should be possible to understand this virtual module in terms of automorphic representations of certain groups. As a first step towards this goal one needs to find a convenient formula for the Lefschetz number

$$\mathrm{Tr}(\Phi_{\mathfrak{p}}^r \circ f\,;\, H_c^\bullet(S_K \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)).$$

Here $\Phi_{\mathfrak{p}}$ denotes the inverse of an arithmetic Frobenius over $\mathfrak{p}$, which acts on the cohomology groups by transport of structure; $f = f^p f_p \in \mathcal{H}$ is spherical at p, and r is a sufficiently high integer (depending on f). In the case where S_K is of PEL-type and G is of type A or C, Kottwitz [11] proved that the Lefschetz number takes the form

$$\sum_{(\gamma_0;\gamma,\delta)} c(\gamma_0;\gamma,\delta)\ O_\gamma(f^p)\ TO_\delta(\phi_r),$$

where the sum is over those equivalence classes of group-theoretic triples $(\gamma_0;\gamma,\delta)$ such that the Kottwitz invariant $\alpha(\gamma_0;\gamma,\delta) = 1$, where $c(\gamma_0;\gamma,\delta)$ is a volume term, $O_\gamma(f^p)$ is the orbital integral of f^p and $TO_\delta(\phi_r)$ is the twisted orbital integral of some function ϕ_r on $G(\mathbb{Q}_{p^r})$ depending on r and the cocharacter μ determined by the datum X.

This expression bears a resemblance to the geometric side of the Arthur-Selberg trace formula. In fact it is possible, assuming some conjectures in harmonic analysis on reductive groups, to rewrite it as a linear combination of the elliptic contributions to the stable trace formulas for certain functions f_H, where H ranges over all the endoscopic groups for G (see [9]). Then one can use the Arthur-Selberg trace formula for the groups H to give an explicit conjectural description of the virtual $\overline{\mathbb{Q}}_\ell[\mathrm{Gal}(\overline{\mathbb{Q}}/E)] \times \mathcal{H}$-module $H_c^\bullet(S_K \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$ in terms of automorphic representations for the group G and the λ-adic representations that conjecturally realize the Langlands correspondence for G (see §7 and §10 of [9]). For certain Shimura varieties, this description has been completely proved: for Shimura varieties attached to GSp_4, in a work of Laumon [19] (Kottwitz' work [9] being made unconditional by the resolution of the problems in harmonic analysis due to Hales [4] and Waldspurger [33], [34], or Weissauer-Schroder [35]); for Shimura varieties attached to "fake" unitary groups (for central division algebras over CM fields) by Kottwitz [10]; for Picard modular surfaces [17]. By taking f to be the trivial Hecke operator in these examples, one can derive an expression for the local zeta function of S_K at $\mathfrak{p}$ in terms of automorphic L-functions of G, as predicted by Langlands (see [14], [15], [16]). This extends some of the earlier work of Shimura on the zeta functions of modular and (quaternion) nonmodular curves (see [29], [30] and [31], and note that Shimura dealt in his works with many Shimura varieties which are not of PEL type), as well as that of Ihara [5] and others. It is also related to more recent work of Reimann [26].

Now let G be the group of symplectic similitudes GSp_{2g}. It is well-known that the Shimura variety S_K is geometrically disconnected. Each connected component is a quasi-projective variety over some number field. In this paper we are concerned with the analogue of the Lefschetz number above for each *connected component* of S_K. To this end we introduce an operator $L(\omega)$ on $H_c^i(S_K \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$ as follows (see §2): we give the set of connected components the structure of a finite abelian group π_p (noncanonically) and consider a character $\omega : \pi_p \to \overline{\mathbb{Q}}_\ell^\times$. Let X_a denote the component of S_K indexed by $a \in \pi_p$. Then we define $L(\omega)$ to act on the direct summand $H_c^i(X_a \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$ of $H_c^i(S_K \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$ by the scalar $\omega(a) \in \overline{\mathbb{Q}}_\ell^\times$. The purpose of introducing the operators $L(\omega)$ is that a suitable linear combination of them gives the projection $H_c^i(S_K \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell) \to H_c^i(X_a \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$ and thus we can isolate the cohomology of any connected component. The main theorem of this paper is the following formula for the "twisted" Lefschetz number (Theorem 8.2).

Theorem 0.1. *Let $p > 2$ be a prime number. Let (GSp_{2g}, X, K) be a Shimura datum for which K_p is a hyperspecial maximal compact subgroup of $GSp_{2g}(\mathbb{Q}_p)$, and let S_K denote the corresponding Shimura variety. Let ω be a character of the group of connected components π_p of S_K. Let $f \in \mathcal{H}$ be the Hecke operator coming from an element $g \in G(\mathbb{A}_f^p)$ (cf. §2). Then for all sufficiently high integers r (depending on f),*

$$Tr(\Phi_p^r \circ f \circ L(\omega)\,;\, H_c^\bullet(S_K \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell))$$

is equal to an expression of the form

$$\sum_{(\gamma_0;\gamma,\delta)} c(\gamma_0;\gamma,\delta)\ \overline{\omega}_p(p^{-1})\ \langle \alpha_1(\gamma_0;\gamma,\delta)\,,\,\kappa_0\rangle\ O_\gamma^\omega(f^p)\ TO_\delta^\omega(\phi_r).$$

where we take the sum over all G-equivalence classes of triples $(\gamma_0;\gamma,\delta)$ such that

1. $\alpha(\gamma_0;\gamma,\delta) = 1$,
2. γ_0 *is ω-special,*

and where $\kappa_0 \in Z(\hat{I}_1)^\Gamma$ is any element which satisfies $\partial(\kappa_0) = \mathbf{a}$.

The notation is too complicated to explain completely here (see §5 and §6 for definitions, and Theorem 8.2 for a more precise statement), so we content ourselves with a brief description of how the formula is related to that of Kottwitz above. The sum is taken over the *ω-special* triples $(\gamma_0;\gamma,\delta)$ which have trivial Kottwitz invariant. The notion of ω-special triples arises because not all isogeny classes of abelian varieties contribute to the "twisted" Lefschetz number. The volume terms $c(\gamma_0;\gamma,\delta)$ are the same as those in Kottwitz' formula. The term $\overline{\omega}_p(p^{-1})$ is a root of unity in $\overline{\mathbb{Q}}_\ell^\times$ depending on ω. The *ω-weighted* orbital integral $O_\gamma^\omega(f^p)$ (resp. twisted orbital integral $TO_\delta^\omega(\phi_r)$) differs from the orbital integral $O_\gamma(f^p)$ (resp. the twisted orbital integral $TO_\delta(\phi_r)$) appearing in Kottwitz' formula by a weighting factor coming from ω in the integrand (see Definition 6.1 (resp. Definition 6.3)). The term $\langle \alpha_1(\gamma_0;\gamma,\delta)\,,\,\kappa_0\rangle$ is root of unity in $\overline{\mathbb{Q}}_\ell^\times$ which results from pairing an element κ_0 (which depends on ω) with a refinement $\alpha_1(\gamma_0;\gamma,\delta)$ of the Kottwitz invariant $\alpha(\gamma_0;\gamma,\delta)$. This refined invariant is defined for any triple $(\gamma_0;\gamma,\delta)$ for which $\alpha(\gamma_0;\gamma,\delta)$ can be defined. Moreover, it lies in a certain finite abelian group $X^*(Z(\hat{I}_1)^\Gamma)/\mathrm{im}(\mathbb{Z}_p^\times)$ which maps to the group $\mathfrak{k}(I_0/\mathbb{Q})^D$ to which $\alpha(\gamma_0;\gamma,\delta)$ belongs. Moreover, under the map

$$X^*(Z(\hat{I}_1)^\Gamma)/\mathrm{im}(\mathbb{Z}_p^\times) \to \mathfrak{k}(I_0/\mathbb{Q})^D,$$

$\alpha_1(\gamma_0;\gamma,\delta)$ maps to $\alpha(\gamma_0;\gamma,\delta)$. It follows from this comparison that if we take ω to be the trivial character in the formula for the twisted Lefschetz number, then we recover the formula of Kottwitz for the untwisted Lefschetz number.

The refined invariant $\alpha_1(\gamma_0;\gamma,\delta)$ is constructed so that the roots of unity $\langle \alpha_1(\gamma_0;\gamma,\delta)\,,\,\kappa_0\rangle$ keep track of how each isogeny class of abelian varieties contributes to the twisted Lefschetz number. If $(\gamma_0;\gamma,\delta)$ comes from the isogeny class of a polarized abelian variety (A,λ) over the finite field $\mathbb{F}_{p^r}$, then the Kottwitz invariant $\alpha(\gamma_0,\gamma,\delta)$ is trivial. However, the refined invariant $\alpha_1(\gamma_0;\gamma,\delta)$ also reflects the choices of symplectic similitudes between the $\mathbb{A}_f^p$-Tate module (resp. the covariant Dieudonné module) and the standard symplectic space:

$$\begin{aligned} \beta^p &: H_1(\overline{A},\mathbb{A}_f^p) \xrightarrow{\sim} V\otimes\mathbb{A}_f^p \\ \beta_r &: H(A) \xrightarrow{\sim} V\otimes L_r. \end{aligned}$$

The refined invariant $\alpha_1(\gamma_0;\gamma,\delta)$ of a triple coming from (A,λ) is trivial if and only if β^p and β_r are *symplectic isomorphisms*, not just symplectic similitudes (relative to a fixed symplectic structure on V of course).

The definition and study of $\alpha_1(\gamma_0;\gamma,\delta)$ is the technical heart of this paper. Both the definition and the proofs of the important properties rest on a theorem of Reimann and Zink [27]. We put their theorem into a group-theoretic framework and thereby construct a refinement of the canonical map $B(G_{\mathbb{Q}_p}) \to X^*(Z(\hat{G})^{\Gamma(p)})$ used in the construction of Kottwitz' invariant $\alpha(\gamma_0;\gamma,\delta)$; this is used to define $\alpha_1(\gamma_0;\gamma,\delta)$. We also use the theorem of Reimann and Zink to prove the crucial vanishing property of $\alpha_1(\gamma_0;\gamma,\delta)$ alluded to above, in effect by proving that it implies the vanishing of a "refined invariant" attached to polarized abelian varieties over finite fields, and then by relating this vanishing to that of $\alpha_1(\gamma_0;\gamma,\delta)$ when the maps β^p (resp β_r) above are symplectic isomorphisms. Theorem 0.1 then follows from a study of how the quantity $\langle\alpha_1(\gamma_0;\gamma,\delta)\,,\,\kappa_0\rangle$ changes when the condition that β^p (resp. β_r) preserves symplectic pairings is relaxed.

The form of the expression for the twisted Lefschetz number of Theorem 0.1 is reminiscent of part of the geometric side of the *twisted* trace formula. In fact the strategy of this paper is modeled on a very special case of the stabilization of the regular elliptic part of the twisted trace formula, due to Kottwitz and Shelstad [13]. This special case results in a formula analogous to Theorem 8.2 of this paper, except that our formula includes as well the terms which correspond to the *singular* elliptic contribution to the twisted trace formula. In particular one sees that the invariant $\alpha_1(\gamma_0;\gamma,\delta)$ is the analogue of the invariant $\mathrm{obs}(\delta)$ from [13]. The reader familiar with [13] will anticipate the next step in the process: one should be able to "stabilize" the formula above, that is, write it as a linear combination of the (elliptic parts of the) stable trace formulas for certain functions f_H, where H ranges over all endoscopic groups for the pair $(G,\mathbf{a})$. This should then yield a description of the cohomology of each connected component of S_K, in a manner analogous to §10 of [9]. We will attend to this aspect of the problem in a future paper.

The results in this paper are related to results of M. Pfau [24], who studied a refinement of the conjecture of Langlands and Rapoport [18] for connected Shimura varieties. They are also related to the book of H. Reimann [26]; in particular, the "mysterious invariant" in Appendix A of [26] should be related to the refined invariant $\alpha_1(\gamma_0;\gamma,\delta)$ of this paper. At the present time, however, we are unable to say exactly what the relationships are.

The paper is organized as follows: in §1 we define notation. In §2 we define the operators $L(\omega)$, and reduce the computation of the virtual trace to the computation of a sum over fixed points of certain roots of unity. In §3 we prove a formula for the sum over the fixed points in a single isogeny class. As a necessary prelude to the definition of the refined invariant in §5, in §4 we introduce a p-adic construction which gives a group-theoretic way to view the theorem of Reimann and Zink, which is also recalled. In §6 we perform further simplifications of the sum in question (essentially determining which isogeny classes contribute, and then grouping them together into "stable" classes). In

§8 the Main Theorem is proved, using a crucial vanishing property of the refined invariant $\alpha_1(\gamma_0;\gamma,\delta)$. This follows from the vanishing of an invariant $\tilde{\alpha}(A,\lambda)$ attached to polarized abelian varieties, which is deduced from the theorem of Reimann-Zink in §7.

1 Notation and terminology

Let $p > 2$ be a prime number. Write k for the finite field $\mathbb{F}_p$. Fix an algebraic closure $\overline{k}$ of k. For each positive integer r let k_r denote the field with p^r elements in $\overline{k}$. We will use v to denote a place of $\mathbb{Q}$. Let ℓ denote a fixed finite place of $\mathbb{Q}$ other than p. We will often use the symbol l to denote an unspecified finite place that is not p (the symbols ℓ and l are always used in different contexts, so this should not cause confusion). Fix an algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$, and let $\mathbb{Q}_p^{un}$ denote the maximal unramified extension of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$. Let $\overline{\mathbb{Q}}_\infty = \mathbb{C}$. Write $W(\overline{k})$ for the ring of Witt vectors of $\overline{k}$. It is the completion of the ring $\mathbb{Z}_p^{un}$ of integers in $\mathbb{Q}_p^{un}$. Let L denote the completion of $\mathbb{Q}_p^{un}$, and write σ for the Frobenius automorphism of L. Fix an algebraic closure $\overline{L}$ of L which contains $\overline{\mathbb{Q}}_p$. For each $r \geq 1$ let L_r denote the fixed field of σ^r in L. Then L_r is the unique unramified extension of $\mathbb{Q}_p$ in L having degree r, and this is also sometimes denoted $\mathbb{Q}_{p^r}$.

If $F \subset \mathbb{R}$, we will use F^+ to denote $F \cap \mathbb{R}_{>0}$.

The crucial constructions in this paper can only be done modulo the choice of the following initial data $\mathcal{D} = \{i\ , j_p,\ j_\infty\}$. Here $i = \sqrt{-1}$ is a choice of square root of -1 in $\mathbb{C}$, and $j_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $j_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ are choices of field embeddings.

By using the data $\mathcal{D}$ and the "reduction modulo p" map $\mathbb{Z}_p^{un} \to \overline{k}$ we get, for each $l \neq p$ and $n \in \mathbb{N}$, identifications of roots of unity:

$$\mu_{l^n}(\overline{k}) \;=\; \mu_{l^n}(\mathbb{Z}_p^{un}) \;=\; \mu_{l^n}(\overline{\mathbb{Q}}_p) \;=\; \mu_{l^n}(\overline{\mathbb{Q}}) \;=\; \mu_{l^n}(\mathbb{C}) \;=\; \mathbb{Z}/(l^n\mathbb{Z}).$$

Here the last identification is defined by the inverse of the exponential map: $x \mapsto \exp((2\pi i x)/l^n)$, where i is the element fixed in $\mathcal{D}$. By taking the limit over n and then tensoring with $\mathbb{Q}$ we get identifications of Tate twists

$$\mathbb{Q}_l(1)(\overline{k}) \;=\; \mathbb{Q}_l(1)(\overline{\mathbb{Q}}_p) \;=\; \mathbb{Q}_l(1)(\overline{\mathbb{Q}}) \;=\; \mathbb{Q}_l(1)(\mathbb{C}) \;=\; \mathbb{Q}_l,$$

and so after taking the restricted product over all $l \neq p$, we get identifications

$$\mathbb{A}_f^p(1)(\overline{k}) \;=\; \mathbb{A}_f^p(1)(\overline{\mathbb{Q}}_p) \;=\; \mathbb{A}_f^p(1)(\overline{\mathbb{Q}}) \;=\; \mathbb{A}_f^p(1)(\mathbb{C}) \;=\; \mathbb{A}_f^p.$$

In a similar way, the data $\mathcal{D}$ give us identifications for the prime p:

$$\mathbb{Q}_p(1)(\overline{\mathbb{Q}}_p) \;=\; \mathbb{Q}_p(1)(\overline{\mathbb{Q}}) \;=\; \mathbb{Q}_p(1)(\mathbb{C}) \;=\; \mathbb{Q}_p.$$

Write Γ for $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. If v is any place of $\mathbb{Q}$ and we have chosen an algebraic closure $\overline{\mathbb{Q}}_v$ of $\mathbb{Q}_v$, then we write $\Gamma(v)$ for $\mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$. Any choice of field embedding $j_v : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_v$ yields an inclusion $j_v^* : \Gamma(v) \hookrightarrow \Gamma$; whenever we are considering choices of such j_v for $v \in \{p, \infty\}$, we will use the choices already specified in the data $\mathcal{D}$. Throughout the paper we will use the symbol j_v^* quite generally to denote any map between objects that is defined in a transparent manner by using the embedding j_v.

Now we describe the Shimura varieties S_{K^p} in this paper, following the introduction of [11]. Let V be a $2g$-dimensional nondegenerate symplectic vector space over $\mathbb{Q}$ (a finite dimensional $\mathbb{Q}$-vector space together with a nondegenerate $\mathbb{Q}$-valued alternating pairing $\langle \cdot, \cdot \rangle$). Let G be the group of automorphisms of the symplectic space V. Since automorphisms are only required to preserve the pairing up to a scalar, G is a realization of the group of symplectic similitudes $\mathrm{GSp}(2g, \mathbb{Q})$. Let $K^p \subset G(\mathbb{A}_f^p)$ be a compact open subgroup which is sufficiently small (the precise meaning of which we describe below). Let $h : \mathbb{C} \to \mathrm{End}(V_\mathbb{R})$ be an $\mathbb{R}$-algebra homomorphism such that $h(\overline{z}) = (h(z))^*$ and $\langle \cdot, h(i) \cdot \rangle$ is positive definite, where $*$ denotes the transpose on $\mathrm{End}(V_\mathbb{R})$ coming from the pairing, and i is from the data $\mathcal{D}$. Let X_∞ denote the $G(\mathbb{R})$-conjugacy class of h. Let K_∞ denote the subgroup of $G(\mathbb{R})$ centralizing h. Let X^+ denote the set of all elements $h_1 \in X_\infty$ such that $\langle \cdot, h_1(i) \cdot \rangle$ is positive definite (again use the i in $\mathcal{D}$).

