ON MATRIX COEFFICIENTS OF THE SATAKE ISOMORPHISM: COMPLEMENTS TO THE PAPER OF M. RAPOPORT

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Abstract. We consider the matrix for the Satake isomorphism with respect to natural bases. We give a simple proof in the case of Chevalley groups that the matrix coefficients which are not obviously zero are in fact positive numbers. We also relate the matrix coefficients to Kazhdan-Lusztig polynomials and to Bernstein functions.

Introduction

M. Rapoport's paper ([10]) is concerned with a matrix $C_{\lambda,\mu}$ which expresses the Satake isomorphism in terms of natural bases (see below for a precise definition of $C_{\lambda,\mu}$). The main theorem is that all of the matrix coefficients $C_{\lambda,\mu}$ which are not obviously zero are in fact positive numbers. The present article is intended to complement that of Rapoport, and consists of three parts. In the first section we present the main result, which is a short proof of the positivity property in the case of p-adic Chevalley groups. The key tools are a theorem of Dabrowski ([2]) which gives a criterion for the intersection of cells in the Iwahori and Iwasawa decompositions of a p-adic Chevalley group, and Macdonald's formula for the Satake transform.

In the second section we relate the matrix $C_{\lambda,\mu}$ to the Kazhdan-Lusztig polynomials for the affine Weyl group attached to a split p-adic group. We use the work of S. Kato ([7]), which is itself an extension of work of G. Lusztig ([8]). For the affine Weyl group of type $\widetilde{A}_1$ the Kazhdan-Lusztig polynomials are very simple, and using this observation we identify the coefficients $C_{\lambda,\mu}$ for the group Gl_2. This provides a direct verification of the positivity property in this case, and indicates that Kazhdan-Lusztig polynomials might provide an explanation of the positivity phenomenon in general.

In the third section we show that the matrix $C_{\lambda,\mu}$ is also the one that relates Bernstein functions to spherical functions, relative to the natural isomorphism between the center of the Iwahori-Hecke algebra and the spherical Hecke algebra of a split p-adic group.

1. A simple proof of Positivity for Chevalley groups

Let F be a p-adic field, with algebraic closure $\overline{F}$. Let $q = p^j$ denote the cardinality of the residue field of F. Let $\mathcal{O}$ denote the ring of integers in F, with ϖ a uniformizer of the maximal ideal. Let G be a Chevalley group, or more generally a split connected reductive group over F whose adjoint group is the Chevalley group corresponding to an irreducible and reduced root system R. The group G is then defined over $\mathcal{O}$. Fix a choice of positive roots R^+ and simple positive roots Δ. For any root $\alpha \in R^+$, let Φ_α denote the corresponding group homomorphism $\Phi_\alpha : Sl_2 \to G$. For any $a,\ c \in \overline{F}$, $c \neq 0$, let

$$x_\alpha(a) = \Phi_\alpha \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix},$$

$$x_{-\alpha}(a) = \Phi_\alpha \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix},$$

$$t_\alpha(c) = \Phi_\alpha \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix},$$

$$\omega_\alpha = \Phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Let Z denote the center of G. Let U, T, and N be respectively the subgroups of G generated by the sets $\{x_\alpha(a) \mid a \in \overline{F},$ and $\alpha \in R^+\}$, $Z \cup \{t_\alpha(c) \mid c \in \overline{F},\ c \neq 0,$ and $\alpha \in R^+\}$, and $T \cup \{\omega_\alpha \mid \alpha \in R^+\}$. Then T is a maximal torus of G, contained in the Borel subgroup $B = TU$, and N is the normalizer of T in G. Let $K = G(\mathcal{O})$; it is a special, good, maximal compact open subgroup of $G(F)$. Let $T_{sc} = T \cap G_{sc}$, the inverse image of T under $G_{sc} \to G$. Let T_0, U_0 and N_0 denote, respectively, the intersections $T(F) \cap K$, $U(F) \cap K$ and $N(F) \cap K$.

