DRINFELD-PLÜCKER RELATIONS FOR HOMOGENEOUS SPACES, (AFFINE) FLAG VARIETIES, AND RAPOPORT-ZINK MODELS

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1. Introduction

Let F be a local field (or any function field $k((\varpi))$), with ring of integers $\mathcal{O}_F$. The main object of this manuscript is to provide a first step in defining Rapoport-Zink "local models" $M_{\check{\mu}}$ attached to an arbitrary split reductive $\mathcal{O}_F$-group G and an arbitrary dominant coweight $\check{\mu}$ for G. As it stands at this time, this paper proposes a definition for $M_{\check{\mu}}$ when $F = k((\varpi))$ and $\mathrm{char}(k) = 0$ (the "equi-char 0" case).

The main idea is to use every representation of G, rather than just one (the "standard representation" is used to define local models attached to classical groups – this does not seem to be adequate for orthogonal groups). Thus we are looking for a "Tannakian" description of $M_{\check{\mu}}$.

Let G denote a linear algebraic group over an algebraically closed field k. Let $\mathrm{Rep}_k G$ denote the category of finite dimensional representations of G over the field k. By a *Tannakian description* of a variety X associated to G (e.g. a flag variety), we mean a bijection between the set of points of X and the set of collections of certain data (e.g. lines, lattices) indexed by the objects of $\mathrm{Rep}_k G$, subject to certain compatibilities with respect to G-morphisms and tensor products. If G is a connected reductive group with a maximal torus T and a Borel subgroup $B = TU$, we let $X^+(T)$ denote the set of B-dominant characters of T. By a *Plücker description* of X we mean a similar bijection where data is only specified for every Weyl module $V(\lambda)$ ($\lambda \in X^+(T)$), rather than for every representation of G.

For H any closed subgroup of a linear algebraic group G, we give a Tannakian description for the affine closure $\overline{G/H}$ of the quotient G/H. Using a theorem of Grosshans [6], the case $H = U$ yields a Plücker description for G/U, where U is a maximal unipotent subgroup of a connected reductive group G. This description of G/U, apparently due to Drinfeld, is announced in work of Kuznetsov [11] (when $\mathrm{char}(k) = 0$) and also Braverman-Gaitsgory [2]. We refer to the tensor compatibility conditions which arise in this and any similar context as *Drinfeld-Plücker relations.*

Letting $B = TU$ be a Borel subgroup, the torus T acts freely on G/U and the quotient is the finite flag variety G/B. Translating this in terms of the Plücker description for G/U we arrive at a Plücker description for G/B, which is also due to Drinfeld (unpublished) and is used in the characteristic zero setting by Kuznetsov [11] and Finkelberg-Mirkovic [4]. The version presented here works equally well in any characteristic.

The first goal of this manuscript is simply to present complete proofs of the aforementioned results (to my knowledge, no proofs have appeared in the literature before now). Also, the description of $\overline{G/H}$ for arbitrary subgroups H does not seem to have appeared before.

The second goal is to give similar Plücker descriptions for the affine Grassmanian and the affine flag variety, and some indications for a Plücker description for Rapoport-Zink local models (we describe "equi-characteristic 0" local models).

More precisely, following [4] we prove a Plücker description for the affine Grassmanian

$$G(k((t)))/G(k[[t]]),$$

when $\mathrm{char}(k) = 0$. (This is apparently due to Beilinson-Drinfeld.) We give a similar description of the closure $\overline{\mathcal{Q}}_{\check{\mu}}$ of the $G(k[[t]])$-orbit $\mathcal{Q}_{\check{\mu}}$ indexed by the dominant coweight $\check{\mu}$.

We also give a Plücker description of the affine flag variety $G(k((t)))/I_k$, where I_k is the Iwahori subgroup $I_k = \pi^{-1}(B)$, where $\pi : G(k[[t]]) \to G(k)$ is the homomorphism given by $t \mapsto 0$. We describe as well the finite-dimensional analogues $\mathcal{F}l_{\leq\check{\mu}}$ of $\overline{\mathcal{Q}}_{\check{\mu}}$.