Now consider the following moduli problem S_{K^p} over $\mathbb{Z}_{(p)}$: for a locally Noetherian $\mathbb{Z}_{(p)}$-scheme S let $S_{K^p}(S)$ denote the set of all isomorphism classes of triples $(A, \lambda, \overline{\eta})$, where A is a projective abelian scheme over S, $\lambda : A \to \hat{A}$ is a polarization of A (a prime-to-p isogeny which is a polarization over all geometric points of S), and $\overline{\eta}$ is a K^p-orbit of symplectic similitudes

$$\eta : V \otimes \mathbb{A}_f^p \to H_1(A, \mathbb{A}_f^p),$$

where the pairing on the right is the Weil pairing coming from the polarization λ. Here we are viewing both sides as smooth $\mathbb{A}_f^p$-sheaves over S with the étale topology, and we are assuming that for each geometric point s of S, the K^p-orbit $\overline{\eta}_s$ of the map on stalks η_s is fixed by the action (on the left) of the algebraic fundamental group $\pi_1(S, s)$. Such an orbit $\overline{\eta}$ will be called a K^p-level structure. We say two triples $(A, \lambda, \overline{\eta})$ and $(A', \lambda', \overline{\eta'})$ are isomorphic if there exists a prime-to-p isogeny $\phi : A \to A'$ such that $\phi^*(\lambda') = c\lambda$, for some $c \in \mathbb{Z}_{(p)}^+$, and which takes $\overline{\eta}$ to $\overline{\eta'}$.

In [23] Mumford used geometric invariant theory to prove that S_{K^p} is represented by a smooth quasiprojective $\mathbb{Z}_{(p)}$-scheme, for any K^p which is small enough so that the moduli problem S_{K^p} has no nontrivial automorphisms (this is what we mean when we say K^p is "sufficiently small"). Kottwitz generalized this result in §5 of [11].

We fix additional data: let $\Lambda_0 \subset V$ be a $\mathbb{Z}_{(p)}$-lattice which is self-dual with respect to the pairing, and let K_p denote the stabilizer of this lattice in $G(\mathbb{Q}_p)$. Then K_p is a hyperspecial good maximal compact subgroup of $G(\mathbb{Q}_p)$ and we can choose an integral structure for $G_{\mathbb{Q}_p}$ such that $K_p = G(\mathbb{Z}_p)$. Let $K = K^p K_p$. We will often write S_K instead of S_{K^p}.

Write G^{sc} for the derived group of G, which is simply connected since it is a realization of $\mathrm{Sp}(2g, \mathbb{Q})$. Let $c : G \to \mathbb{G}_m$ denote the homomorphism given by the equation $\langle gv, gw\rangle = c(g)\langle v, w\rangle$.

We denote by π_0 the scheme of connected components of S_K. It is a finite étale $\mathbb{Z}_{(p)}$-scheme.

If G is any connected reductive group over a field F, we denote by $\hat{G}$ the connected reductive group over $\mathbb{C}$ of type dual to G, together with the action of $\mathrm{Gal}(F^{sep}/F)$ on $\hat{G}$ (which is well-defined up to an inner automorphism of $\hat{G}$). Denote by $Z(\hat{G})$ the center of $\hat{G}$; it carries a canonical action of $\mathrm{Gal}(F^{sep}/F)$.

2 The twisted Lefschetz number

2.1 Definition of the Operator $L(\omega)$

Let π_p denote the finite abelian group

$$\mathbb{Z}_{(p)}^{+}\backslash(\mathbb{A}_f^p)^{\times}/c(K^p) \;\cong\; \mathbb{Q}^{+}\backslash\mathbb{A}_f^{\times}/c(K^p)\mathbb{Z}_p^{\times} \;\cong\; \mathbb{Q}^{\times}\backslash\mathbb{A}^{\times}/c(K)c(K_\infty).$$

(One can show without difficulty that $c(K_p) = \mathbb{Z}_p^{\times}$ and that $c(K_\infty) = \mathbb{R}^{+}$.) Fix a character $\overline{\omega} : \pi_p \to \mathbb{C}^{\times}$. Sometimes (but not always) we will need to consider this character as $\overline{\mathbb{Q}}_\ell$-valued. To do so we choose an isomorphism of fields $\mathbb{C} \xrightarrow{\backsim} \overline{\mathbb{Q}}_\ell$; the context will dictate in which field we are considering $\overline{\omega}$ to take values.

Next we will construct maps (depending on $\mathcal{D}$) $c : S_K(F) \to \pi_p$, where F is any of the rings $\overline{k}$, $\mathbb{Z}_p^{un}$, $\overline{\mathbb{Q}}_p$, $\overline{\mathbb{Q}}$, or $\mathbb{C}$, and then we will denote by ω any of the resulting maps $\overline{\omega}\circ c : S_K(F) \to \mathbb{C}^{\times}$. Because π_0 is finite and étale, $\mathcal{D}$ yields the following commutative diagram

$$\begin{tikzcd}
S_K(\overline{k}) \arrow[d] & S_K(\mathbb{Z}_p^{un}) \arrow[l] \arrow[r] \arrow[d] & S_K(\overline{\mathbb{Q}}_p) \arrow[d] & S_K(\overline{\mathbb{Q}}) \arrow[l] \arrow[r] \arrow[d] & S_K(\mathbb{C}) \arrow[r, "\backsim"] \arrow[d] & G(\mathbb{Q})\backslash G(\mathbb{A})/KK_\infty \arrow[d, "c"] \\
\pi_0(\overline{k}) \arrow[r, equal] & \pi_0(\mathbb{Z}_p^{un}) \arrow[r, equal] & \pi_0(\overline{\mathbb{Q}}_p) \arrow[r, equal] & \pi_0(\overline{\mathbb{Q}}) \arrow[r, equal] & \pi_0(\mathbb{C}) \arrow[r, "\backsim"] & \pi_p.
\end{tikzcd}$$

The rightmost square results from the fact that if we specify $h \in X^{+}$ as a base point, then we can identify the map

$$c : G(\mathbb{Q})\backslash G(\mathbb{A})/KK_\infty \longrightarrow \pi_p$$

induced by the homomorphism $c : G(\mathbb{A}) \to \mathbb{A}^{\times}$ with the canonical map

$$S_K(\mathbb{C}) \longrightarrow \pi_0(\mathbb{C}),$$

see [1]. Using this diagram we can define $c : S_K(F) \to \pi_p$ for any of the rings F.

Definition 2.1. Let F be any of the rings $\overline{k}$, $\mathbb{Z}_p^{un}$, $\overline{\mathbb{Q}}_p$, $\overline{\mathbb{Q}}$, or $\mathbb{C}$. For each character $\overline{\omega} : \pi_p \to \overline{\mathbb{Q}}_\ell^{\times}$, define the operator $L(\omega)$ on $H_c^i(S_K \otimes F\,, \overline{\mathbb{Q}}_\ell)$ to be

$$\bigoplus_{a\in\pi_p} \overline{\omega}(a) : \bigoplus_{a\in\pi_p} H_c^i(c^{-1}(a)\,, \overline{\mathbb{Q}}_\ell) \longrightarrow \bigoplus_{a\in\pi_p} H_c^i(c^{-1}(a)\,, \overline{\mathbb{Q}}_\ell).$$

Next we discuss Hecke correspondences. Let $g \in G(\mathbb{A}_f^p)$, and define $K_g^p = K^p \cap gK^pg^{-1}$. We denote by f the Hecke correspondence on the scheme S_{K^p}

$$S_{K^p} \xleftarrow{R(g)} S_{K_g^p} \xrightarrow{R(1)} S_{K^p}$$

where $R(1)$ is the canonical projection coming from the inclusion $K_g^p \subset K^p$, and $R(g)$ comes from right translation by g; on points $R(g)$ is given by $[A, \lambda, \overline{\eta}] \mapsto [A, \lambda, \overline{\eta g}]$.

Let F be one of the rings $\overline{k}$, $\mathbb{Z}_p^{un}$, $\overline{\mathbb{Q}}_p$, $\overline{\mathbb{Q}}$, or $\mathbb{C}$. If we view $\overline{\mathbb{Q}}_\ell$ as the trivial local system on $S_K \otimes F$, there is a canonical morphism $R(g)^*(\overline{\mathbb{Q}}_\ell) \to R(1)^!(\overline{\mathbb{Q}}_\ell)$ of ℓ-adic local systems on $S_{K_g^p} \otimes F$. Then the correspondence f together with this choice of morphism $R(g)^*(\overline{\mathbb{Q}}_\ell) \to R(1)^!(\overline{\mathbb{Q}}_\ell)$ determines an operator f^{can} on cohomology groups with compact support

$$f = f^{\mathrm{can}} : H_c^i(S_K \otimes F\,,\, \overline{\mathbb{Q}}_\ell) \to H_c^i(S_K \otimes F\,,\, \overline{\mathbb{Q}}_\ell).$$

We can also replace the canonical morphism $\mathrm{can} : R(g)^*(\overline{\mathbb{Q}}_\ell) \to R(1)^!(\overline{\mathbb{Q}}_\ell)$ with the composition

$$R(g)^*(\overline{\mathbb{Q}}_\ell) \xrightarrow{\overline{\omega} \circ c} R(g)^*(\overline{\mathbb{Q}}_\ell) \xrightarrow{\mathrm{can}} R(1)^!(\overline{\mathbb{Q}}_\ell)\ ,$$

where the second map is the canonical one and the first is the map of sheaves over $S_{K_g^p}$ which is given on the stalk over $x \in S_{K_g^p}(F)$ by multiplication by $\overline{\omega} \circ c(R(g)(x))$, where $c : S_K(F) \to \pi_p$ is as above. Thus f together with ω determines a map on cohomology with compact supports which we denote by f^ω.

Lemma 2.2. *As operators on cohomology groups with compact supports* $H_c^i(S_K \otimes F\,,\, \overline{\mathbb{Q}}_\ell)$*, we have*

$$f^\omega \;=\; f \circ L(\omega),$$

where F is $\overline{k}$, $\mathbb{Z}_p^{un}$, $\overline{\mathbb{Q}}_p$, $\overline{\mathbb{Q}}$, or $\mathbb{C}$.

Proof. This follows directly from the definition of how (finite, flat) correspondences induce maps on cohomology with compact supports. See [2], p. 210 for the definition. □

Now let σ_p denote a choice of arithmetic Frobenius in $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and suppose F is $\mathbb{Z}_p^{un}$, $\overline{\mathbb{Q}}_p$, or $\overline{\mathbb{Q}}$. By transport of structure σ_p^{-1} defines an operator on $H_c^i(S_K \otimes F\,,\, \overline{\mathbb{Q}}_\ell)$. Since S_K has good reduction at p, the inertia subgroup at p acts trivially on the cohomology groups $H_c^i(S_K \otimes F\,,\, \overline{\mathbb{Q}}_\ell)$, and so the operator is independent of the choice of σ_p. Thus we have a well-defined operator $\sigma_p^{-1} \circ f^\omega = \sigma_p^{-1} \circ f \circ L(\omega)$ acting on $H_c^i(S_K \otimes F\,,\, \overline{\mathbb{Q}}_\ell)$.

Now suppose $F = \overline{k}$. Let Φ_p be the Frobenius morphism for the k-scheme S_K. We get the correspondence $f_r := \Phi_p^r \circ f$ by replacing the morphism $R(1)$ in the definition of f with $R(1)_r := \Phi_p^r \circ R(1)$. If we use the canonical morphism can : $R(g)^*(\overline{\mathbb{Q}}_\ell) \to R(1)^!_r(\overline{\mathbb{Q}}_\ell)$ (viewing $\overline{\mathbb{Q}}_\ell$ as the trivial local system on S_K), then we get an operator denoted by the same symbol

$$f_r = \Phi_p^r \circ f : H^i_c(S_K \otimes F\,,\, \overline{\mathbb{Q}}_\ell) \longrightarrow H^i_c(S_K \otimes F\,,\, \overline{\mathbb{Q}}_\ell),$$

(the r-th "twist" of f). We can also replace the morphism can : $R(g)^*(\overline{\mathbb{Q}}_\ell) \to R(1)^!_r(\overline{\mathbb{Q}}_\ell)$ with the composition

$$R(g)^*(\overline{\mathbb{Q}}_\ell) \xrightarrow{\overline{\omega}\circ c} R(g)^*(\overline{\mathbb{Q}}_\ell) \xrightarrow{\text{can}} R(1)^!_r(\overline{\mathbb{Q}}_\ell)\ ,$$

where the second map is the canonical one and the first is the map of sheaves over $S_{K^p_g}$ which is defined above. As in the "untwisted" case the pair (f_r, ω) determines an operator on cohomology with compact supports which we denote by $(f_r)^\omega = (\Phi_p^r \circ f)^\omega$. Performing the "twists" by ω and Φ_p^r in the other order yields an operator $(f^\omega)_r = \Phi_p^r \circ f^\omega$, and it is easy to see that $(f_r)^\omega = (f^\omega)_r$. To summarize we have the following consequence of our definitions and the reasoning in the proof of Lemma 2.2.

Lemma 2.3. *As operators on cohomology groups with compact supports* $H^i_c(S_K \otimes \overline{k}\,,\, \overline{\mathbb{Q}}_\ell)$*, we have*

$$\Phi_p^r \circ f^\omega \;=\; (\Phi_p^r \circ f)^\omega \;=\; \Phi_p^r \circ f \circ L(\omega).$$

Remark 2.4. In the case where $F = \mathbb{Z}_p^{un}$, $\overline{\mathbb{Q}}_p$, or $\overline{\mathbb{Q}}$, we can view σ_p^{-1} as a morphism $S_K \otimes F \to S_K \otimes F$ (determined by $\sigma_p^{-1} : F \to F$ via base-change). Thus it makes sense to think of the operator on cohomology $\sigma_p^{-1} \circ f$ as coming from a correspondence, in a way analogous to the case over the finite field. *From now on we will abuse notation and denote this correspondence and its action on* $H^i_c(S_K \otimes F\,,\, \overline{\mathbb{Q}}_\ell)$ *by the symbol* $\Phi_p^r \circ f$. By reasoning analogous to that preceding Lemma 2.3, we therefore have the formulas $\sigma_p^{-r} \circ f = \Phi_p^r \circ f$ and $\sigma_p^{-r} \circ f^\omega = \Phi_p^r \circ f^\omega = (\Phi_p^r \circ f)^\omega$. With our notational conventions the above lemma holds also when $\overline{k}$ is replaced by $F = \mathbb{Z}_p^{un}$, $\overline{\mathbb{Q}}_p$, or $\overline{\mathbb{Q}}$.

We have taken care in the above discussion in order to insure that the base change theorems for étale cohomology with compact supports of S_K can be applied to the situation where the operator $L(\omega)$ is present. These are usually stated for a correspondence such as $\Phi_p^r \circ f$ above, the morphism $R(g)^*(\overline{\mathbb{Q}}_\ell) \to R(1)^!_r(\overline{\mathbb{Q}}_\ell)$ being tacitly assumed to be the canonical one. However, everything works just as well when the morphism determined by ω is used instead, i.e, for the "correspondence" $(\Phi_p^r \circ f)^\omega$ (see the Appendix of [21]). Thus after interpreting our original operator $\sigma_p^{-r} \circ f \circ L(\omega)$ as a kind of correspondence we are able to pass to the special fiber and "count fixed points" of the correspondence over the finite field, in the usual way. This is the content of the next section.

2.2 A first reduction

The base change theorems of étale cohomology with compact supports (see [20], VI, Thm. 3.2) applied to the correspondence $(\Phi_p^r \circ f)^\omega$ imply that

$$\mathrm{Tr}(\Phi_p^r \circ f \circ L(\omega)\ ;\ H^\bullet_c(S_K \otimes \overline{\mathbb{Q}}\,,\, \overline{\mathbb{Q}}_\ell)) \;=\; \mathrm{Tr}(\Phi_p^r \circ f \circ L(\omega)\ ;\ H^\bullet_c(S_K \otimes \overline{k}\,,\, \overline{\mathbb{Q}}_\ell)).$$

Recall that on the right hand side Φ_p denotes a geometric Frobenius. To evaluate the right hand side for sufficiently large r, we will use Deligne's conjecture, which has been proved by Fujiwara [3]. It follows from the statement of Deligne's conjecture on p.212 of [2] that the right hand side of the above equality is given, for sufficiently high values of r (depending on g), by the sum

$$\sum_{[\overline{A},\lambda,\overline{\eta}]} \omega[\overline{A}, \lambda, \overline{\eta g}].$$

Here we sum over fixed points of $\Phi_p^r \circ f$ in $S_{K_g^p}(\overline{k})$ the value of $[\overline{A}, \lambda, \overline{\eta g}] \in S_{K^p}(\overline{k})$ under the map ω. It is easy to see that this can be rewritten as

$$\omega(g) \sum_{[\overline{A},\lambda,\overline{\eta}]} \omega[\overline{A}, \lambda, \overline{\eta}],$$

where here we sum over the same set of fixed points the value of $[\overline{A}, \lambda, \overline{\eta}] \in S_{K_g^p}(\overline{k})$ under ω.

Lemma 2.5. *Without loss of generality, we can assume $\omega(g) = 1$.*

Proof. It is easy to prove by direct calculation on the level of de Rham cohomology classes that $f \circ L(\omega) = \omega(g)L(\omega) \circ f$. The same relation holds on étale cohomology groups. Therefore the virtual trace of $\Phi_p^r \circ f \circ L(\omega)$ is zero unless $\omega(g) = 1$. □

Remark. Looking ahead to the final expression for this sum (Theorem 8.2) we can easily see that if $\omega(g) \neq 1$, then $O_\gamma^\omega(f^p) = 0$, for each orbital integral that appears in the sum. To prove this, note that $f^p = \mathrm{char}(K^p g^{-1} K^p)$ and assume that this orbital integral is nonzero. Then there exists $y \in G(\mathbb{A}_f^p)$ such that $y^{-1}\gamma y \in K^p g^{-1} K^p$. Since $\omega(\gamma) = 1$ (as ω is trivial on $G(\mathbb{A}_f^p)_\gamma$), and $\omega(K^p) = 1$, we see immediately that $\omega(g) = 1$.

We have now achieved the first reduction of our problem: in order to compute the virtual trace of $\Phi_p^r \circ f \circ L(\omega)$ on $H_c^\bullet(S_K \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$ (for sufficiently high r), we need to find a formula for the sum

$$\sum_{[\overline{A},\lambda,\overline{\eta}]} \omega[\overline{A}, \lambda, \overline{\eta}],$$

where we are summing over the fixed point set Fix in $S_{K_g^p}(\overline{k})$ of the correspondence $\Phi_p^r \circ f$. We may group these fixed points together according to $\mathbb{Q}$-isogeny class, and then the first step is to find a formula for the portion of the sum corresponding to a single isogeny class. This is done in the next section.

3 The Fixed Points in a Given Isogeny Class

We need to recall some definitions from §10 of [11], in particular the notion of a *c-polarized virtual abelian variety (A, λ) over k_r up to prime-to-p isogeny.*

A virtual abelian variety A over k_r up to prime-to-p isogeny consists of a pair $A = (\overline{A}, u)$ where $\overline{A}$ is an abelian variety over $\overline{k}$ up to prime-to-p isogeny and $u : \sigma^r(\overline{A}) \to \overline{A}$ is a prime-to-p isogeny (here $\sigma : \overline{k} \to \overline{k}$ is $x \mapsto x^p$ and $\sigma^r(\overline{A})$ is defined by base-changing $\overline{A}$ via σ^r). We define the Frobenius element $\pi_A \in \mathrm{End}(\overline{A})$ by $\pi_A = u \circ \Phi_r$, where $\Phi_r : \overline{A} \to \sigma^r(\overline{A})$ is the relative Frobenius morphism. Thus if $\overline{A}$ is defined over k_r then the canonical u is an isomorphism and π_A is the absolute Frobenius morphism coming from the k_r-structure of $\overline{A}$.