Let $W = N(F)/T(F)$ be the finite Weyl group of G, and let $\widetilde{W} = N(F)/T_0$ denote the extended affine Weyl group. There is a canonical isomorphism $\widetilde{W} \xrightarrow{\sim} X_*(T) \rtimes W$. Let $W_a = X_*(T_{sc}) \rtimes W$; this is a Coxeter group with generators consisting of the simple affine reflections $S_a = \{s_\alpha \mid \alpha \in \Delta\} \cup \{s_a = t_{-\check{\alpha}_0} s_{\alpha_0}\}$, where $t_\lambda \in \widetilde{W}$ is the translation corresponding to the cocharacter $\lambda \in X_*(T)$, and α_0 is the highest root of R^+. Let Ω denote the subgroup of $\widetilde{W}$ which preserves the set of simple reflections $S_a \subset \widetilde{W}$ under conjugation. Then there is a canonical isomorphism $\widetilde{W} \xrightarrow{\sim} \Omega \ltimes W_a$. The length function $l : W_a \to \mathbb{N}$ is extended to a length function on $\widetilde{W}$ by setting the length of ωx ($\omega \in \Omega,\ x \in W_a$) to be $l(x)$.

We fix an Iwahori subgroup for $G(F)$ as follows. Let I denote the subgroup of K generated by U_0, T_0, and the subgroup V generated by the set $\{x_{-\alpha}(a) \mid a \in \varpi\mathcal{O},\ \alpha \in R^+\}$. Then I is an Iwahori subgroup of $G(F)$ in good position relative to K, meaning that $IWI = K$. Note that both I and K contain T_0, so for any $w \in \widetilde{W}$ it makes sense to write $wI = nI$ (resp. $wK = nK$), where $n \in N(F)$ maps to w under the projection $N(F) \to \widetilde{W}$. For $\lambda \in X_*(T)$, let ϖ^λ denote the image of $\lambda \otimes \varpi$ under the canonical isomorphism $X_*(T) \otimes F^\times \xrightarrow{\sim} T(F)$. The element $\varpi^\lambda \in T(F)$ maps to $t_\lambda \in \widetilde{W}$ under the projection $N(F) \to \widetilde{W}$, so with the conventions above we can write $Kt_\lambda K = K\varpi^\lambda K$, $It_\lambda I = I\varpi^\lambda I$, and $U(F)t_\lambda I = U(F)\varpi^\lambda I$, etc. (The notation in the main theorem below thus differs slightly from that of Rapoport [10].)

Although the groups $G(F)$, K, $T(F)$, T_0, $U(F)$ and I change if F is replaced by a finite extension field, the groups W, $\widetilde{W}$, and W_a do not; in particular they are independent of the quantity q.

Next we will summarize some definitions and results of R. Dabrowski ([2]). If $w \in \widetilde{W}$, denote by $\overline{w}$ the image of w in the finite Weyl group W under the canonical projection $\widetilde{W} \to W$. Fix an element $\tau \in \widetilde{W}$ and a reduced expression $\tau = \omega s_1 \dots s_k$, where the s_i are simple affine reflections, and $\omega \in \Omega$. Following Deodhar ([4]) we call a sequence of elements $\{\sigma_j\}_{j=0}^k$ a *subexpression* if $\sigma_0 = \omega$ and for each $j \geq 1$, $\sigma_j \in \{\sigma_{j-1}, \sigma_{j-1}s_j\}$. Following Dabrowski ([2]) we call the subexpression *good* if the following hold:

- $\overline{\sigma}_{j-1}s_j \leq \overline{\sigma}_j$ for all j such that $s_j \in S$,
- $\overline{\sigma}_{j-1}s_{\alpha_0} \geq \overline{\sigma}_j$ for all j such that $s_j = s_a$.

Here α_0 denotes the highest root of R^+ and $s_a = t_{-\check{\alpha}_0}s_{\alpha_0}$ is the corresponding simple affine reflection. The ordering $\leq$ denotes the Bruhat order on the Coxeter group (W, S), where S is the set $W \cap S_a$. Let $\mathcal{G}$ denote the set of all good subexpressions (relative to a fixed reduced expression of a fixed τ). Denote by $G(\tau)$ the set $\{\sigma_k \mid \{\sigma_j\}_{j=0}^k \in \mathcal{G}\}$. We will use the following result of Dabrowski:

Proposition 1.1. *Let τ and ω be arbitrary elements of $\widetilde{W}$. Then the following are equivalent:*

(1) $U(F)\omega I \cap I\tau I \neq \emptyset$,
(2) $\omega \in G(\tau)$.

In particular, the set $G(\tau)$ is independent of the choice of reduced expression for τ.

Proof. This is Proposition 3.2 of [2]. □

Let $b : C_c(K\backslash G(F)/K) \xrightarrow{\backsim} \mathbb{C}[X_*(T)]^W$ denote the Satake isomorphism, as defined in ([1]). In particular, this means that Haar measure on $G(F)$ is normalized so that K has volume 1. For $\mu \in X_*(T)$ a dominant coweight, let $f_\mu = \mathrm{char}(Kt_\mu K)$ and $m_\mu = |W_\mu|^{-1}\sum_{w\in W} w(\mu)$ (here W_μ is the stabilizer of μ in W). We define the matrix $C_{\lambda,\mu}$ by the following equation

$$b(f_\lambda) = \sum_\mu C_{\lambda,\mu} m_\mu.$$

For μ and λ in $X_*(T)$, we write $\mu \overset{!}{\leq} \lambda$ if $\lambda - \mu$ is a (possibly empty) sum of simple coroots. When in addition $\mu \neq \lambda$, we write $\mu \overset{!}{<} \lambda$.
The main result of this paper is the following theorem.

Theorem 1.2. *Let μ and λ be dominant coweights in $X_*(T)$. Then the following statements are equivalent:*

(1) $C_{\lambda,\mu} > 0$,
(2) $U(F)t_\mu K \cap Kt_\lambda K \neq \emptyset$,
(3) $\mu \overset{!}{\leq} \lambda$,
(4) $U(F)t_\mu I \cap I\tau I \neq \emptyset$, *for some* $\tau \in Wt_\lambda W$,
(5) $t_\mu \in G(\tau)$, *for some* $\tau \in Wt_\lambda W$.

Proof. The equivalence of (1) and (2) is well-known ([1]). The equivalence of (4) and (5) is a special case of Proposition 1.1 above. The implication (4) ⇒ (2) holds because $I \subset K$ and $K\tau K = Kt_\lambda K$ (the group $K \cap N(F)$ contains a set of representatives of W). The implication (2) ⇒ (4) follows from $Kt_\lambda K = IWIt_\lambda IWI = IWt_\lambda WI$. We now prove (5) ⇒ (3): It is clear from the definition that $G(\tau) \subset \{x \in \widetilde{W} \mid x \leq \tau\}$, where $\leq$ denotes the Bruhat order on $\widetilde{W}$. Therefore if (5) holds, then $t_\mu \leq \tau$ and so $Wt_\mu W \leq W\tau W = Wt_\lambda W$ in the Bruhat order on $W\backslash\widetilde{W}/W$. It is well-known that this implies $\mu \overset{!}{\leq} \lambda$. Finally, we show (3) ⇒ (4): Suppose $\mu \overset{!}{\leq} \lambda$. Then by the Lemma below we see that (2) holds for sufficiently large q. Then as above it follows that (4) holds for sufficiently large q. But (5) and hence (4) is independent of q; hence (4) holds for all q. □

Lemma 1.3. *If $\mu \overset{!}{\leq} \lambda$, then $U(F)t_\mu K \cap Kt_\lambda K \neq \emptyset$ for $q \gg 0$.*

Proof. The intersection is nonempty if and only if the matrix coefficient $C_{\lambda,\mu}(q) > 0$, where

$$b(f_\lambda) = \sum_\mu C_{\lambda,\mu}(q) m_\mu.$$

We write $C_{\mu,\lambda}(q)$ to emphasize the dependence of this matrix on q.