Let c be a rational number which we write as $c = c_0 p^r$. By a c-polarization of A we mean a $\mathbb{Q}$-polarization $\lambda \in \mathrm{Hom}(\overline{A}, \check{\overline{A}}) \otimes \mathbb{Q}$ such that $\pi_A^*(\lambda) = c\lambda$. Equivalently, $u^*(\lambda) = c_0 \sigma^r(\lambda)$. In order for λ to exist it is necessary that $c \in \mathbb{Q}^+$ and that c_0 is a p-adic unit.

As in §10 of [11], attached to (A, λ) we have the $\mathbb{A}_f^p$-Tate module $H_1(\overline{A}, \mathbb{A}_f^p)$, on which π_A acts. Also, we have an L_r-isocrystal $(H(A), \Phi)$ which is defined to be the u-fixed points in the L-isocrystal $(H(\overline{A}), \Phi)$ dual to $H^1_{crys}(\overline{A}/W(\overline{k})) \otimes L$ together with its σ-linear bijection. The morphism π_A also induces endomorphisms of $H(\overline{A})$ and $H(A)$. On $H(\overline{A})$ we have the identification $\Phi^r = \pi_A^{-1} \circ u$, hence on $H(A)$ we have $\Phi^r = \pi_A^{-1}$.

Each fixed point $[\overline{A}', \lambda', \overline{\eta}'] \in S_{K_g^p}(\overline{k})$ of the correspondence $\Phi_p^r \circ f$ gives rise to a c-polarized virtual abelian variety over k_r up to prime-to-p isogeny (A', λ') (see §16 of [11]). Let us fix one (A, λ) coming from a fixed point and let S denote the set of those fixed points $[\overline{A'}, \lambda', \overline{\eta'}] \in S_{K_g^p}(\overline{k})$ which have the property that (A', λ') and (A, λ) are $\mathbb{Q}$-isogenous. The purpose of this section is to prove a formula for

$$\sum_{[\overline{A'}, \lambda', \overline{\eta'}] \in S} \omega[\overline{A'}, \lambda', \overline{\eta'}].$$

Recall that in the statement of the moduli problem for S_{K^p} we are given V with a symplectic pairing $\langle \cdot, \cdot \rangle$ and a self-dual $\mathbb{Z}_p$-lattice $\Lambda_0 \subseteq V \otimes \mathbb{Q}_p$. Now we choose symplectic similitudes

$$\begin{aligned} \beta^p \quad &: \quad H_1(\overline{A}, \mathbb{A}_f^p) \xrightarrow{\sim} V \otimes \mathbb{A}_f^p \\ \beta_r \quad &: \quad H(A) \xrightarrow{\sim} V \otimes L_r. \end{aligned}$$

Here is what we mean by the term "symplectic similitudes": $H_1(\overline{A}, \mathbb{A}_f^p)$ is the $\mathbb{A}_f^p$-Tate module for the $\overline{k}$-abelian variety $\overline{A}$. We can regard the pairing $(\cdot, \cdot)_\lambda$ on this space induced by the polarization λ as having values in $\mathbb{A}_f^p$ as follows: the pairing originally takes values in $\mathbb{A}_f^p(1)(\overline{k})$. Using the data $\mathcal{D}$ we can identify this with $\mathbb{A}_f^p$ as in §1.4, and thus we can consider the pairings on both sides as $\mathbb{A}_f^p$-valued. We call β^p a symplectic similitude if it preserves the pairings up to a scalar: $(\beta^p)^* \langle \cdot, \cdot \rangle = c(\beta^p)(\cdot, \cdot)_\lambda$, for some $c(\beta^p) \in (\mathbb{A}_f^p)^\times$. The property of being a symplectic similitude is independent of the choice of data $\mathcal{D}$, although of course the value of $c(\beta^p)$ is not. Furthermore, the polarization λ induces an L-valued pairing $(\cdot, \cdot)_\lambda$ on the L-isocrystal $(H(\overline{A}), \Phi)$ (see §10 of [11]). Once we choose $d \in \mathcal{O}_L^\times$ such that $d^{-1}\sigma^r(d) = c_0$, then the pairing $d(\cdot, \cdot)_\lambda$ is L_r-valued on the L_r-isocrystal $(H(A), \Phi)$, and this pairing is well-defined up to an element of $\mathcal{O}_{L_r}^\times$. We call β_r a symplectic similitude if there exists $c(\beta_r) \in L_r^\times$ such that

$$\beta_r^*\langle\cdot,\cdot\rangle \;=\; c(\beta_r) d(\cdot,\cdot)_\lambda.$$

Again the propery of being a symplectic similitude is independent of the choice of d, but the value of $c(\beta_r) \in L_r^\times$ is not. However, the image of $c(\beta_r)$ in $L_r^\times/\mathcal{O}_{L_r}^\times$ is independent of the choice of d. We will often simply write β for the pair (β^p, β_r).

Let $I = \mathrm{Aut}(A, \lambda)$, and define X^p to be the set of K_g^p-orbits of symplectic similitudes

$$\eta : V \otimes \mathbb{A}_f^p \;\longrightarrow\; H_1(\overline{A}, \mathbb{A}_f^p)$$

such that $\pi_A \eta \equiv \eta g \pmod{K^p}$. Similarly we define X_p to be the set of lattices $\Lambda \subset H(A)$ which are self-dual up to a scalar in $L_r^\times$, and for which we have

$$p^{-1}\Lambda \supset \Phi\Lambda \supset \Lambda.$$

We will consider the following diagram:

$$\begin{tikzcd}
S \arrow[r, "\sim"] \arrow[d, "c"'] & I(\mathbb{Q})\backslash[X^p \times X_p] \arrow[d, "c^p \times c_p"'] \\
\mathbb{Z}_{(p)}^+\backslash(\mathbb{A}_f^p)^\times/c(K^p) \arrow[r, "\sim"] & \mathbb{Q}^+\backslash\mathbb{A}_f^\times/c(K^p)\mathbb{Z}_p^\times
\end{tikzcd}$$

We will need to discuss each of the four maps in turn.
1. The top map may be viewed as the first step in the usual process of writing the fixed points of a correspondence in a single isogeny class in terms of cosets for the groups $G(\mathbb{A}_f^p)$ and $G(L_r)$. We describe it explicitly in the following way. First choose a $\mathbb{Q}$-isogeny $\phi : (A', \lambda') \to (A, \lambda)$. Then we define the map to be

$$[\overline{A'}, \lambda', \overline{\eta'}] \;\mapsto\; [H_1(\phi) \circ \overline{\eta'}, H(\phi) D(A')].$$

Here $D(A') \subseteq H(A')$ is the usual lattice inside the L_r-vector space $H(A')$ (see §10 of [11]). This map is clearly independent of the choice of ϕ.
2. The left map is given as follows: We define $c([\overline{A'}, \lambda', \overline{\eta'}])$ by considering the element c of $\mathbb{Z}_{(p)}^+\backslash(\mathbb{A}_f^p)^\times/c(K_g^p)$ such that

$$\eta'^*(\cdot,\cdot)_{\lambda'} \;=\; c\langle\cdot,\cdot\rangle.$$

Here $(\cdot,\cdot)_{\lambda'}$ is the $\mathbb{A}_f^p$-valued pairing (using $\mathcal{D}$ to consider it as such) on $H_1(\overline{A'}, \mathbb{A}_f^p)$ coming from λ', and $\langle\cdot,\cdot\rangle$ is the given pairing on V. Take $c([\overline{A'}, \lambda', \overline{\eta'}])$ to be the image of c in $\mathbb{Z}_{(p)}^+\backslash(\mathbb{A}_f^p)^\times/c(K^p)$.
3. The right map: Consider a K_g^p-orbit of symplectic similitudes

$$\overline{\eta} : V \otimes \mathbb{A}_f^p \;\longrightarrow H_1(\overline{A}, \mathbb{A}_f^p),$$

and a lattice $\Lambda \subset H(A)$ which is self-dual up to a scalar. Then $c_g^p(\overline{\eta}) \in (\mathbb{A}_f^p)^\times/c(K_g^p)$ is defined by the equality

$$\overline{\eta}^*(\cdot,\cdot)_\lambda \;=\; c_g^p(\overline{\eta})\langle\cdot,\cdot\rangle,$$

and $c^p(\overline{\eta})$ is the image of this element in $(\mathbb{A}_f^p)^\times/c(K^p)$. In a similar manner we define $c_p(\Lambda) \in L_r^\times/\mathcal{O}_{L_r}^\times \;=\; \mathbb{Q}_p^\times/\mathbb{Z}_p^\times$ by

$$\Lambda^\perp \;=\; c_p(\Lambda)^{-1}\Lambda.$$

4. The bottom isomorphism is induced by the canonical inclusion of $\mathbb{A}_f^{p\times}$ into $\mathbb{A}_f^\times$.
One can check easily that this diagram commutes. Note also that the entire diagram is independent of the choice of β.

Remark 3.1. The commutativity of this diagram is related to the commutativity of the diagram at the beginning of §2 which was used to define the map $c : S_K(\overline{k}) \to \pi_p$. In fact under the indentification of $\pi_0(S_K)(\overline{k}) \cong \pi_p$, the left vertical map $c : S \to \pi_p$ above is simply the restriction to S of the map $S_{K_g^p}(\overline{k}) \to \pi_0(S_{K_g^p})(\overline{k}) \to \pi_0(S_{K^p})(\overline{k})$. This provides an interpretation of $c : S_K \to \pi_0(S_K)$ "on points" and explains why we used the same symbol c for the vertical map in the above diagram.

Next we will consider a second diagram wherein the effect of altering β comes clearly into view. Using β^p we can transport the action (by symplectic similitudes) of π_A^{-1} on $H_1(\overline{A}, \mathbb{A}_f^p)$ over to $V \otimes \mathbb{A}_f^p$. Denote the resulting element of $G(\mathbb{A}_f^p)$ by γ. Similarly use β_r and the σ-linear bijection $\Phi : H(A) \to H(A)$ to define $\delta\sigma$, where $\delta \in G(L_r)$. Note that γ and δ depend on our choice of β, but altering β does not change the $G(\mathbb{A}_f^p)$-conjugacy class of γ or the σ-conjugacy class of δ in $G(L_r)$.
Define the sets:

$$\begin{aligned} Y^p &= \left\{ \overline{y} \in G(\mathbb{A}_f^p)/K_g^p \;\middle|\; y^{-1}\gamma y \in K^p g^{-1} \right\}, \\ Y_p &= \left\{ \overline{x} \in G(L_r)/K_r \;\middle|\; x^{-1}\delta\sigma(x) \in K_r\sigma(a)K_r \right\}. \end{aligned}$$

Here K_r is the stabilizer in $G(L_r)$ of the lattice $\Lambda_0 \otimes L_r$, and $a = \mu_1(p)^{-1}$, where μ_1 comes from $h \in X_\infty$ as on p.430 of [11]. The double coset $K_r\sigma(a)K_r$ does not depend on the choice of h. We note that the sets Y^p and Y_p both depend on the choice of β, since γ and δ do. We need to explain the following diagram:

$$\begin{tikzcd} I(\mathbb{Q})\backslash[X^p \times X_p] \arrow[r, "\sim"] \arrow[d, "c^p \times c_p"'] & \beta I(\mathbb{Q})\beta^{-1}\backslash[Y^p \times Y_p] \arrow[d, "c"'] \\ \mathbb{Q}^+\backslash\mathbb{A}_f^\times/c(K^p)\mathbb{Z}_p^\times \arrow[r, "\sim", "c(\beta^p)\overline{c}(\beta_r)"'] & \mathbb{Q}^+\backslash\mathbb{A}_f^\times/c(K^p)\mathbb{Z}_p^\times \end{tikzcd}$$

1. The top isomorphism is $[\overline{\eta}, \Lambda] \mapsto [yK_g^p, xK_r]$. Here y is any representative for the K_g^p-orbit of symplectic similitudes

$$\beta^p\overline{\eta} \;:\; V \otimes \mathbb{A}_f^p \;\longrightarrow V \otimes \mathbb{A}_f^p.$$

Note that $\beta_r(\Lambda)$ is a lattice in $V \otimes L_r$ which is self-dual up to a scalar. By Corollary 7.3 of [11], there is a uniquely determined $\overline{x} \in G(L_r)/K_r$ such that $\beta_r(\Lambda) = x(\Lambda_0)$. This gives us the coset xK_r.
2. The right map is $[yK_g^p, xK_r] \mapsto [c(y)\overline{c}(x)]$. Here $c(y) \in \mathbb{A}_f^{p\times}/c(K_g^p)$ and $\overline{c}(x) \in \mathbb{Q}_p^\times/\mathbb{Z}_p^\times$, the image of $c(x) \in L_r^\times/\mathcal{O}_{L_r}^\times$ under the identification

$$L_r^\times/\mathcal{O}_{L_r}^\times \;=\; \mathbb{Z} \;=\; \mathbb{Q}_p^\times/\mathbb{Z}_p^\times.$$

The brackets $[\ldots]$ mean take the class of $(c(y), \overline{c}(x))$ in $\mathbb{Q}^+\backslash\mathbb{A}_f^p/c(K^p)\mathbb{Z}_p^\times$.
3. The bottom isomorphism is multiplication by $c(\beta^p)\overline{c}(\beta_r)$. Here $\overline{c}(\beta_r)$ denotes the image of $c(\beta_r) \in L_r^\times/\mathcal{O}_{L_r}^\times$ under the identification $L_r^\times/\mathcal{O}_{L_r}^\times = \mathbb{Q}_p^\times/\mathbb{Z}_p^\times$.
One can easily check that this second diagram commutes, keeping in mind the identities

$$\begin{aligned} (x\Lambda_0)^\perp &= c(x)^{-1}(x\Lambda_0) \quad (\forall\, x \in G(L_r)), \\ (\beta_r\Lambda)^\perp &= c(\beta_r)^{-1}c_p(\Lambda)^{-1}(\beta_r\Lambda). \end{aligned}$$

Since $x\Lambda_0 = \beta_r\Lambda$ (by definition of x), we see that $c(x) = c(\beta_r)c_p(\Lambda)$ in $L_r^\times/\mathcal{O}_{L_r}^\times$. The rest of the verification is easy.

Now as in §16 of [11], we let dy denote the Haar measure on $G(\mathbb{A}_f^p)$ giving K_g^p measure 1. Let dx denote the Haar measure on $G(L_r)$ giving K_r measure 1. Let $\tilde{f}^p$ denote the characteristic function of $K^p g^{-1}$, and let $\tilde{\phi}_r$ denote the characteristic function of $K_r\sigma(a)K_r$. We use the Haar measure on the discrete group $\beta I(\mathbb{Q})\beta^{-1}$ that give points measure 1. Now using the commutativity of the two diagrams above and standard calculations (keeping in mind that K^p is sufficiently small that the moduli problem S_{K^p} has no nontrivial automorphisms) we can conclude with the following theorem:

Theorem 3.2. *In the notation above the sum*

$$\sum_{[\overline{A'}, \lambda', \overline{\eta'}] \in S} \omega[\overline{A'}, x, \overline{\eta'}]$$

can be written as

$$\overline{\omega}\left(c(\beta^p)\overline{c}(\beta_r)\right)^{-1} \int_{\beta I(\mathbb{Q})\beta^{-1}\backslash[G(\mathbb{A}_f^p)\times G(L_r)]} \overline{\omega}\left(c(y)\overline{c}(x)\right) \tilde{f}^p(y^{-1}\gamma y)\tilde{\phi}_r(x^{-1}\delta\sigma(x))\, dydx.$$

To avoid overly complicated notation from this point on, we write the integral more simply as

$$\omega(\beta^p\beta_r)^{-1} \int_{I(\mathbb{Q})\backslash[G(\mathbb{A}_f^p)\times G(L_r)]} \omega(y)\omega(x)\tilde{f}^p(y^{-1}\gamma y)\tilde{\phi}_r(x^{-1}\delta\sigma(x))\, dydx,$$

and we write the sum above as $T^\omega(A, \lambda)$.

4 Construction of Some Useful Maps

4.1

For this section we make a slight change in our notation. Let F denote a p-adic field, and L the completion of the maximal unramified extension of F in some algebraic closure $\overline{F}$. Let $\overline{L}$ be an algebraic closure of L which contains $\overline{F}$. Let $\Gamma = \mathrm{Gal}(\overline{F}/F)$, and denote by σ the Frobenius automorphism of L over F. If G is any connected reductive F-group let $B(G)$ denote the pointed set of σ-conjugacy classes in $G(L)$, as defined by Kottwitz in [7].

Let T be any F-torus. In §2 of [7] there is the construction of a canonical functorial isomorphism

$$B(T) \xrightarrow{\sim} X_*(T)_\Gamma.$$

This is canonical up to the choice of generator of $\mathbb{Z}$. Note that $B(\mathbb{G}_m) = \mathbb{Z}$ and that the isomorphism is determined by functoriality once we set the convention that $X_*(\mathbb{G}_m) \to B(\mathbb{G}_m)$ be $\mu \mapsto [\mu(p)]$. To get the other isomorphism we would have used the other generator of $B(\mathbb{G}_m)$, namely the σ-conjugacy class of elements of $L^\times$ having normalized valuation -1. This would give a different functorial isomorphism, and the map $X_*(\mathbb{G}_m) \to B(\mathbb{G}_m)$ would then be $\mu \mapsto [\mu(p^{-1})]$.

The aim of this section is to define groups $B^\pi(T_1 \to T)$ and $X(T_1 \to T)$ attached to any exact sequence of F-tori

$$1 \longrightarrow T_1 \longrightarrow T \xrightarrow{c} \mathbb{G}_m \longrightarrow 1$$

and to construct an isomorphism $B^\pi(T_1 \to T) \longrightarrow X(T_1 \to T)$ that will depend on a tower of uniformizers Π. This construction was inspired by a similar one due to Reimann and Zink [27], and the point here is to put their construction into a group-theoretic framework.

We proceed to the relevant definitions. Fix a uniformizer π of L. Let us consider the collection of all finite Galois extensions E of L in $\overline{L}$. Suppose $\Pi = \{\pi_E\}$ is a collection of uniformizers π_E of E, one for each E.

Definition 4.1. We call Π a tower over π if

1. $E' \supseteq E \Rightarrow Nm_{E'/E}(\pi'_E) = \pi_E$,

2. $\pi_L = \pi$.

Lemma 4.2. *For each π, there is a tower Π over π.*

Proof. If we are given $E' \supset E$ and a uniformizer π_E of E, then there exists a uniformizer $\pi_{E'}$ of E' such that $Nm_{E'/E}(\pi_{E'}) = \pi_E$. To see this use the fact that $Nm_{E'/E}(U_{E'}) = U_E$, a consequence of the fact that E'/E is totally ramified (see V§6 of [28]). Therefore we will be done if we can show that $\overline{L}/L$ is exhausted by a countable family of finite Galois extensions E_i/L. But F, being p-adic, has only a finite number of Galois extensions of given degree, so $\overline{F}/F$ is exhausted by a countable collection of finite Galois extensions F_i/F, and thus so is $\overline{F}/F^{un}$ (use

F_iF^{un}). But now our lemma follows from the observation that $\overline{L}/L$ and $\overline{F}/F^{un}$ have isomorphic lattices of intermediate fields, since $\mathrm{Gal}(\overline{L}/L) = \mathrm{Gal}(\overline{F}/F^{un})$. $\square$

Consider next an exact sequence of F-tori

$$1 \longrightarrow T_1 \longrightarrow T \xrightarrow{c} \mathbb{G}_m \longrightarrow 1\,.$$

This is still exact on L-points, as $H^1(L, T_1) = (1)$.