Now recall Macdonald's formula (see Theorem 2.4 of [7]):

$$b(f_\lambda) = \frac{q^{l(\lambda)/2}}{W_\lambda(q^{-1})} \sum_{w \in W} e^{w(\lambda)} \prod_{\alpha>0} \frac{1 - q^{-1}e^{-w(\alpha)}}{1 - e^{-w(\alpha)}}.$$

As $q \to \infty$, the sum approaches

$$\sum_{w \in W} e^{w(\lambda)} \prod_{\alpha>0} (1 - e^{-w(\alpha)})^{-1}.$$

By Weyl's character formula, this last expression is the character of the highest weight module for λ:

$$\chi_\lambda = \sum_\mu m_\lambda(\mu) m_\mu.$$

One knows that $m_\lambda(\mu) > 0$ if and only $\mu \stackrel{!}{\leq} \lambda$ (see [5], p.204), and if so it follows that $(W_\lambda(q^{-1})/q^{l(\lambda)/2})C_{\mu,\lambda}(q)$ and therefore $C_{\mu,\lambda}(q)$ is > 0 for $q \gg 0$. □

2. The Relation with Kazhdan-Lusztig polynomials

Let G be as above (or more generally an almost simple, connected reductive group, defined and split over F). Define a matrix $M(q)$ by

$$M(q)_{\mu,\lambda} = q^{-l(\lambda)/2} P_{w_0\mu, w_0\lambda}(q).$$

Here μ and λ range over dominant coweights in $X_*(T)$ (in a suitable ordering compatible with $\stackrel{!}{\leq}$), w_0 denotes the longest element of W, and $P_{x,y}(q)$ are the Kazhdan-Lusztig polynomials for the group $\widetilde{W}$.

Note first that Theorem 1.5 of [7] with $q = 1$ gives

$$\chi_\lambda = \sum_\mu M(1)_{\mu,\lambda} m_\mu.$$

On the other hand, Lemma 2.7, (3.5), and Theorem 1.8 of (loc. cit.) imply that

$$\chi_\lambda = \sum_\mu M(q)_{\mu,\lambda} b(f_\mu).$$

It follows that

Proposition 2.1. $C_{\lambda,\mu}(q) = [M(1)M(q)^{-1}]_{\mu,\lambda}$.

We can use this to illustrate the positivity property in the case of Gl_2. Since Gl_2 is of type A_1, we know that the Kazhdan-Lusztig polynomials for $\widetilde{W}$ are particularly simple: $P_{x,y}(q) = 1$ if $x \leq y$ and 0 otherwise. It follows that

$$M(1)_{\mu,\lambda} = \begin{cases} 1, & \text{if } \mu \stackrel{!}{\leq} \lambda, \\ 0, & \text{otherwise,} \end{cases}$$

and thus

$$M(1)^{-1}_{\mu,\lambda} = \begin{cases} -1, & \text{if } \mu = \lambda - \check{\alpha}, \\ 1, & \text{if } \mu = \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Here $\check{\alpha} = (1, -1)$ is the unique simple coroot. Moreover $M(q) = M(1)\mathrm{diag}(\ldots, q^{-l(\lambda)/2}, \ldots)$. This implies

$$M(q)^{-1}_{\sigma,\lambda} = \begin{cases} -q^{(l(\lambda)/2)-1}, & \text{if } \sigma = \lambda - \check{\alpha}, \\ q^{l(\lambda)/2}, & \text{if } \sigma = \lambda, \\ 0, & \text{otherwise,} \end{cases}$$

and thus

$$C_{\lambda,\mu}(q) = \sum_{\mu \overset{!}{\leq} \sigma \overset{!}{\leq} \lambda} M(1)_{\mu,\sigma} M(q)^{-1}_{\sigma,\lambda} = \begin{cases} q^{l(\lambda)/2} - q^{(l(\lambda)/2)-1}, & \text{if } \mu \overset{!}{<} \lambda, \\ q^{l(\lambda)/2}, & \text{if } \mu = \lambda, \\ 0, & \text{otherwise.} \end{cases}$$

The positivity of $C_{\lambda,\mu}(q)$ for $\mu \overset{!}{\leq} \lambda$ is therefore evident in this case.