Definition 4.3. For each uniformizer π of L, define

1. $T(L)_\pi \ = \ c^{-1}(\pi^{\mathbb{Z}})$,
2. $B^\pi(T_1 \to T) \ = \ T(L)_\pi / (A\langle\sigma\rangle T_1(L))$.

Here $A\langle\sigma\rangle$ denotes the augmentation ideal for the infinite cyclic group $\langle\sigma\rangle$. Therefore we get an exact sequence

$$1 \longrightarrow T_1(L) \longrightarrow T(L)_\pi \xrightarrow{c} \pi^{\mathbb{Z}} \longrightarrow 1\,,$$

and by dividing out each of the first two groups by $A\langle\sigma\rangle T_1(L)$, we get the exact sequence

$$1 \longrightarrow B(T_1) \longrightarrow B^\pi(T_1 \to T) \xrightarrow{c} \pi^{\mathbb{Z}} \longrightarrow 1\,.$$

Definition 4.4.

$$X(T_1 \to T) \ = \ \frac{X_*(T)}{A(\Gamma)X_*(T_1)}.$$

Here $A(\Gamma)$ denotes the augmentation ideal for the group Γ. Now an argument similar to the one above shows that there is an exact sequence

$$0 \longrightarrow X_*(T_1)_\Gamma \longrightarrow X(T_1 \to T) \xrightarrow{c} X_*(\mathbb{G}_m) \longrightarrow 0\,.$$

Claim 4.5. For any tower Π over π and exact sequence of F-tori as above, we can define a group homomorphism

$$K_\Pi \ : \ X_*(T) \ \longrightarrow \ B^\pi(T_1 \to T)$$

as follows: Choose any E/L that splits T (and thus T_1) and let K_Π send $\mu \in X_*(T)$ to the class of $Nm_{E/L}(\mu(\pi_E))$ in $B^\pi(T_1 \to T)$. It is easy to see that this is well-defined using the properties (1) and (2) in Definition 4.1.

Proposition 4.6. *K_Π has the following properties:*

1. *K_Π factors through $X(T_1 \to T)$.*

2. *The following diagram commutes:*

$$\begin{tikzcd}
0 \arrow[r] & X_*(T_1)_\Gamma \arrow[r] \arrow[d] & X(T_1 \to T) \arrow[r, "c"] \arrow[d, "K_\Pi"] & X_*(\mathbb{G}_m) \arrow[r] \arrow[d] & 0 . \\
1 \arrow[r] & B(T_1) \arrow[r] & B^\pi(T_1 \to T) \arrow[r, "c"] & \pi^{\mathbb{Z}} \arrow[r] & 1
\end{tikzcd}$$

where the left vertical arrow is Kottwitz' map in [7], and the right vertical arrow is $n \mapsto \pi^n$*.*

Proof. Restricting K_Π to $X_*(T_1)$ gives exactly Kottwitz' map $X_*(T_1) \to B(T_1)$, as is seen easily from the definition. Since Kottwitz' map is trivial on $A(\Gamma)X_*(T_1)$, so is K_Π. This proves (1) and also the commutativity of the left square of (2). The commutativity of the right follows from the functoriality (with respect to T) of the map $X_*(T) \to T(L)$ given by Π. □

Corollary 4.7.

$$K_\Pi : X(T_1 \to T) \to B^\pi(T_1 \to T)$$

is an isomorphism.

Proof. The other two vertical arrows above are isomorphisms. □

Now we want to construct a generalization of K_Π^{-1} that works for connected reductive groups instead of just tori. Consider an exact sequence of connected reductive F-groups

$$1 \longrightarrow I_1 \longrightarrow I \xrightarrow{c} \mathbb{G}_m \longrightarrow 1.$$

Definition 4.8.

$$I(L)_\pi \;=\; c^{-1}\left(\pi^{\mathbb{Z}}\right).$$

Definition 4.9.

$$B^\pi(I_1 \to I) \;=\; \frac{I(L)_\pi}{A\langle\sigma\rangle I_1(L)}.$$

Definition 4.10.

$$X(I_1 \to I) \;=\; \frac{X^*(Z(\hat{I}))}{A(\Gamma)X^*(Z(\hat{I}_1))}.$$

In the second definition we are abusing notation, and the quotient is meant to denote the equivalence relation

$$x \;\sim\; y \Longleftrightarrow x \;=\; z^{-1}y\sigma(z)$$

for some $z \in I_1(L)$. Thus $B^\pi(I_1 \to I)$ is only a pointed set and not a group in general. In the third definition $\hat{I}$ denotes the connected dual group (over $\mathbb{C}$). If I is a torus this definition agrees with the earlier one.

We want to construct a functorial surjection of pointed sets

$$M_\Pi : B^\pi(I_1 \to I) \longrightarrow X(I_1 \to I)$$

which agrees with K_Π^{-1} when I is a torus. To do so, first assume that $I^{\mathrm{der}} = I^{\mathrm{sc}}$. Let $D = I/I^{\mathrm{der}}$, and $D_1 = I_1/I^{\mathrm{der}}$. Note that D and D_1 are both tori. Also, c induces a map $\overline{c} : D \to \mathbb{G}_m$. We have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
1 \arrow[r] & I_1 \arrow[r] \arrow[d] & I \arrow[r, "c"] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d, "\mathrm{id}"'] & 1 \\
1 \arrow[r] & D_1 \arrow[r] & D \arrow[r, "\overline{c}"] & \mathbb{G}_m \arrow[r] & 1
\end{tikzcd}$$

On L-points the sequences are still exact, and $I(L)_\pi \to D(L)_\pi$ is a surjection. Since I^{der} is simply connected, we know that the restriction map $X^*(Z(\hat{I})) \to X^*(\hat{D})$ is an isomorphism, and it follows from this that the same is true with I_1 replacing I and D_1 replacing D. Thus $X(I_1 \to I) = X(D_1 \to D)$ and we can define

$$M_\Pi : I(L)_\pi \longrightarrow X(I_1 \to I)$$

to be the unique map making the following diagram commute

$$\begin{tikzcd}
I(L)_\pi \arrow[r] \arrow[d] & X(I_1 \to I) \arrow[d, equal] \\
D(L)_\pi \arrow[r, "K_\Pi^{-1}"] & X(D_1 \to D).
\end{tikzcd}$$

The map M_Π is obviously functorial in exact sequences like the one containing I_1 and I, and it is equally clear that it is a surjective group homomorphism which agrees with K_Π^{-1} when I is a torus. By properties of K_Π^{-1} already mentioned we can deduce that M_Π factors through $B^\pi(I_1 \to I)$.

Remark. In the applications of this construction we will be concerned with the case where I is the centralizer of a semisimple element of $G(\mathbb{Q})$ and where $I_1 = I \cap G^{\mathrm{sc}}$. Since $G = \mathrm{GSp}(2g)$ we know not only that I and I_1 are connected reductive groups but also that $I^{\mathrm{der}} = I_1^{\mathrm{der}}$ is simply connected, so that the above construction will suffice for our purposes. However, the construction can be made to work for I_1 and I arbitrary connected reductive groups by passing to a z-extension I' of I: we end up replacing the sequence with I and I_1 with one involving I' and $I_1' = \ker(I' \longrightarrow I \xrightarrow{c} \mathbb{G}_m)$, and then using the usual argument.

4.2 The Theorem of Reimann and Zink

In this section we will recall the main result of [27]. We will first change our notation once again. We will now denote by F a CM-algebra with involution

$*$. By this we mean that F is a product of totally imaginary extensions of totally real number fields, and that $*$ induces complex conjugation on each factor of $F \otimes \mathbb{R} \cong \mathbb{C} \times \cdots \times \mathbb{C}$. Let L denote the completion of a maximal unramified extension of $\mathbb{Q}_p$ as in §1. Recall that we have fixed $i = \sqrt{-1} \in \mathbb{C}$ and embeddings j_p, j_∞. Given these data, Reimann and Zink define an element $g \in L(\sqrt{p})$ as follows. Let $\gamma \in \mathbb{C}$ be the unique solution in $\mathbb{R}^+ \amalg i\mathbb{R}^+$ for the equation $\gamma^2 = (-1)^{(p-1)/2}p$. Using j_p and j_∞ we regard γ as an element of $\overline{L}$. Let

$$\left[\left(\frac{p-1}{2}\right)!\right] \in W(\overline{k})$$

be the unique root of unity which is congruent modulo p to $((p-1)/2)!$. We set

$$g = -\left[\left(\frac{p-1}{2}\right)!\right]^{-1} \gamma.$$

Note that $g^2 = -p$, so that $Nm_{L(\sqrt{p})/L}(g) = p$.
Now to our CM-algebra F we associate the $\mathbb{Q}$-torus $T = \{x \in F^\times | xx^* \in \mathbb{Q}^\times\}$. Let T_1 be the $\mathbb{Q}$-torus $\{x \in F^\times | xx^* = 1\}$. Let

$$S \subset \mathrm{Hom}_{\mathbb{Q}}(F, \overline{\mathbb{Q}}_p)$$

be a CM-type for F, with associated cocharacter $\mu_S \in X_*(T_{\overline{\mathbb{Q}}_p})$. Using j_p, j_∞ we can also consider this to be in $X_*(T_{\mathbb{C}})$.

Lemma 4.11. *For any such F, T, μ_S as above,*

$$K_\Pi(\mu_S) \in B^p(T_1 \to T)$$

is independent of the choice of tower Π over g.

Proof. This is proved, line by line, by following the argument of Reimann and Zink on p.472 of [27]. □

Remark. In fact an inspection of the argument cited above shows that we may replace g with any uniformizer in $L(\sqrt{p})$. This yields the following:

Corollary 4.12. *Fix $\pi_{L(\sqrt{p})}$ a uniformizer of $L(\sqrt{p})$ with $Nm(\pi_{L(\sqrt{p})}) = \pi$, where π is a uniformizer of L. Let F and T be as above. Then*

$$K_\Pi(\mu) \in B^\pi(T_1 \to T)$$

is independent of the choice of tower Π over $\pi_{L(\sqrt{p})}$, for all $\mu \in X_*(T_{\overline{\mathbb{Q}}_p})$.

Proof. Consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & X_*(T_1)_{\Gamma(p)} \arrow[r] \arrow[d] & X(T_1 \to T) \arrow[r, "c"] \arrow[d, "K_\Pi"'] & X_*(\mathbb{G}_m) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B(T_1) \arrow[r] & B^\pi(T_1 \to T) \arrow[r, "c"] & (\pi)^{\mathbb{Z}} \arrow[r] & 1
\end{tikzcd}$$

Note that the unlabelled vertical arrows are both isomorphisms and are independent of the choice of Π over $\pi_{L(\sqrt{p})}$. Also, both these exact sequences split, and there are particular elements that give the splitting, namely μ_S and $K_\Pi(\mu_S)$, since $c[\mu_S] = id$, and $c[K_\Pi(\mu_S)] = \pi$. Because these elements are independent of Π (over $\pi_{L(\sqrt{p})}$), the splittings are independent of the choice Π, and every K_Π preserves the splittings. Thus K_Π, as a map, is independent of the choice of Π over $\pi_{L(\sqrt{p})}$. $\square$

Remark. ¿From now on, when considering tori T as above and some uniformizer $g' = \pi_{L(\sqrt{p})} \in L(\sqrt{p})$, we will use the notation $K_{g'}$ in place of K_Π, where Π is a tower over g'. The corollary above justifies this notation.

We now recall the statement of the main theorem of [27]. Suppose A is an abelian variety over the integers of a p-adic field K, and let $T_p(A) = H_1(A_{\overline{\mathbb{Q}}_p}, \mathbb{Z}_p)$, the p-adic Tate module of $A_{\overline{\mathbb{Q}}_p}$. Let $\overline{A}$ denote the geometric special fiber $A_{\overline{k}}$. We can associate to $\overline{A}$ an L-isocrystal $(H(\overline{A}), \Phi)$; by definition this is the dual of the L-isocrystal $(H^1_{crys}(\overline{A}/W(\overline{k})) \otimes L, Fr)$, where Fr denotes the Frobenius map on crystalline cohomology. Suppose that A has complex multiplication by the CM-algebra F, and that μ_{CM} is the associated cocharacter of the $\mathbb{Q}$-torus $T = \{x \in F^\times | xx^* \in \mathbb{Q}^\times\}$. Suppose also that A has a polarization $\lambda : A \to \hat{A}$ which commutes with the F-action; this gives pairings $(\cdot,\cdot)_\lambda$ on $H(\overline{A})$ and $T_p(A) \otimes L$. The pairing on $T_p(A) \otimes \mathbb{Q}$ is a priori $\mathbb{Q}_p(1)(\overline{\mathbb{Q}}_p)$-valued, but it can viewed as $\mathbb{Q}_p$-valued using our data $\mathcal{D}$ and $\exp(2\pi i x)$ for our fixed choice of $i \in \mathbb{C}$, as explained in §1. Choose a symplectic isomorphism over L

$$H(\overline{A}) \xrightarrow{\sim} T_p(A) \otimes L$$

and use it to define $b_0 \in B^p(T_1 \to T)$ by the equality

$$\Phi^{-1} \ = \ \sigma^{-1} b_0.$$

Theorem 4.13. *(***Reimann-Zink***) With the particular element g and the other notation as above, $K_g[\mu_{CM}] = [b_0]$.*

Proof. See Satz 1.8 of [27]. $\square$

We remark that the hypothesis $p > 2$ is necessary in the results of this paper only because the same restriction occurs in the theorem of Reimann and Zink above.

4.3

Suppose I is the centralizer of a semisimple element of $G(\mathbb{Q})$ and that T is a maximal $\mathbb{Q}$-torus in I. We showed in the previous section (§4.2) that if the torus is the group of automorphisms of a CM-algebra with involution $*$, then the maps K_Π for T depend only on the choice of $\pi_{L(\sqrt{p})}$. We will need to know that the same is true for the maps M_Π defined for the groups I. First we remark that I always contains such a torus T; the reader not familiar with this fact can look at the proof of the Main Theorem in §8, where such a torus inside I is constructed. Next consider the diagram

$$\begin{tikzcd} T(L)_\pi \arrow[r, hook] \arrow[d, "K_\Pi^{-1}"'] & I(L)_\pi \arrow[d, "M_\Pi"'] \\ X(T_1 \to T) \arrow[r] & X(I_1 \to I) \end{tikzcd}$$

Proposition 4.14. *M_Π is uniquely determined on such I by $\pi_{L(\sqrt{p})}$.*

Proof. Let $x \in I(L)_\pi$. Choose $t \in T(L)_\pi$ having the same image in $\pi^{\mathbb{Z}}$ as x. Thus $t^{-1}x \in I_1(L)$. Note that restricting M_Π to $I_1(L)$ gives the canonical map

$$w_{I_1} : I_1(L) \longrightarrow X^*(Z(\hat{I}_1))_\Gamma$$

constructed in §7 of [12]. (To check this we use that $I_1/I_1^{\text{der}} \to D_1$, in the notation used earlier.) Now our result follows from the equation

$$M_\Pi(x) = K_\Pi^{-1}(t) + w_{I_1}(t^{-1}x).$$

□

Remark. From now on, when we consider such a centralizer I, a uniformizer $g' \in L(\sqrt{p})$, and a tower Π over g', we will write $M_{g'}$ instead of M_Π. The proposition above justifies this notation.

We will use the maps M_g, for the particular element g of Reimann-Zink constructed above, to define the invariants $\alpha_1(\gamma_0; \gamma, \delta)$ in section 5.

5 The Definition of the Refined Invariant

5.1

The purpose of this section is to explain how to attach to any triple $(\gamma_0; \gamma, \delta)$ satisfying the conditions in §2 of [9] an invariant $\alpha_1(\gamma_0; \gamma, \delta)$. This will be called the refined invariant because it will be related to the invariant $\alpha(\gamma_0; \gamma, \delta)$ defined by Kottwitz in [9] in the following way: it will be an element of a certain group which maps naturally to the group $\mathfrak{k}(I_0/\mathbb{Q})^D$ to which $\alpha(\gamma_0; \gamma, \delta)$ belongs, and furthermore $\alpha(\gamma_0; \gamma, \delta)$ will be the image of $\alpha_1(\gamma_0; \gamma, \delta)$.

We will recall first the conditions imposed on the triples. A triple $(\gamma_0; \gamma, \delta)$ is supposed to satisfy the following conditions:

1. $\gamma_0 \in G(\mathbb{Q})$ is semisimple and $\mathbb{R}$-elliptic.
2. $\gamma = (\gamma_l)_l \in G(\mathbb{A}_f^p)$ and for every $l \neq p, \infty$, γ_l is stably conjugate to γ_0.
3. $\delta \in G(L_r)$ and $N\delta = \delta\sigma(\delta)\ldots\sigma^{r-1}(\delta)$ is stably conjugate to γ_0.

Further we assume that the image of $[\delta]$ under the canonical map

$$B(G_{\mathbb{Q}_p}) \longrightarrow X^*(Z(\hat{G})^{\Gamma(p)})$$

is the negative of the restriction to $Z(\hat{G})^{\Gamma(p)}$ of $\mu_1 \in X^*(Z(\hat{G}))$, where μ_1 is the element attached to any $h \in X_\infty$ as in [9]. It is easy to see that since $G = \mathrm{GSp}(2g)$ this condition on δ can be rephrased as: $c(\delta) = p^{-1}u^{-1}$, where $u \in \mathcal{O}_{L_r}^\times$.

To which group should the refined invariant belong? To answer this write $I_0 = G_{\gamma_0}$ and $I_1 = I_0 \cap G^{sc}$. The reader who is familiar with the stabilization of the twisted trace formula in [13] might guess that this group should be $X^*(Z(\hat{I}_1)^\Gamma)$, because the element $\alpha_1(\gamma_0;\gamma,\delta)$ should play the same role as $\mathrm{obs}(\delta)$ of that paper, which for δ regular with centralizer the maximal torus T_0 and with $T_1 = T_0 \cap G^{sc}$, belongs to the group $H^1(\mathbb{A}/\mathbb{Q}, T_1) \cong X^*(\hat{T}_1^\Gamma)_{\mathrm{tor}}$. However, we were not able to get a well-defined element of this group from an arbitrary triple $(\gamma_0;\gamma,\delta)$, but only for those for which $c(\delta) = p^{-1}$. This seems to be due to the fact that we only consider the case of Shimura varieties with good reduction at p, meaning that in particular K_p is a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$. There are several noncanonical ways to produce an element of $X^*(Z(\hat{I}_1)^\Gamma)$ from a triple $(\gamma_0;\gamma,\delta)$, but probably one would need to consider also Shimura varieties where K_p is allowed to be an arbitrary parahoric subgroup in order to determine which of these ways is the correct one. Since such varieties are not considered in this paper, we content ourselves to produce an element of a certain quotient group of $X^*(Z(\hat{I}_1)^\Gamma)$, and it turns out that this will be sufficient for the purposes of this paper.

To be precise, we will associate to any triple $(\gamma_0;\gamma,\delta)$ an element $\alpha_1(\gamma_0;\gamma,\delta)$ of the group

$$X^*(Z(\hat{I}_1)^\Gamma)/\mathrm{im}(\mathbb{Z}_p^\times),$$

and this element will depend on our choice of data $\mathcal{D}$. Here $\mathrm{im}(\mathbb{Z}_p^\times)$ denotes the image of the map

$$\mathbb{Z}_p^\times \hookrightarrow \mathbb{Q}_p^\times \xrightarrow{\partial} H^1(\mathbb{Q}_p, I_1) \xrightarrow{\mathrm{can}} X^*(Z(\hat{I}_1)^{\Gamma(p)})_{\mathrm{tor}} \xrightarrow{j_p^*} X^*(Z(\hat{I}_1)^\Gamma).$$

The second map is the boundary map coming from the exact sequence

$$1 \longrightarrow I_1 \longrightarrow I_0 \xrightarrow{c} \mathbb{G}_m \longrightarrow 1,$$

the third map is the canonical isomorphism extending the Tate-Nakayama isomorphism for tori that is described in §1 of [8], and the fourth is the obvious map induced by the field embedding j_p in our data $\mathcal{D}$. In a similar way we can consider the image of $\mathbb{Z}_p^\times$ in $B(I_1)$, $B^p(I_1 \to I_0)$, $X^*(Z(\hat{I}_1)^{\Gamma(p)})$, and in $X(I_1 \to I_0)_{\mathbb{Q}_p}$, or even $X(I_1 \to I_0)$, where these last two groups are defined in the obvious way by considering the connected reductive groups I_0 and I_1 over $\mathbb{Q}_p$ and $\mathbb{Q}$, respectively (see §4.1). We use the same symbol $\mathrm{im}(\mathbb{Z}_p^\times)$ to denote any of these images. Note that none of them actually depends on the choice of the field embedding j_p. Furthermore, in each case the set $\mathrm{im}(\mathbb{Z}_p^\times)$ is a *subgroup*; this follows easily in the case where I_0 and I_1 are tori. The general case follows from the special case by noting that the image $\mathrm{im}(\mathbb{Z}_p^\times)$ does not change if we replace I_0 with a maximal $\mathbb{Q}$-torus $T \subset I_0$ and I_1 with $T_1 = T \cap I_1$.

To define $\alpha_1(\gamma_0; \gamma, \delta)$, first choose field embeddings $j_v : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}_v$ for each place v of $\mathbb{Q}$. For $v = p$ and $v = \infty$, use the embeddings already fixed in our data $\mathcal{D}$. These give us inclusions $\Gamma(v) \to \Gamma$, for every place v. We will first define, for each place v of $\mathbb{Q}$, an element

$$\begin{cases} \alpha_1(l) \in X^*(Z(\hat{I}_1)^{\Gamma(l)}), \text{ if } l \neq p, \infty, \\ \alpha_1(p) \in X(I_1 \to I_0)_{\mathbb{Q}_p}/\mathrm{im}(\mathbb{Z}_p^\times), \\ \alpha_1(\infty) \in X(I_1 \to I_0)_{\mathbb{C}}. \end{cases}$$

Here, $X(I_1 \to I_0)_{\mathbb{C}} = X^*(Z(\hat{I}_0))/A(\infty)X^*(Z(\hat{I}_1))$, and $A(\infty)$ is the augmentation ideal for the group $\Gamma(\infty)$.

Definition of $\alpha_1(l)$, for $l \neq p, \infty$: by assumption the elements γ_0 and γ_l are conjugate in $G(\overline{\mathbb{Q}}_l)$. Choose an element $g_l \in G^{sc}(\overline{\mathbb{Q}}_l)$ such that $g_l \gamma_0 g_l^{-1} = \gamma_l$. Then the 1-cocycle $\tau \to g_l^{-1}\tau(g_l)$ of $\Gamma(l)$ with values in $I_1(\overline{\mathbb{Q}}_l)$ gives us a well-defined element of $H^1(\mathbb{Q}_l, I_1)$. Using the canonical (extension of the) Tate-Nakayama isomorphism mentioned above, we then get an element $\alpha_1(l) \in X^*(Z(\hat{I}_1)^{\Gamma(l)})$. We denote by $j_l^*(\alpha_1(l))$ the image (under the obvious map induced by j_l) of this element in the group $X^*(Z(\hat{I}_1)^{\Gamma})/\mathrm{im}(\mathbb{Z}_p^\times)$. Note that this element does not actually depend on the choice of embedding j_l.

Next, we define $\alpha_1(\infty)$. By assumption γ_0 is $\mathbb{R}$-elliptic, so we may choose an elliptic maximal $\mathbb{R}$-torus T of G containing γ_0. Then T is also a maximal $\mathbb{R}$-torus of I_0. We may choose $h \in X^+$ which factors through $T(\mathbb{R})$ (use that any two elliptic maximal $\mathbb{R}$-tori of G are conjugate under $G^{sc}(\mathbb{R})$). Let $T^{sc} = T \cap G^{sc}$. It is not hard to see that h is well-defined up to an element of the Weyl group $\Omega(G^{sc}(\mathbb{R}), T^{sc}(\mathbb{R}))$, and thus using Lemma 5.1 of [9], we see that the class of the cocharacter μ_h gives us a well-defined element in $X(T^{sc} \to T)_{\mathbb{C}}$. Then we define $\alpha_1(\infty)$ to be the image of $[\mu_h]$ under the restriction map

$$X(T^{sc} \to T)_{\mathbb{C}} \to X(I_1 \to I_0)_{\mathbb{C}}$$

defined by the canonical inclusions $Z(\hat{I}_1) \to \hat{T}^{sc}$ and $Z(\hat{I}_0) \to \hat{T}$. This element does not depend on the choices of T and h made in its construction, as one can check by using the fact that any two elliptic maximal $\mathbb{R}$-tori in I_0 are conjugate under $I_1(\mathbb{R})$. Finally, we denote by $j_\infty^*(\alpha_1(\infty))$ the image of this element in the group $X(I_1 \to I_0)/\mathrm{im}(\mathbb{Z}_p^\times)$. The element $\alpha_1(\infty)$ depends on the embedding j_∞, and thus on our data $\mathcal{D}$.

We now define the element $\alpha_1(p)$. We have assumed that γ_0 and $N\delta$ are conjugate under $G(\overline{\mathbb{Q}}_p)$, and that $c(\delta) = p^{-1}u^{-1}$, for some $u \in \mathcal{O}_{L_r}^\times$. Make a preliminary choice of $d \in \mathcal{O}_L^\times$ such that $d^{-1}\sigma(d) = u$. Using Steinberg's theorem that $H^1(L, I_1) = (1)$, it is not hard to find a $y \in G(L)$ such that

1. $y\gamma_0 y^{-1} = N\delta$.

2. $c(y) = d$.

By applying σ to the first equation and noting that y is unique up to right translation by an element of $I_1(L)$ we see that we get a well-defined (but depending on d) element $[y^{-1}\delta\sigma(y)] \in B^p(I_1 \to I_0)$. The purpose of the second condition on y is to insure that $c(y^{-1}\delta\sigma(y)) = p^{-1}$. It is easy to see what happens to this element when we change the choice of d. The result is that $[y^{-1}\delta\sigma(y)]$ is multiplied by an element of $\mathrm{im}(\mathbb{Z}_p^\times)$. Therefore, $[y^{-1}\delta\sigma(y)]$ is in fact well-defined (independently of our choice of d) in the quotient group $B^p(I_1 \to I_0)/\mathrm{im}(\mathbb{Z}_p^\times)$. Now recall the maps M_g constructed in §4, using the particular element g defined by Reimann and Zink. We define $\alpha_1(p)$ to be the element

$$M_g([y^{-1}\delta\sigma(y)]) \in X(I_1 \to I_0)_{\mathbb{Q}_p}/\mathrm{im}(\mathbb{Z}_p^\times).$$

Further, we denote by $j_p^*(\alpha_1(p))$ the image of this element in $X(I_1 \to I_0)/\mathrm{im}(\mathbb{Z}_p^\times)$. The element $\alpha_1(p)$ depends on g and thus on the data $\mathcal{D}$.

Definition 5.1.

$$\alpha_1(\gamma_0; \gamma, \delta) \;=\; \sum_v j_v^* \alpha_1(v).$$

We need to show that we get a well-defined element of $X^*(Z(\hat{I}_1)^\Gamma)/\mathrm{im}(\mathbb{Z}_p^\times)$. At all places $l \neq p, \infty$ it is clear that $j_l^*\alpha_1(l)$ lies in this group. Moreover, by imitating Proposition 7.1 of [8] it is not hard to prove that almost all of these elements vanish. Consider the exact sequence

$$0 \longrightarrow X^*(Z(\hat{I}_1)^\Gamma)/\mathrm{im}(\mathbb{Z}_p^\times) \longrightarrow X(I_1 \to I_0)/\mathrm{im}(\mathbb{Z}_p^\times) \xrightarrow{c} X_*(\mathbb{G}_m) \longrightarrow 0$$

The image of $j_p^*(M_g(\gamma^{-1}\delta\sigma(y)))$ under c is -1 (use $c(y^{-1}\delta\sigma(y)) = p^{-1}$). The image of $j_\infty^*([\mu_h])$ under c is 1 (we use $\mu_h\mu_h^* = id$). Therefore

$$j_p^*(\alpha_1(p)) + j_\infty^*(\alpha_1(\infty)) \;\in\; X^*(Z(\hat{I}_1)^\Gamma)/\mathrm{im}(\mathbb{Z}_p^\times).$$

This completes the construction of the invariant $\alpha_1(\gamma_0; \gamma, \delta)$ depending on the data $\mathcal{D}$.

Remark 5.2. It is immediate from the definition of the local terms $\alpha_1(v)$ above and those going into the definition of Kottwitz' invariant $\alpha(\gamma_0; \gamma, \delta)$, that $\alpha_1(\gamma_0; \gamma, \delta)$ maps to $\alpha(\gamma_0, \gamma, \delta)$ under the map

$$X^*(Z(\hat{I}_1)^\Gamma)/\mathrm{im}(\mathbb{Z}_p^\times) \to \mathfrak{k}(I_0/\mathbb{Q})^D.$$

5.2 Transformation Laws for $\alpha_1(\gamma_0; \gamma, \delta)$.

In this section we will explain the effect on $\alpha_1(\gamma_0; \gamma, \delta)$ of changing the triple $(\gamma_0; \gamma, \delta)$.

Proposition 5.3. *If γ_0' is stably conjugate to γ_0, then*

$$\alpha_1(\gamma_0'; \gamma, \delta) \;=\; \alpha_1(\gamma_0; \gamma, \delta).$$

Proof. Write I_0' (resp. I_0) for the centralizer of γ_0' (resp. γ_0). Then the $\mathbb{Q}$-groups I_1' and I_1 (with the obvious meaning) are inner twists of each other, so that we may identify $Z(\hat{I}_1)$ with $Z(\hat{I}_1')$, and thus $X^*(Z(\hat{I}_1)^\Gamma)/\mathrm{im}(\mathbb{Z}_p^\times)$ with $X^*(Z(\hat{I}_1')^\Gamma)/\mathrm{im}(\mathbb{Z}_p^\times)$, and it is via this identification that we understand the equality to be proved.

We proceed as in §2 of [9]. Choose $g \in G^{sc}(\overline{\mathbb{Q}})$ such that $g\gamma_0 g^{-1} = \gamma_0'$. One can show that for all v,

$$\alpha_1'(v) \;=\; \alpha_1(v) - t_v.$$

Here, for $v \neq p$, t_v denotes the image under

$$H^1(\mathbb{Q}, I_1) \to H^1(\mathbb{Q}_v, I_1) \to X^*(Z(\hat{I}_1)^{\Gamma(v)})$$

of the class of the 1-cocycle $[\tau \mapsto g^{-1}\tau(g)]$ of Γ in $I_1(\overline{\mathbb{Q}})$. For $v = p$, t_p is the image of the same 1-cocycle under the map

$$H^1(\mathbb{Q}, I_1) \to H^1(\mathbb{Q}_p, I_1) \to X^*(Z(\hat{I}_1)^{\Gamma(p)})/\mathrm{im}(\mathbb{Z}_p^\times).$$

Actually, when tracing through the definitions at the place ∞, the reader will find that $\alpha_1'(\infty) = \alpha_1(\infty) + t_\infty$, but this is equivalent to what we want, because $X^*(Z(\hat{I}_1)^{\Gamma(\infty)})$ is an elementary abelian 2-group. Now the Proposition follows from Proposition 2.6 of [9]. □

To state the next transformation law, we need to introduce some more notation. Suppose that $(\gamma_0; \gamma', \delta)$ and $(\gamma_0; \gamma, \delta)$ are two triples such that γ_l and γ_l' are conjugate in $G(\overline{\mathbb{Q}}_l)$, for each $l \neq p, \infty$. Then for each $l \neq p, \infty$, we can find $x_l \in G^{sc}(\overline{\mathbb{Q}}_l)$ such that $x_l \gamma_l x_l^{-1} = \gamma_l'$. We can identify $Z(\hat{G}^{sc}_{\gamma_l})$ with $Z(\hat{I}_1)$. We denote by $\mathrm{inv}_1(\gamma_l, \gamma_l')$ the image of the 1-cocycle $\tau \mapsto x_l^{-1}\tau(x_l)$ under the Tate-Nakayama isomorphism (as extended by Kottwitz)

$$H^1(\mathbb{Q}_l, G^{sc}_{\gamma_l}) \to X^*(Z(\hat{I}_1)^{\Gamma(l)})_{\mathrm{tor}}.$$

Proposition 5.4. *For any choice of field embeddings j_l, ($l \neq p, \infty$), we have*

$$\alpha_1(\gamma_0; \gamma', \delta) \;=\; \alpha_1(\gamma_0; \gamma, \delta) + \sum_{l \neq p, \infty} j_l^*(\mathrm{inv}_1(\gamma_l, \gamma_l')).$$

Proof. We first note that by the obvious analogue of Proposition 7.1 of [8], almost all of the terms $\mathrm{inv}_1(\gamma_l, \gamma_l')$ are trivial, and therefore the sum exists. Then the result follows from the local result

$$\alpha_1'(l) \;=\; \mathrm{inv}_1(\gamma_l, \gamma_l') + \alpha_1(l),$$

which is an easy consequence of the definitions, using Lemma 1.4 of [8]. □

Now we come to the most important transformation laws, which will be directly used in the proof of the Main Theorem in §8. However, these are only relevant (indeed they can only be stated) for certain triples, namely those triples $(\gamma_0; \gamma, \delta)$ for which γ_0 satisfies a certain condition, which we now explain. The element

γ_0 gives rise to the groups I_0 and I_1 and, by considering dual groups, to the exact sequence

$$1 \longrightarrow Z(\hat{G}) \longrightarrow Z(\hat{I}_0) \longrightarrow Z(\hat{I}_1) \longrightarrow 1,$$

which induces the boundary map $\partial : Z(\hat{I}_1)^\Gamma \to H^1(\mathbb{Q}, Z(\hat{G}))$. We have fixed the character ϖ, which corresponds to the global Langlands parameter $\mathbf{a} \in H^1(\mathbb{Q}, Z(\hat{G}))/\ker^1(\mathbb{Q}, Z(\hat{G})) \cong H^1(\mathbb{Q}, Z(\hat{G}))$ (since G is the group of symplectic similitudes).

Definition 5.5. We say that γ_0 is ω-special if there exists $\kappa_0 \in Z(\hat{I}_1)^\Gamma$ such that $\partial(\kappa_0) = \mathbf{a}$.

Remark 5.6. It is easy to show that if γ_0' and γ_0 are $G(\overline{\mathbb{Q}})$-conjugate, then one is ω-special if and only if the other one is. Therefore it makes sense to call a pair (γ, δ) ω-special if the element γ_0 constructed from this pair is ω-special, since γ_0 is uniquely determined up to stable conjugacy by (γ, δ).

If γ_0 is ω-special with $\partial(\kappa_0) = \mathbf{a}$, then one can show that κ_0 satisfies $\langle \mathrm{im}(\mathbb{Z}_p^\times), \kappa_0 \rangle = 1$ with respect to the canonical pairing

$$\langle \cdot, \cdot \rangle : X^*(Z(\hat{I}_1)^\Gamma) \times Z(\hat{I}_1)^\Gamma \ \longrightarrow \ \mathbb{C}^\times.$$

First one proves the equality $\langle \mathrm{im}(x), \kappa_0 \rangle = \varpi_p(x)$, for any $x \in \mathbb{Q}_p^\times$, by routine but tedious diagram chases which relate ϖ and ϖ_p to the global and local Tate-Nakayama pairings defined on the torus $I_0^{ab} = I_0/I_0^{\mathrm{der}}$, and thus to κ_0. Then the result follows from the fact that $\varpi_p(\mathbb{Z}_p^\times) = 1$. Thus it makes sense to consider the pairing $\langle \alpha_1(\gamma_0; \gamma, \delta), \kappa_0 \rangle$, in the case where γ_0 is ω-special. Note that we are using in an essential way that we are in the case of good reduction here, i.e, K_p is a hyperspecial maximal compact subgroup and $c(K_p) = \mathbb{Z}_p^\times$.

Theorem 5.7. *Suppose that $(\gamma_0; \gamma, \delta)$ is a triple such that γ_0 is ω-special, and suppose that $\partial(\kappa_0) = \mathbf{a}$. Then the following transformation laws hold:*

1. *If $g = (g_l)_{l \neq p, \infty} \in G(\mathbb{A}_f^p)$ and $\gamma' = g\gamma g^{-1}$, then*

$$\langle \alpha_1(\gamma_0; \gamma', \delta), \kappa_0 \rangle \ = \ \langle \alpha_1(\gamma_0, \gamma, \delta), \kappa_0 \rangle \omega(g)^{-1}.$$

2. *If $h \in G(L_r)$ and $\delta' = h\delta\sigma(h)^{-1}$, then*

$$\langle \alpha_1(\gamma_0; \gamma, \delta'), \kappa_0 \rangle \ = \ \langle \alpha_1(\gamma_0; \gamma, \delta), \kappa_0 \rangle \omega(h)^{-1}.$$

Proof. (1). Using the preceding Proposition, we are reduced to proving a purely local statement:

$$\langle \mathrm{inv}_1(\gamma_l, g_l\gamma_l g_l^{-1}), j_l^*(\kappa_0) \rangle \ = \ \omega_l(g_l)^{-1},$$

where we are using j_l to define the inclusion $j_l^* : Z(\hat{I}_1)^\Gamma \to Z(\hat{I}_1)^{\Gamma(l)}$. This is easy to prove, once again using standard diagram chases to relate $j_l^*(\kappa_0)$ to ω_l.