3. The Relation with Bernstein Functions

We retain the notation of the previous section. Consider the following diagram:

$$\begin{array}{ccc} \mathbb{C}[X_*(T)]^W & \xleftarrow[\sim]{b} & \mathrm{C}_c(K\backslash G(F)/K) \\ \Big\| & & \Big\uparrow e_K * - \\ \mathbb{C}[X_*(T)]^W & \xrightarrow[\sim]{Bern} & Z(\mathrm{C}_c(I\backslash G(F)/I)) \end{array}$$

Here the right vertical map is multiplication by $e_K = \mathrm{char}(K)$. The convolution product in $C_c(K\backslash G(F)/K)$ (resp. $C_c(I\backslash G(F)/I)$) is defined using the Haar measure on $G(F)$ for which $\mathrm{vol}(K) = 1$ (resp. $\mathrm{vol}(I) = 1$). The Satake isomorphism is defined using the same normalizations as in [1]. The map *Bern* was discovered by Bernstein and is constructed as follows (see [8], [9]). For any coweight $\nu \in X_*(T)$, write $\nu = \nu_1 - \nu_2$ where the elements ν_i are dominant. Then define an element of the Iwahori-Hecke algebra

$$\Theta_\nu = q^{-(l(\nu_1)-l(\nu_2))/2} T_{t_{\nu_1}} T^{-1}_{t_{\nu_2}},$$

where T_w denotes the generator of the Iwahori-Hecke algebra $C_c(I\backslash G(F)/I)$ given by the characteristic function of the double coset IwI, for any $w \in \widetilde{W}$. Then the element Θ_ν is independent of the choice of ν_1 and ν_2. Moreover it is proved in [8] that if $\mu \in X_*(T)$ is dominant and we set

$$z_\mu = \sum_{\nu \in W(\mu)} \Theta_\nu,$$

then the element z_μ is in $Z(C_c(I\backslash G(F)/I)$, and moreover the map which associates to $m_\mu \in \mathbb{C}[X_*(T)]^W$ the element z_μ defines an isomorphism of $\mathbb{C}$-algebras $Bern : \mathbb{C}[X_*(T)]^W \xrightarrow{\sim} Z(C_c(I\backslash G(F)/I)$. We call the elements z_μ the *Bernstein functions*.

The next proposition seems to be well-known to the experts. It implies that the center of the Iwahori-Hecke algebra is naturally isomorphic to the spherical Hecke algebra. A proof can be found in a paper of J.-F. Dat ([3]). It can also be deduced easily from known results of Lusztig ([8]) and S. Kato ([7]), as is explained in [6].

Proposition 3.1. *The Satake and Bernstein isomorphisms are compatible, i.e., the diagram above is commutative.*

We deduce that the matrix $C_{\lambda,\mu}$ relates the Bernstein functions to the spherical functions.

Proposition 3.2. $f_\lambda = \sum_\mu C_{\lambda,\mu} e_K * z_\mu$.

Proof. Apply the Satake isomorphism b to both sides, then use the commutativity of the above diagram and the definition of the matrix $C_{\lambda,\mu}$. □

From this we get an interesting relation for the matrix coefficients $C_{\lambda,\mu}$:

Corollary 3.3. *Let ρ denote half the sum of the positive roots. Let $W(q) = \sum_{w\in W} q^{l(w)}$. Then for any dominant $\lambda \in X_*(T)$,*

$$W(q)^{-1} \sum_{w\in Wt_\lambda W} q^{l(w)} = \sum_{\mu \overset{!}{\leq} \lambda} C_{\lambda,\mu}(q) \Big(\sum_{\nu\in W(\mu)} q^{\langle\rho,\nu\rangle} \Big).$$

Proof. Consider the $\mathbb{C}[q^{1/2}, q^{-1/2}]$-algebra homomorphism $\chi : C_c(I\backslash G(F)/I) \to \mathbb{C}[q^{1/2}, q^{-1/2}]$ defined by $\chi(T_w) = q^{l(w)}$, where T_w denotes the generator of the Iwahori-Hecke algebra corresponding to $w \in \widetilde{W}$. We have $f_\lambda = \sum_{w\in Wt_\lambda W} T_w$ and $e_K = \sum_{w\in W} T_w$. Moreover, if $\nu \in W(\mu)$ and $\nu = \nu_1 - \nu_2$ (ν_i dominant), one has $\chi(\Theta_\nu) = q^{-(l(\nu_1)-l(\nu_2))/2} \chi(T_{\nu_1})\chi(T_{\nu_2})^{-1} = q^{(\langle 2\rho,\nu_1\rangle - \langle 2\rho,\nu_2\rangle)/2} = q^{\langle\rho,\nu\rangle}$. The result now follows by applying χ to the equation in the preceeding proposition. □

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