(2). Here we have not introduced the p-adic analogue of $\mathrm{inv}_1(\gamma_l, \gamma'_l)$, and so we must argue directly. To define $\alpha_1(p)$, write $c(\delta) = p^{-1}d\sigma(d)^{-1}$, where $d \in \mathcal{O}_L^\times$. Then we can find $y \in G(L)$ such that (i) $y\gamma_0 y^{-1} = N\delta$, and (ii) $c(y) = d$. Note that $c(\delta') = p^{-1}(c(h)d)\sigma(c(h)d)^{-1}$. We use y to define $\alpha_1(p) = M_g[y^{-1}\delta\sigma(y)]$. But we cannot necessarily use the element $hy \in G(L)$ (the natural guess!) to define $\alpha'_1(p)$, because although (i) $(hy)\gamma_0(hy)^{-1} = N\delta'$, and (ii) $c(hy) = c(h)d$, the element $c(h)d \in L^\times$ might not be in $\mathcal{O}_L^\times$. To remedy this, write $c(h) = p^n v$, where $v \in \mathcal{O}_{L_r}^\times$ and $n \in \mathbb{Z}$. Using Steinberg's theorem we can find $\lambda \in I_0(\mathbb{Q}_p^{un})$ such that $c(\lambda) = p^{-n}$. Then it follows easily that we can now use the element $hy\lambda \in G(L)$ to define $\alpha'_1(p)$, and moreover that we have

$$\alpha'_1(p) \ = \ \alpha_1(p) + M_g([\lambda^{-1}\sigma(\lambda)]).$$

Note that $M_g[\lambda^{-1}\sigma(\lambda)]$ is precisely the image of p^{-n} under the map

$$\mathbb{Q}_p^\times \longrightarrow H^1(\mathbb{Q}_p, I_1) \longrightarrow X^*(Z(\hat{I}_1)^{\Gamma(p)}) \longrightarrow X(I_1 \to I_0)_{\mathbb{Q}_p}/\mathrm{im}(Z_p^\times).$$

We see then that we are reduced to proving the local statement

$$\langle M_g[\lambda^{-1}\sigma(\lambda)], j_p^*(\kappa_0)\rangle \ = \ \overline{\omega}_p c(h)^{-1}.$$

which is is just the equality

$$\overline{\omega}_p(p^{-n}) \ = \ \overline{\omega}_p c(h)^{-1}.$$

This in turn follows from $c(h) = p^n v$ and the fact that $\overline{\omega}_p$ is trivial on $\mathcal{O}_{L_r}^\times$. $\square$

Remark 5.8. As stated, the equalities above are between complex numbers, since we have regarded the pairing $\langle\cdot,\cdot\rangle$ and the character $\overline{\omega}$ (and thus $\omega = \overline{\omega} \circ c$ as having values in $\mathbb{C}^\times$. However, it is important to note that later in this paper (especially in the statement of the Main Theorem of §8), we shall be regarding both sides of the equalities as elements of $\overline{\mathbb{Q}}_\ell^\times$. This we do by using the isomorphism $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$ that we chose in §2, and which has already been used there to regard $\overline{\omega}$ as having values in $\overline{\mathbb{Q}}_\ell^\times$.

6 Further Reductions

Recall that we are denoting by Fix the set of points $[\overline{A'}, \lambda', \overline{\eta'}] \in S_{K_g^p}(\overline{k})$ which are fixed by the correspondence $\Phi_p^r \circ f$. As stated in §2, we want to give a formula for

$$\sum_{[\overline{A'}, \lambda', \overline{\eta'}] \in \mathrm{Fix}} \omega[\overline{A'}, \lambda', \overline{\eta'}].$$

To so so we recall several facts from [11]. In §16 of that paper it is shown that each fixed point $[\overline{A'}, \lambda', \overline{\eta'}]$ gives rise to a positive rational number $c = c_0 p^r$ (c_0 a p-adic unit) and a c-polarized virtual abelian variety over k_r up to isogeny (A', λ'). Also, (A', λ') automatically satisfies the conditions in §14 of [11], and so gives rise to a triple $(\gamma_0'; \gamma', \delta')$, which is well-defined up to an equivalence relation we will call G-equivalence. (The triples $(\gamma_0; \gamma, \delta)$ and $(\gamma_0'; \gamma', \delta')$ are G-equivalent if γ_0 and γ_0' are $G(\overline{\mathbb{Q}})$-conjugate, γ and γ' are conjugate under $G(\mathbb{A}_f^p)$, and δ and δ' are σ-conjugate under $G(\mathbb{Q}_{p^r})$.) Note that $(\gamma_0')(\gamma_0')^* = c^{-1}$. So by these remarks and using the notation from the end of §3, we see that the sum above can be written as

$$\sum_c \sum_{(A,\lambda)} T^\omega(A, \lambda).$$

The outer sum ranges over all numbers c of the form $c_0 p^r$, and for each c the pairs (A, λ) range over those which are c-polarized. We could just as well write our sum as

$$\sum_c \sum_{(\gamma_0;\gamma,\delta)} \sum_{(A,\lambda)} T^\omega(A, \lambda).$$

Now c ranges as before. For each c, we then sum over those G-equivalence classes $(\gamma_0; \gamma, \delta)$ with $\gamma_0\gamma_0^* = c^{-1}$. Given $(\gamma_0; \gamma, \delta)$ the inner sum is over the pairs (A, λ) which give rise to a triple which is G-equivalent to $(\gamma_0; \gamma, \delta)$.

The first step in simplifying the above sum is to determine which pairs (A, λ) have $T^\omega(A, \lambda) \neq 0$. To do this fix (A, λ) and set $I = \mathrm{Aut}(A, \lambda)$. Choose β^p and β_r for (A, λ) and use them to define γ and δ as in §3. Recall that $T^\omega(A, \lambda)$ is then

$$\omega(\beta^p\beta_r)^{-1} \int_{I(\mathbb{Q})\backslash[G(\mathbb{A}_f^p)\times G(L_r)]} \omega(y)\omega(x)\widetilde{f}^p(y^{-1}\gamma y)\widetilde{\phi}_r(x^{-1}\delta\sigma(x))\ dydx.$$

Thus we want to know when this integral does not vanish. We choose Haar measures on $G(\mathbb{A}_f^p)_\gamma$ and $G_{\delta\sigma}(\mathbb{Q}_p)$ as follows: Choose Haar measures on $I(\mathbb{A}_f^p)$ and $I(\mathbb{Q}_p)$. Tate's Theorem (as generalized in §10 of [11]) and our choice of $\beta = (\beta^p, \beta_r)$ give us isomorphisms

$$\begin{array}{rcl} I(\mathbb{A}_f^p) & \xrightarrow{\sim} & G(\mathbb{A}_f^p)_\gamma \\ I(\mathbb{Q}_p) & \xrightarrow{\sim} & G_{\delta\sigma}(\mathbb{Q}_p), \end{array}$$

which we use to transport the measures to the right hand side. Since all these groups are unimodular, the measures on the right do not depend on the choice of β.

We also want to transport the character ω to the groups $I(\mathbb{A}_f^p)$, $I(\mathbb{Q}_p)$, and $I(\mathbb{A})$. We know that $I(\mathbb{Q})$ is discrete in $I(\mathbb{A}_f)$ and has finite co-volume. Note also that ω is already defined on the groups on the right (it is defined on $G_{\delta\sigma}(\mathbb{Q}_p)$ because $G_{\delta\sigma,\mathbb{Q}_p}$ is an inner form of $G_{\gamma_0,\mathbb{Q}_p}$). Therefore we can use the isomorphisms to define ω on $I(\mathbb{A}_f)$ (and it does not depend on the choice of β). We could also transport the character ω to the group $I(\mathbb{A})$ using global Langlands parameters. Namely, the restriction of ω to $I_0(\mathbb{A})$ (where $I_0 = G_{\gamma_0}$) comes from a global Langlands parameter for the group I_0. Since the $\mathbb{Q}$-groups I_0 and I are inner forms, we may identify the centers of their dual groups and via this identification we get a global Langlands parameter for I, and thus a

character on $I(\mathbb{A})$. On $I(\mathbb{A}_f)$ this agrees with the one constructed above, but now we have a character at the infinite place as well as the finite ones. However, since the involution $*$ on $\mathrm{End}(V_{\mathbb{R}})$ is positive, the characters on $I(\mathbb{R})$ and $I_0(\mathbb{R})$ are both trivial anyway.
We need some more definitions.

Definition 6.1. If $\omega(G(\mathbb{A}_f^p)_\gamma) = 1$, let $O_\gamma^\omega(f^p)$ denote the integral

$$\int_{G(\mathbb{A}_f^p)_\gamma \backslash G(\mathbb{A}_f^p)} \overline{\omega}\ c(y)\ f^p(y^{-1}\gamma y)\ dy,$$

where dy is determined by the measure on $G(\mathbb{A}_f^p)$ giving K^p measure 1 and the measure on $G(\mathbb{A}_f^p)_\gamma$ specified above. Also, f^p is the characteristic function for $K^p g^{-1} K^p$.

Definition 6.2. If $\omega(G(\mathbb{A}_f^p)_\gamma) = 1$, let $O_\gamma^\omega(\widetilde{f}^p)$ denote the integral above with f^p replaced by $\widetilde{f}^p$, the characteristic function for $K^p g^{-1}$, and dy replaced with the measure giving $K_g^p = K^p \cap gK^p g^{-1}$ measure 1.

Remark. It is straightforward to check that

$$O_\gamma^\omega(f^p) \ = \ O_\gamma^\omega(\widetilde{f}^p)$$

(Remember $\overline{\omega}\ c(K^p) = 1$.)

Definition 6.3. If $\omega(G_{\delta\sigma}(\mathbb{Q}_p)) = 1$, let $TO_\delta^\omega(\phi_r)$ denote the integral

$$\int_{G_{\delta\sigma}(\mathbb{Q}_p)\backslash G(L_r)} \overline{\omega}\ \overline{c}(x)\phi_r\left(x^{-1}\delta\sigma(x)\right)\ dx,$$

where dx is determined by the measure on $G(L_r)$ giving K_r measure 1 and the measure on $G_{\delta\sigma}(\mathbb{Q}_p)$ specified above. Here $\overline{c}(x)$ is the image of $c(x) \in L_r^\times/\mathcal{O}_{L_r}^\times$ in $\mathbb{Q}_p^\times/\mathbb{Z}_p^\times$, and ϕ_r is the characteristic function for $K_r a K_r$, where $a = \mu_h(p^{-1})$ for any $h \in X_\infty$, as defined in §3.

Remark. If we replace ϕ_r in the integral with $\widetilde{\phi}_r$ from §3, then one can check that

$$TO_\delta^\omega(\widetilde{\phi}_r) = TO_{\sigma^{-1}\delta}^\omega(\phi_r) = \overline{\omega}\ \overline{c}(\delta) TO_\delta^\omega(\phi_r).$$

These equalities are straightforward to prove when one keeps in mind that for all $x \in G(L_r)$, $\overline{\omega}\ \overline{c}(x) = \overline{\omega}\ \overline{c}(\sigma(x))$. Also, since the triples $(\gamma_0;\gamma,\delta)$ we are considering have $c(\delta) = p^{-1}u^{-1}$ for some $u \in \mathcal{O}_{L_r}^\times$, we can write $\overline{\omega}\ \overline{c}(\delta) = \overline{\omega}_p(p^{-1})$, $\overline{\omega}_p$ being the p-component of $\overline{\omega}$. So the last term above can be written

$$\overline{\omega}_p(p^{-1}) TO_\delta^\omega(\phi_r).$$

For future reference we record here the transformation laws for these (twisted) orbital integrals.

Lemma 6.4. *For $g \in G(\mathbb{A}_f^p)$ and $h \in G(L_r)$ we have*

$$\begin{aligned} O^{\omega}_{g\gamma g^{-1}}(f^p) &= \omega(g)\ O^{\omega}_{\gamma}(f^p) \\ TO^{\omega}_{h\delta\sigma(h^{-1})}(\phi_r) &= \omega(h)\ TO^{\omega}_{\delta}(\phi_r) \end{aligned}$$

Proof. Straightforward computations. Note also that for each equation, the left hand side is defined if and only if the right hand side is. □

We can now finish the first step by proving the following result.

Proposition 6.5. *$T^{\omega}(A,\lambda) = 0$ unless $\omega(I(\mathbb{A}_f)) = 1$, in which case $T^{\omega}(A,\lambda)$ is given by*

$$\mathrm{vol}(I(\mathbb{Q})\backslash I(\mathbb{A}_f))\ \overline{\omega}_p(p^{-1})\ \omega(\beta^p\beta_r)^{-1}\ O^{\omega}_{\gamma}(f^p)\ TO^{\omega}_{\delta}(\phi_r).$$

Proof. Suppose $J \subseteq H \subseteq G$ are three unimodular groups. We have the integration formula

$$\int_{J\backslash G} \phi(g)\ dg\ =\ \int_{H\backslash G}\left(\int_{J\backslash H}\phi(hg)\ dh\right) dg.$$

If we apply this to the integral expression for $T^{\omega}(A,\lambda)$ (take $J = I(\mathbb{Q}), H = G(\mathbb{A}_f^p)_{\gamma} \times G_{\delta\sigma}(\mathbb{Q}_p)$, $G = G(\mathbb{A}_f^p) \times G(L_r)$), then the inner integral contains the factor

$$\int_{I(\mathbb{Q})\backslash[G(\mathbb{A}_f^p)_{\gamma}\times G_{\delta\sigma}(\mathbb{Q}_p)]} \omega(h^p y)\ \omega(h_p x)\ dh^p \times dh_p,$$

which is 0 unless $\omega(G(\mathbb{A}_f^p)_{\gamma} \times G_{\delta\sigma}(\mathbb{Q}_p)) = 1$ (that is, $\omega(I(\mathbb{A}_f)) = 1$), in which case it is

$$\mathrm{vol}(I(\mathbb{Q})\backslash I(\mathbb{A}_f))\ \omega(y)\omega(x).$$

Now considering the outer integral shows that in this case $T^{\omega}(A,\lambda)$ is

$$\mathrm{vol}(I(\mathbb{Q})\backslash I(\mathbb{A}_f))\ \omega(\beta^p\beta_r)^{-1}\ O^{\omega}_{\gamma}(\widetilde{f^p})\ TO^{\omega}_{\delta}(\widetilde{\phi}_r),$$

and the proposition is a consequence of this and the preceding remarks. □

The second step in simplifying our sum is to fix a (A,λ) such that $T^{\omega}(A,\lambda) \neq 0$, let I, β^p, β_r and $(\gamma_0;\gamma,\delta)$ be as above, and consider the part of the sum over all (A',λ') which give rise to a triple G-equivalent to $(\gamma_0;\gamma,\delta)$. Let $I' = \mathrm{Aut}(A',\lambda')$. Let $I_0 = G_{\gamma_0}$.

Lemma 6.6.

$$\omega(I_0(\mathbb{A})) = \omega(I(\mathbb{A})) = \omega(I'(\mathbb{A})) = 1.$$

Proof. By §14 of [11] we know that there is an inner twisting over $\overline{\mathbb{Q}}$

$$I_0 \xrightarrow{\backsim} I$$

which is unique up to inner automorphisms of $I_0(\overline{\mathbb{Q}})$. Therefore I is a connected reductive $\mathbb{Q}$-group because I_0 is, by a theorem of Steinberg. It follows that $I(\mathbb{Q})$ is dense in $I(\mathbb{R})$, and so

$$I(\mathbb{A}) = I(\mathbb{Q})[I(\mathbb{A}_f)I(\mathbb{R})^0].$$

Since $\omega(I(\mathbb{A}_f)) = 1$, we see from this that $\omega(I(\mathbb{A})) = 1$. (Alternatively, we can avoid invoking the real approximation theorem for this point by using the fact that the involution $*$ on $\mathrm{End}(V_{\mathbb{R}})$ is positive.) The groups I_0 and I' are each inner forms of I, and so $\omega(I'(\mathbb{A})) = \omega(I_0(\mathbb{A})) = \omega(I(\mathbb{A})) = 1$. $\square$

Remark. This type of result is easy to see directly if we assume that γ_0 is a regular semisimple element. In that case I_0, I, and I' are all tori, isomorphic over $\mathbb{Q}$. Moreover, we note that in this case the $\mathbb{Q}$-isomorphism $I_0 \xrightarrow{\sim} I$ is unique, and over $\mathbb{Q}_l$ for any $l \neq p, \infty$ it is the isomorphism constructed by using Tate's theorem, our choice of β_l, and the fact that γ_0 and γ_l are stably conjugate. A similar argument works for the finite place p and that is sufficient because the characters are automatically trivial at ∞.

Let $c_2(\gamma_0; \gamma, \delta) = \mathrm{vol}(I(\mathbb{Q})\backslash I(\mathbb{A}_f))$, and $c_1(\gamma_0) = |\ker^1(\mathbb{Q}, I)|$. Let $c(\gamma_0; \gamma, \delta) = c_1(\gamma_0)c_2(\gamma_0; \gamma, \delta)$. Note that these numbers really do depend only on $(\gamma_0; \gamma, \delta)$ and remain unchanged if we replace I by I'. The set of pairs (A', λ') we are considering is in bijective correspondence with $\ker^1(\mathbb{Q}, I)$, by Theorem 17.2 of [11]. We describe this explicitly: Any $\overline{\mathbb{Q}}$-isogeny $\theta : (A', \lambda') \to (A, \lambda)$ gives an element $[\theta \circ \tau(\theta^{-1})] \in H^1(\mathbb{Q}, I)$, which lies in $\ker^1(\mathbb{Q}, I)$ if and only if (A', λ') gives a triple G-equivalent to $(\gamma_0; \gamma, \delta)$. If this element is in $\ker^1(\mathbb{Q}, I)$, choose a finite Galois extension $K/\mathbb{Q}$ such that $[\theta \circ \tau(\theta^{-1})] \in H^1(K/\mathbb{Q}, I)$. Then there exists an element

$$\psi = (\psi_v)_v \in \prod_v I(\mathbb{Q}_v \otimes K),$$

such that

1. $\psi_v \in I(\mathbb{Z}_v \otimes K)$ for almost every finite place v of $\mathbb{Q}$, and
2. $\theta \circ \tau(\theta^{-1}) \;=\; \psi_v^{-1} \circ \tau(\psi_v), \quad \forall\, \tau \in \Gamma(v)$, for every place v.

Then it follows that we can use $\beta^p\psi^p\theta$ and $\beta_r\psi_p\theta$ to define γ' and δ' for (A', λ'). When we do so we find that $\gamma' = \gamma$ and $\delta = \delta'$. We thus see that

$$T^{\omega}(A', \lambda') = \omega(\psi^p\theta)^{-1}\omega(\psi_p\theta)^{-1}T^{\omega}(A, \lambda).$$

Now recall $\mathbf{a} \in H^1(\mathbb{Q}, Z(\hat{G}))/\ker^1(\mathbb{Q}, Z(\hat{G}))$, the global Langlands parameter corresponding to $\omega : G(\mathbb{A}) \to \mathbb{C}^{\times}$. Let $I_1 = I_0 \cap G^{\mathrm{SC}}$. We have the exact sequence

$$1 \longrightarrow Z(\hat{G}) \longrightarrow Z(\hat{I}_0) \longrightarrow Z(\hat{I}_1) \longrightarrow 1$$

which gives rise to maps (see §6.2 of [13])

$$\alpha : H^1(\mathbb{Q}, Z(\hat{G}))/\ker^1(\mathbb{Q}, Z(\hat{G})) \longrightarrow H^1(\mathbb{Q}, Z(\hat{I}_0))/\ker^1(\mathbb{Q}, Z(\hat{I}_0)),$$

$$\beta : \ker\alpha \longrightarrow \mathrm{coker}[\ker^1(\mathbb{Q}, Z(\hat{G})) \to \ker^1(\mathbb{Q}, Z(\hat{I}_0))].$$

Note that in our situation $\ker^1(\mathbb{Q}, Z(\hat{G})) = (1)$, since G is the group of symplectic similitudes. Because $\omega(I_0(\mathbb{A})) = 1$ we know from the theory of global Langlands parameters that $\mathbf{a} \in \ker\alpha$, and thus we get an element $\beta(\mathbf{a}) \in$

$\ker^1(\mathbb{Q}, Z(\hat{I}_0))$. We denote by $\mathrm{inv}((A', \lambda'), (A, \lambda))$ the element $[\theta \cdot \tau(\theta^{-1})]$ in $\ker^1(\mathbb{Q}, I_0)$ (here we can identify $\ker^1(\mathbb{Q}, I_0)$ with $\ker^1(\mathbb{Q}, I)$, since I and I_0 are inner forms).

Proposition 6.7.

$$\langle \mathrm{inv}\,((A', \lambda'), (A, \lambda)), \beta(\mathbf{a})\rangle \;=\; \omega(\psi^p\theta)\ \omega(\psi_p\theta),$$

where

$$\langle \cdot, \cdot \rangle : \ker^1(\mathbb{Q}, I_0) \times \ker^1(\mathbb{Q}, Z(\hat{I}_0)) \;\longrightarrow \mathbb{C}^\times$$

is the Tate-Nakayama pairing (as extended by Kottwitz) on these finite abelian groups (see §4 of [6]).

Proof. This involves a straightforward unwinding of the definition of the aforementioned Tate-Nakayama pairing. It is very similar to the argument given in §6.2 of [13], where the required diagram chase is carried out in a much more general situation. We omit the details. □

Now we can complete the second step of the simplification.

Corollary 6.8. *The sum of $T^\omega(A', \lambda')$ over all pairs (A', λ') which give rise to the G-equivalence class of $(\gamma_0; \gamma, \delta)$ is zero unless $\omega(I_0(\mathbb{A})) = 1$ and $\beta(\mathbf{a}) \in \ker^1(\mathbb{Q}, Z(\hat{I}_0))$ is trivial. When these conditions hold, the sum is given by*

$$c(\gamma_0; \gamma, \delta)\ \overline{\omega}_p(p^{-1})\ \omega(\beta^p\beta_r)^{-1}\ O^\omega_\gamma(f^p)\ TO^\omega_\delta(\phi_r).$$

Proof. If $\omega(I_0(\mathbb{A})) \neq 1$ then we have already seen that each term $T^\omega(A', \lambda') = 0$. Suppose $\omega(I_0(\mathbb{A})) = 1$, so that $\beta(\mathbf{a})$ exists. As (A', λ') varies, $\mathrm{inv}((A', \lambda'), (A, \lambda))$ ranges over the elements x of $\ker^1(\mathbb{Q}, I_0)$. So using the preceeding two Propositions, our assertion follows from the fact that

$$\sum_x \langle x, \beta(\mathbf{a})\rangle$$

is 0 if $\beta(\mathbf{a})$ is nontrivial, and is $|\ker^1(\mathbb{Q}, I_0)|$ if $\beta(\mathbf{a})$ is trivial. □

Now recalling Theorem 18.1 of [11] and imitating the reasoning in §19 of that paper, we see that our sum

$$\sum_c \ \sum_{(\gamma_0;\gamma,\delta)} \ \sum_{(A,\lambda)} T^\omega(A, \lambda)$$

can now be written as

$$\sum_{(\gamma_0;\gamma,\delta)} c(\gamma_0; \gamma, \delta)\ \overline{\omega}_p(p^{-1})\ \omega(\beta^p\beta_r)^{-1}\ O^\omega_\gamma(f^p)\ TO^\omega_\delta(\phi_r),$$

where $(\gamma_0; \gamma, \delta)$ ranges over G-equivalence classes of triples such that

1. The Kottwitz invariant $\alpha(\gamma_0; \gamma, \delta)$ is trivial,

2. $\omega(I_0(\mathbb{A})) = 1$,

3. $\beta(\mathbf{a})$ is trivial,

and where, for any pair (A, λ) giving rise to the G-equivalence class of $(\gamma_0; \gamma, \delta)$, β^p and β_r are chosen for (A, λ) and are used to get γ and δ. Note that if the summand indexed by $(\gamma_0; \gamma, \delta)$ in this last sum is nonzero, then there is some c and a c-polarized virtual abelian variety over k_r up to isogeny (A, λ) that gives rise to the G-equivalence class of $(\gamma_0; \gamma, \delta)$. Moreover, this summand does not then depend upon the choice of $(A, \lambda), \beta^p, \beta_r$ giving rise to the triple. Therefore the notation makes sense.

Lemma 6.9. *Conditions 2. and 3. above hold if and only if γ_0 is ω-special.*

Proof. This is an easy consequence of the definitions. $\square$

By the Lemma, we see that the index set of the second sum is precisely the set of G-equivalence classes of triples $(\gamma_0; \gamma, \delta)$ such that

1. The Kottwitz invariant $\alpha(\gamma_0; \gamma, \delta)$ is trivial,

2. γ_0 is ω-special.

Note that $\omega(\beta^p \beta_r)^{-1}$ is the only term appearing in the summand indexed by $(\gamma_0; \gamma, \delta)$ that is not purely group-theoretic in nature: one needs to make reference to an abelian variety to define this number. It turns out that the refined invariant $\alpha_1(\gamma_0; \gamma, \delta)$ is precisely what is needed in order to remedy this:

Theorem 6.10. *Let (A, λ) be a c-polarized virtual abelian variety over k_r up to isogeny, and suppose that β^p and β_r are symplectic similitudes used to define γ and δ. Suppose that (γ, δ) is ω-special, and that $\kappa_0 \in Z(\hat{I}_1)^\Gamma$ is any element with $\partial(\kappa_0) = \mathbf{a}$. Then*

$$\langle \alpha_1(\gamma_0; \gamma, \delta), \kappa_0 \rangle \;=\; \omega(\beta^p \beta_r)^{-1}.$$

Note that we can regard this as an equality of elements in either $\mathbb{C}$ or $\overline{\mathbb{Q}}_\ell$. The preceding simplication of our sum and this theorem immediately imply the Main Theorem of §8. But before we prove this theorem we must use the theorem of Reimann and Zink to prove a result about abelian varieties with complex multiplication. This is the crucial step, and is done in the next section.

7 Study of an Invariant Attached to Abelian Varieties

In order to prove the necessary properties of $\alpha_1(\gamma_0; \gamma, \delta)$, we need to introduce and study another invariant which is attached to a polarized abelian variety over $\overline{k}$ with an action by a CM-algebra. We will in fact prove that this invariant is always trivial, and this will turn out to yield key information about $\alpha_1(\gamma_0; \gamma, \delta)$.

Let F be a CM-algebra with involution $*$. By this we mean F is a product of totally imaginary quadratic extensions of totally real number fields, and $*$ induces complex conjugation on each factor of $F \otimes \mathbb{R} \cong \mathbb{C} \times \cdots \times \mathbb{C}$. Let $(V, \langle \cdot, \cdot \rangle)$ be a symplectic vector space over $\mathbb{Q}$ with CM by F. This means that V is a free F-module of rank 1 and the relation $\langle xv, w \rangle = \langle v, x^* w \rangle$ holds for $x \in F$ and $v, w \in V$. Let T be the $\mathbb{Q}$-torus $\{x \in F^\times \mid xx^* \in \mathbb{Q}^\times\}$, and let T_1 be the $\mathbb{Q}$-torus $\{x \in F^\times \mid xx^* = 1\}$. Recall from §13 of [11] that a morphism $f : (V_1; \langle \cdot, \cdot \rangle_1) \to (V_2; \langle \cdot, \cdot \rangle_2)$ between two symplectic vector spaces with CM by F is by definition an F-linear map such that $f^* \langle \cdot, \cdot \rangle_2 = c \langle \cdot, \cdot \rangle_1$ for some $c \in \mathbb{Q}^\times$. We call such a morphism *strict* if $c = 1$. Therefore we have

$$\begin{aligned} T_1 &= \mathrm{Aut}_{\mathrm{strict}}(V, \langle \cdot, \cdot \rangle), \\ T &= \mathrm{Aut}(V, \langle \cdot, \cdot \rangle). \end{aligned}$$

When we want to regard $(V, \langle \cdot, \cdot \rangle)$ as an object in the category with strict morphisms, we call it a strict symplectic space.

Let A be an abelian variety over $\overline{k}$ up to $\mathbb{Q}$-isogeny, and suppose that there is an injection $i : F \hookrightarrow \mathrm{End}(A)$. Further suppose that $2 \dim(A) = [F : \mathbb{Q}]$, and that there is a $\mathbb{Q}$-polarization $\lambda : A \to \hat{A}$ such that i is a $*$-homomorphism for the involution on F and the Rosati involution $\imath_\lambda$ on $\mathrm{End}(A)$. For such a pair (A, λ) we will define an invariant $\widetilde{\alpha}(A, \lambda) \in X_*(T_1)_\Gamma$.

To do this we choose embeddings $j_v : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_v$ for every place v; for $v = p, \infty$ use the embeddings in our data $\mathcal{D}$. These allow us to regard each local Galois group as a subgroup of the global Galois group: $\Gamma(v) \hookrightarrow \Gamma$. Using $\mathcal{D}$ we get an element g as in §1 of [27] and §4.2 above. Thus we get an isomorphism

$$K_g^{-1} \ : \ B^p(T_1 \to T) \longrightarrow X(T_1 \to T)$$

from the exact sequence

$$1 \longrightarrow T_1 \longrightarrow T \xrightarrow{x \mapsto x^* x} \mathbb{G}_m \longrightarrow 1.$$

(Recall that $Nm_{L(\sqrt{p})/L}(g) = p$ and that we have proved in §4.2 that K_Π is independent of the choice of tower Π over g for tori of this form; this justifies the notation K_g.) To define $\widetilde{\alpha}(A, \lambda)$ we will define $\widetilde{\alpha}(v) \in X_*(T_1)_{\Gamma(v)}$ for each place v:

For $l \neq p, \infty$, note that the l-adic Tate module $H_1(A, \mathbb{Q}_l)$ is a strict symplectic space with CM by F (the pairing comes from the polarization λ, and we view the $\mathbb{Q}_l(1)(\overline{k})$-valued pairing as $\mathbb{Q}_l$-valued using $\mathcal{D}$ as in §1). Note that $H_1(A, \mathbb{Q}_l)$ and $V \otimes \mathbb{Q}_l$ become isomorphic over $\overline{\mathbb{Q}}_l$ as strict symplectic spaces with CM by F. Therefore their difference is measured by an element of $H^1(\mathbb{Q}_l, T_1)$. Now using the Tate-Nakayama isomorphism

$$H^1(\mathbb{Q}_l, T_1) \xrightarrow{\backsim} X_*(T_1)_{\Gamma(l), \mathrm{tor}} \ ,$$

we get an element $\widetilde{\alpha}(l)$ of $X_*(T_1)_{\Gamma(l)}$. Using

$$j_l^* : X_*(T_1)_{\Gamma(l)} \longrightarrow X_*(T_1)_\Gamma$$

we get an element $j_l^*(\widetilde{\alpha}(l)) \in X_*(T_1)_\Gamma$.

Now consider the place p. Let (D, Φ) denote the dual of the L-isocrystal $(H^1_{crys}(A/W(\overline{k})) \otimes L, Fr)$. Then D is a strict symplectic space with CM by F, and Φ commutes with the action of F. Note that D and V become strictly isomorphic over $\overline{L}$ and so by Steinberg's Theorem that $H^1(L, T_1) = (1)$, they become isomorphic over L. Choose a strict isomorphism

$$D \xrightarrow{\sim} V \otimes L$$

and use it to transport the σ-linear bijection Φ to $b\sigma$ on the right hand side. Thus $b \in T(L)$ and $bb^* = p^{-1}$. Changing the choice of the above strict isomorphism only changes b by an element of $A\langle\sigma\rangle T_1(L)$, so we have a well-defined element $[b] \in B^p(T_1 \to T)$. We define $\widetilde{\alpha}(p) = K_g^{-1}([b])$, an element of $X(T_1 \to T)_{\mathbb{Q}_p}$. Then $j_p^*(\widetilde{\alpha}(p))$ is an element of $X(T_1 \to T)_{\mathbb{Q}}$.

Finally at the place ∞ we let

$$h : \mathbb{C} \to \mathrm{End}_F(V_{\mathbb{R}}) \;=\; F \otimes \mathbb{R}$$

be the unique $*$-homomorphism such that $\langle \cdot, h(i) \cdot \rangle$ is positive definite on $V_{\mathbb{R}} \times V_{\mathbb{R}}$. Then we get a cocharacter $\mu_h \in X_*(T_{\mathbb{C}})$. We define $\widetilde{\alpha}(\infty)$ to be the class of μ_h in $X(T_1 \to T)_{\mathbb{C}}$. Thus $j_\infty^*(\widetilde{\alpha}(\infty)) \in X(T_1 \to T)_{\mathbb{Q}}$.

Definition 7.1.

$$\widetilde{\alpha}(A, \lambda) \;=\; \sum_v j_v^*(\widetilde{\alpha}(v)).$$

Remark. We have to show this gives us a well-defined element of $X_*(T_1)_\Gamma$. For $v \neq p, \infty$, it is clear that $j_v^*(\widetilde{\alpha}(v))$ lies in this group. Moreover, it is not too difficult to show that for almost all v, $j_v^*(\widetilde{\alpha}(v)) = 0$. Recall that we have the exact sequence

$$0 \longrightarrow X_*(T_1)_\Gamma \longrightarrow X(T_1 \to T)_{\mathbb{Q}} \xrightarrow{c} X_*(\mathbb{G}_m) \longrightarrow 0$$

where we use c for the map $x \mapsto x^* x$. Tracing through the definitions shows that $\mu_h^* \mu_h = \mathrm{id}$, so that the image of $j_\infty^*(\widetilde{\alpha}(\infty))$ under c is 1. The image of $j_p^*(\widetilde{\alpha}(p))$ under c is -1 since $K_g^{-1}[p^{-1}] = -1$. Therefore

$$j_p^*(\widetilde{\alpha}(p)) + j_\infty^*(\widetilde{\alpha}(\infty)) \in X_*(T_1)_\Gamma.$$

It still appears that $\widetilde{\alpha}(A, \lambda)$ depends on the choices of the field embeddings j_v. The following result shows that we do not need to worry about this.

Theorem 7.2. $\widetilde{\alpha}(A, \lambda) = 0$.

Proof. The strategy is similar to that of the proof of Lemma 13.2 in [11]. The first step is to notice that $\widetilde{\alpha}(A, \lambda)$ is independent of the choice of symplectic space $(V, \langle \cdot, \cdot \rangle)$ with CM by F we made at the outset. This is proved by imitating the proof of the analogous step in Lemma 13.2 of [11], but since we are concerned with strict symplectic spaces we use the torus T_1 instead of T throughout (see also the proof of Proposition 5.4). Next we can verify that $\widetilde{\alpha}(A,\lambda)$ is independent of the choice of polarization λ. Again this is tedious but not hard. Using the same arguments, we can show that $\widetilde{\alpha}(A,\lambda)$ is not changed if we replace A by an abelian variety in the same $\mathbb{Q}$-isogeny class. Now by a theorem of Tate [32], A is $\mathbb{Q}$-isogenous to the geometric special fiber of an abelian scheme A_1 over $\mathcal{O}_K$ with CM by F, $\mathcal{O}_K$ being the integers of some p-adic field K, which we can choose large enough so that $|\mathrm{Hom}_{\mathbb{Q}-\mathrm{alg}}(F,K)| = [F:\mathbb{Q}]$; thus K splits $T_{\mathbb{Q}_p}$. Moreover, there is a $\mathbb{Q}$-polarization λ_1 of the F-abelian scheme $A_1/\mathcal{O}_K$, by the argument of §5 in [11], where the valuative criterion of properness is verified for certain Shimura varieties.

Therefore for the purposes of proving the theorem we may assume (A,λ) is the reduction modulo p of (A_1,λ_1). We change notation and write (A,λ) for the abelian variety over $\mathcal{O}_K$ and $(\overline{A},\overline{\lambda})$ for the geometric special fiber. We choose $j:\overline{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$ such that $j\circ j_p = j_\infty$, and by base changing via j we can consider the abelian variety over $\mathbb{C}$, which we denote by $A_{\mathbb{C}}$. The polarization λ can also be regarded over $\mathbb{C}$ in this way. By our remarks above, in defining $\widetilde{\alpha}(\overline{A},\overline{\lambda})$ we are free to use the strict symplectic space

$$(H_1(A_{\mathbb{C}},\mathbb{Q}),\ \ -E^{\lambda}(\cdot,\cdot)),$$

where $E^{\lambda}(\cdot,\cdot)$ denotes the Riemann form attached to λ, as in [22] for example.

Claim 7.3.

$$\widetilde{\alpha}(l) \ = \ 0 \quad \forall\ l \neq p,\infty.$$

Proof. We have isomorphisms of strict symplectic spaces with CM by F:

$$H_1(\overline{A},\mathbb{Q}_l) \ = \ H_1(A_{\mathbb{Q}_p},\mathbb{Q}_l) \ = H_1(A_{\mathbb{C}},\mathbb{Q}_l).$$

The first equality holds because the base change isomorphisms for étale (co-)homology preserve the first Chern classes of line bundles, and therefore preserve the $\mathbb{Q}_l(1)$-valued pairings $(\cdot,\cdot)_\lambda$ on both sides. (Here we identify $\mathbb{Q}_l(1)(\overline{k})$ with $\mathbb{Q}_l(1)(\overline{\mathbb{Q}}_p)$ via reduction modulo $p:\mathbb{Z}_p^{un}\to\overline{k}$.) For the second equality we consider the pairing on the left hand side to be $\mathbb{Q}_l$-valued by first using j to identify $\mathbb{Q}_l(1)(\overline{\mathbb{Q}_p})$ with $\mathbb{Q}_l(1)(\mathbb{C})$ and then by using $\exp(2\pi i x)$ to identify this group with $\mathbb{Q}_l$ (the same $i\in\mathbb{C}$ as always). Now the theorem on p.237 of [22] shows that this pairing is precisely the pairing $-E^{\lambda}(\cdot,\cdot)$ on the right hand side, and thus the second equality holds. This proves the claim. □

Now we need to describe $\widetilde{\alpha}(p)$. Choose an isomorphism of strict symplectic spaces with CM by F over L:

$$(D,(\cdot,\cdot)_\lambda) \xrightarrow{f} (H_1(A_{\mathbb{C}},L), -E^{\lambda}(\cdot,\cdot)). \tag{1}$$

This is what we use to define $\widetilde{\alpha}(p)$, and it must be compared to the situation Reimann and Zink consider in [27]. To do this consider the p-adic Tate module $T_p(A) = H_1(A_{\overline{\mathbb{Q}}_p},\mathbb{Z}_p)$. Reimann and Zink choose a strict isomorphism

$$D \ \xrightarrow{\sim}\ T_p(A)\otimes L \tag{2}$$

where the right hand side has L-valued pairing by using j and $\exp(2\pi ix)$ to identify $\mathbb{Q}_p(1)(\overline{\mathbb{Q}}_p)$ with $\mathbb{Q}_p$. They define $b' \in T(L)$ by the equality

$$\Phi^{-1} \;=\; \sigma^{-1}b'.$$

Now recall that the action of F on $\mathrm{Lie}(A)$ (over K) provides a cocharacter $\mu_{\mathrm{CM}} \in X_*(T_{\mathbb{Q}_p})$. (Use j and $\mathrm{Lie}A_{\mathbb{C}} \oplus \overline{\mathrm{Lie}A_{\mathbb{C}}} = H_1(A_{\mathbb{C}}, \mathbb{Q}) \otimes \mathbb{C}$ to prove the existence and the compatibility of μ_{CM} with the complex structure on $\mathrm{Lie}(A_{\mathbb{C}})$.) The theorem of Reimann and Zink (Theorem 4.13) says that if i, j_p and j_∞ are used in defining g, then

$$K_g[\mu_{CM}] \;=\; [b'],$$

where $[b']$ is the class of b' in $B^p(T_1 \to T)$. But now again by [22], p.237 we note that we can identify the right hand side of (1) and (2) above, and so in defining $\widetilde{\alpha}(p)$ we may use the map in (2). Thus we see that $\widetilde{\alpha}(p) = K_g^{-1}[b']^{-1}$, so that the theorem of Reimann and Zink can be interpreted as $\widetilde{\alpha}(p) = -[\mu_{CM}]$.

Finally, we need a description of $\widetilde{\alpha}(\infty)$. We have to find the unique $*$-homomorphism $h : \mathbb{C} \to \mathrm{End}_F(V_{\mathbb{R}})$ such that $\langle \cdot, h(i)\cdot\rangle$ is positive definite.

Claim 7.4. The desired h is given by the complex structure on $\mathrm{Lie}(A_{\mathbb{C}}) = V_{\mathbb{R}}$.

Proof. Recall that $\langle \cdot, \cdot\rangle = -E^\lambda(\cdot, \cdot)$. The relation $-E^\lambda(iv, iw) = -E^\lambda(v, w)$ shows that h is a $*$-homomorphism. It is obvious that the F-action commutes with the complex structure, so $h(i)$ is an element of $\mathrm{End}_F V_{\mathbb{R}}$. Finally, the first chapter of [22] shows that $E^\lambda(i\cdot, \cdot)$ is positive definite on $V_{\mathbb{R}}$, and this shows that $\langle \cdot, h(i)\cdot\rangle = -E^\lambda(\cdot, i\cdot)$ is positive definite. This proves the claim. $\square$

The aforementioned compatibility of μ_{CM} with the complex structure on $\mathrm{Lie}(A_{\mathbb{C}})$ implies that $j^*(\mu_h) = \mu_{CM}$. Now $\widetilde{\alpha}(\overline{A}, \overline{\lambda}) = 0$ follows from the equalities

$$\begin{aligned} j_p^*\widetilde{\alpha}(p) + j_\infty^*\widetilde{\alpha}(\infty) &= -j_p^*[\mu_{CM}] + j_\infty^*[\mu_h] \\ &= -j_p^* j^*[\mu_h] + j_\infty^*[\mu_h] \\ &= 0. \end{aligned}$$

$\square$

8 Proof of the Main Theorem

In this section we give the proof of Theorem 6.10. The first step is to show that both sides of the desired equality are trivial when β^p and β_r are chosen in a particular way. First we need a preliminary construction. Suppose that (A, λ) and the pair (β^p, β_r) are used to define the elements γ, δ, and thus also the element γ_0 (which is only determined up to $\overline{\mathbb{Q}}$-conjugacy). As before, write $I_0 = G_{\gamma_0}$, and $I = \mathrm{Aut}(A, \lambda)$. As in §14 of [11], write $M = \mathrm{End}(A)$ and $M_0 = \mathrm{End}_{\gamma_0}(V)$, and consider the functor $\mathbf{\Psi}$ of $\mathbb{Q}$-algebras to sets where for each $\mathbb{Q}$-algebra R, $\mathbf{\Psi}(R)$ is the set of all $*$-isomorphisms $\psi : M_0 \to M$ over R taking γ_0 to π_A^{-1}. We know from §14 of [11] that $\mathbf{\Psi}(\overline{\mathbb{Q}})$ is nonempty, and any two of its elements differ from each other by an inner automorphism of $I_0(\overline{\mathbb{Q}})$. Moreover any element of $\mathbf{\Psi}(\overline{\mathbb{Q}})$ restricts to an inner twisting of $\mathbb{Q}$-groups $I_0 \to I$. Choose $\psi \in \mathbf{\Psi}(\overline{\mathbb{Q}})$, and write $\psi_v = \psi \otimes 1 \in \mathbf{\Psi}(\overline{\mathbb{Q}}_v)$ for the map resulting from ψ by extension of scalars from $\overline{\mathbb{Q}}$ to $\overline{\mathbb{Q}}_v$, for each place v of $\mathbb{Q}$. For every finite place v of $\mathbb{Q}$, our choice of β_v and Tate's theorem gives us another element $\psi(v) \in \mathbf{\Psi}(\overline{\mathbb{Q}}_v)$, and moreover the elements ψ_v and $\psi(v)$ differ from each other by an inner automorphism of $I_0(\overline{\mathbb{Q}}_v)$.

Now choose a maximal $\mathbb{Q}$-torus T of I which is elliptic at the infinite place and at all finite places of $\mathbb{Q}$ where I_0 is not quasisplit (the existence of T follows from the existence of the local tori, together with weak approximation for the variety of maximal tori in I; see Cor. 3, p.405 of [25]). The torus T transfers to I_0 locally everywhere. Further, by the argument of §14 of [11], we see that the obstruction to T tranferring to I_0 globally vanishes, because T is elliptic at the infinite place; hence T transfers to I_0 globally. Therefore the element $[\psi \circ \tau(\psi)^{-1}]$ of $H^1(\mathbb{Q}, I_{ad})$ which defines the inner twist I_0 of I is the image of an element $[t_\tau] \in H^1(\mathbb{Q}, T_{ad})$, where T_{ad} denotes the image of T in I_{ad}. Therefore by making a different choice of ψ we can assume

$$\mathrm{Int}(t_\tau) \ = \ \psi \circ \tau(\psi)^{-1},$$

for every $\tau \in \Gamma$. Now let $N = \mathrm{Cent}_M T$. Then N is a maximal commutative semisimple subalgebra of M, and is free of rank two over $N_0 = \{x \in N | x^* = x\}$. It follows that N is a CM-algebra of rank $2(\dim A)$, in the sense discussed earlier. The equation above implies that $\phi = \psi^{-1} : M \to M_0$ restricts to give an embedding $\phi : N \hookrightarrow M_0$, *which is defined over* $\mathbb{Q}$. We also get a $\mathbb{Q}$-embedding $\phi : T \hookrightarrow I_0$, because

$$T(R) \ = \ \{x \in N_R \mid x^* x \in R^\times\}.$$

We have just constructed a $\mathbb{Q}$-torus inside I_0 which is the group of automorphisms of the CM-algebra $F = \phi(N)$. From now on write $\phi(T) = T_0$ and $T_1 = T_0 \cap G^{sc}$. This torus T_0 is used in the proof of the next proposition: if γ_0 were regular, the proof of this proposition would be simpler because then I_0 would itself be the torus of automorphisms of a CM-algebra. We need the torus T_0 in the general case.

Proposition 8.1. *Let $c = c_0 p^r$ be a positive rational number, with c_0 a p-adic unit. Let $d \in \mathcal{O}_L^\times$ be such that $d^{-1}\sigma^r(d) = c_0$. Let (A, λ) be a c-polarized virtual abelian variety over k_r up to isogeny which satisfies the conditions in §14 of [11]. Suppose*

$$\beta^p : (H_1(\overline{A}, \mathbb{A}_f^p)\, , \, (\cdot,\cdot)_\lambda) \longrightarrow (V \otimes \mathbb{A}_f^p \, , \, \langle \cdot, \cdot \rangle)$$

$$\beta_r : (H(A)\, , \, d(\cdot,\cdot)_\lambda) \longrightarrow (V \otimes L_r \, , \, \langle \cdot, \cdot \rangle)$$

are symplectic isomorphisms. *Suppose that we use β^p and β_r to define γ and δ. Then*

$$\alpha_1(\gamma_0; \gamma, \delta) \ = \ 0.$$

Proof. Note that the statement to be proved is independent of the choice of γ_0, by Proposition 5.3. Therefore we fix one choice of γ_0 and define I_0, M_0 and the functor $\mathbf{\Psi}$ as above. This yields T, N, ϕ, F, T_0, T_1 just as before. It is clear that $(V, \langle\cdot,\cdot\rangle)$ is a symplectic space with CM by F and that

$$\begin{aligned} T_1 &= \mathrm{Aut}_{\mathrm{strict}}(V, \langle\cdot,\cdot\rangle), \\ T_0 &= \mathrm{Aut}(V, \langle\cdot,\cdot\rangle). \end{aligned}$$

We see that $\overline{A}$ has CM by the CM-algebra F (action is through that of N via ϕ^{-1}), and furthermore that the $\mathbb{Q}$-polarization $\lambda : \overline{A} \to \check{\overline{A}}$ commutes with the action of F. Therefore we can define $\widetilde{\alpha}(\overline{A}, \lambda) \in X^*(\hat{T}_1^{\Gamma})$, as in §7. Since we have proved in Theorem 7.2 that this element is always trivial, to prove that $\alpha_1(\gamma_0; \gamma, \delta)$ is trivial it suffices to show that for every place v of $\mathbb{Q}$, $\widetilde{\alpha}(v) \mapsto \alpha_1(v)$ under the maps

$$\begin{cases} X^*(\hat{T}_1^{\Gamma(l)}) \longrightarrow X^*(Z(\hat{I}_1)^{\Gamma(l)}),\ l \neq p, \infty, \\ X(T_1 \to T_0)_{\mathbb{Q}_p} \longrightarrow X(I_1 \to I_0)_{\mathbb{Q}_p}/\mathrm{im}(\mathbb{Z}_p^{\times}), \\ X(T_1 \to T_0)_{\mathbb{C}} \longrightarrow X(I_1 \to I_0)_{\mathbb{C}} \end{cases}$$

induced by the canonical inclusions $Z(\hat{I}_1) \hookrightarrow \hat{T}_1$ and $Z(\hat{I}_0) \hookrightarrow \hat{T}_0$. Recall that we have chosen in our preliminary construction an element $\psi \in \mathbf{\Psi}(\overline{\mathbb{Q}})$. Write $\phi = \psi^{-1}$, $\phi_v = \psi_v^{-1}$, and $\phi(v) = \psi(v)^{-1}$. Then for every finite place v there exists $h_v \in I_1(\overline{\mathbb{Q}}_v)$ such that $\mathrm{Int}(h_v) \circ \phi(v) = \phi_v$. We will use these elements h_v to prove that $\widetilde{\alpha}(v) \mapsto \alpha_1(v)$.

First consider the case $l \neq p, \infty$. Then to define $\alpha_1(l)$ we chose $g_l \in G^{sc}(\overline{\mathbb{Q}}_l)$ such that $g_l \gamma_0 g_l^{-1} = \gamma_l$. Thus to define the element $\widetilde{\alpha}(l)$ we may use the map

$$H_1(\overline{A}, \mathbb{Q}_l) \xrightarrow{h_l g_l^{-1} \beta_l} V \otimes \mathbb{Q}_l.$$

To see this, note that this map is a symplectic isomorphism (as β_l is), and preserves the CM-algebra actions because N is transported by this map over to

$$\mathrm{Int}(h_l)\phi(l)(N) \ = \ \phi_l(N) \ = \ F.$$

Since $h_l \in I_1(\overline{\mathbb{Q}}_l)$, it follows that the 1-cocycle $[h_l g_l^{-1} \beta_l \tau (h_l g_l^{-1} \beta_l)^{-1}]$ maps to $[g_l^{-1} \tau(g_l)]$ under

$$H^1(\mathbb{Q}_l, T_1) \longrightarrow H^1(\mathbb{Q}_l, I_1),$$

and thus via the Tate-Nakayama isomorphisms we see that $\widetilde{\alpha}(l) \mapsto \alpha_1(l)$.

For the case $v = \infty$, recall that $\alpha_1(\infty)$ was defined by choosing an elliptic maximal $\mathbb{R}$-torus T' of G containing γ_0 and an $h' \in X^+$ such that h' factors through $T'(\mathbb{R})$. Then we defined $\alpha_1(\infty)$ using $\mu_{h'}$. On the other hand, $\widetilde{\alpha}(\infty)$ is defined using μ_h, where

$$h : \mathbb{C} \longrightarrow \mathrm{End}_F(V \otimes \mathbb{R}) = F \otimes \mathbb{R}$$

is the unique $*$-homomorphism such that $\langle \cdot, h(i) \cdot \rangle$ is positive definite on $V \otimes \mathbb{R}$. It follows that we may take $T' = T_0$ and $h' = h$, and so $\widetilde{\alpha}(\infty) \mapsto \alpha_1(\infty)$.

Finally, consider the case $v = p$. To define $\alpha_1(p)$, recall that we choose $y \in G(L)$ such that (i) $y\gamma_0 y^{-1} = N\delta = \gamma_p$, and (ii) $c(y) = d$. (We can use d here because $c(\beta_r) = d \Rightarrow c(\delta) = p^{-1}d\sigma(d)^{-1}$.) Then $\alpha_1(p) = M_g[y^{-1}\delta\sigma(y)]$. Now consider the set

$$\{g \in I_1 \mid \mathrm{Int}(g) \circ \phi(p) = \phi_p\},$$

where we are considering equality only as maps $T \to I_0$. This set is a left T_1-torsor, defined over L. Moreover, it has a point over $\overline{L}$, namely the element h_p constructed above. Steinberg's Theorem then implies that this set has a point h over L. Now we can define $\widetilde{\alpha}(p)$ using the map

$$H(\overline{A}) \xrightarrow{hy^{-1}\beta_r} V \otimes L.$$

To see this, note that this map preserves pairings, since $c(y) = d = c(\beta_r)$. Also, by our choice of h, T is transported over to

$$\mathrm{Int}(h)\phi(p)(T) \;=\; \phi_p(T) \;=\; T_0.$$

Therefore N is transported over to $\mathrm{Cent}_{M_0}T_0 = F$. Finally, note that Φ is transported over to $[hy^{-1}\delta\sigma(y)\sigma(h^{-1})\sigma]$, and thus $\widetilde{\alpha}(p) = M_g[hy^{-1}\delta\sigma(y)\sigma(h^{-1})]$. From this it is obvious that $\widetilde{\alpha}(p) \mapsto \alpha_1(p)$. □

We can now give the *Proof of Theorem 6.10:*

Proof. Choose $d \in \mathcal{O}_L^\times$ such that $d^{-1}\sigma^r(d) = c_0$. We first note that because we are assuming that the c-polarized virtual abelian variety (A, λ) comes from a fixed point of our correspondence, there exist maps

$$\beta^p : (H_1(\overline{A}, \mathbb{A}_f^p), (\cdot,\cdot)_\lambda) \longrightarrow (V \otimes \mathbb{A}_f^p, \langle\cdot,\cdot\rangle)$$

$$\beta_r : (H(A), d(\cdot,\cdot)_\lambda) \longrightarrow (V \otimes L_r, \langle\cdot,\cdot\rangle),$$

for which $(\beta^p)^*\langle\cdot,\cdot\rangle = (\cdot,\cdot)_\lambda$ and $\beta_r^*\langle\cdot,\cdot\rangle = d(\cdot,\cdot)_\lambda$. For β^p this is a consequence of the existence of *some* symplectic similitude β^p (because there is a level structure η) and the fact that $c : G(\mathbb{A}_f^p) \to (\mathbb{A}_f^p)^\times$ is surjective. For β_r this follows from Lemma 7.2 of [11]. Let γ and δ be constructed using β^p and β_r. It is obvious that in this case the right hand side of the equation in Theorem 6.10 is trivial, and we have just proved in Proposition 8.1 that the left hand side is also trivial. Thus the conclusion of Theorem 6.10 holds in this special case. The general case (where β^p and β_r are arbitrary symplectic similitudes) follows from the special case and the fact that both sides of the equation in Theorem 6.10 transform in the same way. Indeed, if we take $g \in G(\mathbb{A}_f^p)$ (resp. $h \in G(L_r)$) and replace a symplectic *isomorphism* β^p with $g\beta^p$ (resp. replace a symplectic *isomorphism* β_r with $h\beta_r$), then the result is that both sides of the equation in Theorem 6.10 are

$$\overline{\varpi}c(g)^{-1}\overline{\varpi}c(h)^{-1},$$

because of Theorem 5.7. □

We conclude with statement of the Main Theorem, which has now been proved:

Theorem 8.2. *Let $G = GSp_{2g}$. Suppose p is an odd prime. Let $K_p \subset G(\mathbb{Q}_p)$ be a hyperspecial maximal compact subgroup, and let $K^p \subset G(\mathbb{A}_f^p)$ be a sufficiently small compact open subgroup, as in §1. Let S_K denote the Shimura variety associated to $K = K^p K_p$ as in §1. Let f denote the Hecke correspondence coming from an element $g \in G(\mathbb{A}_f^p)$. Fix a character $\varpi : \pi_p \to \overline{\mathbb{Q}}_\ell^\times$ (the group π_p is identified with the set of connected components of S_K as in §2). Set $\omega = \varpi \circ c : G(\mathbb{A}) \to \overline{\mathbb{Q}}_\ell^\times$. Let $L(\omega)$ denote the operator on $H_c^i(S_K \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell)$ as in §2. Then the virtual trace*

$$Tr(\Phi_p^r \circ f \circ L(\omega) \ ; \ H_c^\bullet(S_K \otimes \overline{\mathbb{Q}}, \overline{\mathbb{Q}}_\ell))$$

is zero unless $\omega(g) = 1$, in which case there exists a natural number $r(f)$ such that for all $r \geq r(f)$ the virtual trace is given by the expression

$$\sum_{(\gamma_0;\gamma,\delta)} c(\gamma_0;\gamma,\delta) \ \overline{\omega}_p(p^{-1}) \ \langle \alpha_1(\gamma_0;\gamma,\delta), \kappa_0 \rangle \ O_\gamma^\omega(f^p) \ TO_\delta^\omega(\phi_r),$$

where we take the sum over all G-equivalence classes of triples $(\gamma_0;\gamma,\delta)$ such that

1. *$\alpha(\gamma_0;\gamma,\delta) = 0$,*
2. *γ_0 is ω-special,*

and where $\kappa_0 \in Z(\hat{I}_1)^\Gamma$ is any element which satisfies $\partial(\kappa_0) = \mathbf{a}$.

Here $\mathbf{a} \in H^1(\mathbb{Q}, Z(\hat{G}))$ is the Langlands parameter corresponding to the character ω, and $\partial : Z(\hat{I}_1)^\Gamma \to H^1(\mathbb{Q}, Z(\hat{G}))$ is the boundary map arising from the exact sequence of Γ-modules

$$1 \longrightarrow Z(\hat{G}) \longrightarrow Z(\hat{I}_0) \longrightarrow Z(\hat{I}_1) \longrightarrow 1,$$

where $I_0 = G_{\gamma_0}$ and $I_1 = G_{\gamma_0}^{sc}$.

Remark 8.3. Recall that the invariant $\alpha_1(\gamma_0;\gamma,\delta)$ was constructed only after we fixed a choice of initial data $\mathcal{D}$. This is not too disturbing, however, because the operators $L(\omega)$ on $H_c^i(S_K \otimes \overline{k}, \overline{\mathbb{Q}}_\ell)$ which we used (via the base change theorems of étale cohomology) to find an expression for the virtual trace, are also defined only after choosing $\mathcal{D}$, and the dependence of $\alpha_1(\gamma_0;\gamma,\delta)$ on $\mathcal{D}$ merely reflects this fact.

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