Adding a parameter ϖ, we give the Plücker relations for Beilinson's deformation of the affine Grassmanian to the affine flag variety, which played a crucial role in Gaitsgory's construction of the center of the affine Hecke algebra via nearby cycles, see [5] and [7]. More precisely, we define a model over $\mathrm{Spec}(k[[\varpi]])$ which on the generic fiber is the affine Grassmanian of G over $k((\varpi))$, and on the special fiber is the affine flag variety of G over k. The finite-dimensional scheme $M_{\check{\mu}}$ corresponding to a minuscule $\check{\mu}$ looks like a local model for a Shimura variety with Iwahori level structure: the generic fiber is the subvariety $\bar{\mathcal{Q}}_{\check{\mu}}$ of the affine Grassmanian, and the special fiber $\mathcal{F}l_{\leq\check{\mu}}$ has a stratification indexed by a set which contains the $\check{\mu}$-*admissible* orbits of the Iwahori subgroup I_k acting on the affine flag variety. This scheme $M_{\check{\mu}}$ is our candidate for the definition of a Rapoport-Zink model attached to G and $\check{\mu}$.

Speculative remarks:

Similar ideas should be useful in giving a definition of $M_{\check{\mu}}$ over a p-adic number ring. (However, I expect that only a Tannakian description, not a Plücker description, will be possible for a general group.) At least when $\check{\mu}$ is minuscule, it should be possible to define the correct object this way (i.e. a Witt-vector description of the affine Grassmanian is not needed in this case). Also, these ideas should give a Tannakian description of the Bruhat-Tits parahoric group schemes. It is conceivable that the Bruhat-Tits building itself has a Tannakian description (a point would be identified with a "Moy-Prasad filtration" on every finite-type rational representation of G over $\mathcal{O}_F$, the filtrations being compatible in a certain sense with tensor products). These things will be studied as this project continues.

2. The affine closure of G/H

In this section G will be a linear algebraic group over $k = \bar{k}$, and $H \subset G$ will be a closed subgroup. By abuse we will often write G instead of the group of k-points $G(k)$. We write $k[G]$ for the ring of regular functions on G. The quotient G/H exists and its ring of regular functions satisfies

$$k[G/H] = k[G]^H$$

where H acts on $k[G]$ via the right regular representation: $h \cdot f(g) = f(gh)$.

2.1. **Affine closure.** We define the affine closure of G/H to be

$$\overline{G/H} = \mathrm{Spec}(k[G]^H).$$

The left action of G on G/H clearly extends to an action on $\overline{G/H}$. We denote the action morphism by $a : G \times \overline{G/H} \to \overline{G/H}$ and its comorphism by $a^* : k[G]^H \to k[G]^H \otimes k[G]$. We will often abbreviate $a(g, x) = g \cdot x$, for $g \in G$ and $x \in \overline{G/H}$. (Similarly with respect to other actions.)

There are canonical dominant G-morphisms

$$G \to G/H \to \overline{G/H}.$$

The composition is denoted $\xi : G \to \overline{G/H}$.

2.2. **Reformulation of Tannaka's theorem.** For any object $(V, a_v) \in \mathrm{Rep}_k G$, the action $a_V : G \times V \to V$ is determined by its comorphism $a_V^* : V \to V \otimes k[G]$.

We let $\mathbb{I}$ denote the 1-dimensionial trivial representation of G.

Theorem 2.1 (Tannaka). *The map $g \mapsto (a_V(g))_V$ defines a bijection between G and the set of collections $\{\alpha_V \in \mathrm{Aut}_k V\}_{V \in \mathrm{Rep}_k G}$ satisfying the following properties:*

(1) $\alpha_{\mathbb{I}} = \mathrm{id}$,
(2) *For any G-morphism $V \to W$, the following diagram commutes*

$$\begin{array}{ccc} V & \xrightarrow{\alpha_V} & V \\ \downarrow & & \downarrow \\ W & \xrightarrow{\alpha_W} & W \end{array}$$

(3) $\alpha_{V \otimes W} = \alpha_V \otimes \alpha_W$, *for any* $V, W \in \mathrm{Rep}_k G$.

We want to interpret the rule $g \mapsto \{a_V(g)\}$ in a different way, which is more amenable to generalization to quotients. Let $e \in G$ denote the identity element.

Given $g \in G$, there is a unique G-equivariant morphism

$$\phi_g : G \to G$$

such that $\phi_g(e) = g$ (here G acts on the left). Its comorphism $\phi_g^* : k[G] \to k[G]$ fits into the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{a_v^*} & V \otimes k[G] \\ {\scriptstyle a_V(g)}\downarrow & & \downarrow{\scriptstyle \mathrm{id} \otimes \phi_g^*} \\ V & \xrightarrow{a_v^*} & V \otimes k[G]. \end{array}$$

Since the map a_V^* is injective, the vertical map $a_V(g) : V \to V$ is the unique map making the diagram commute, hence is determined by $\mathrm{id} \otimes \phi_g^*$.

Tannaka's map can thus be described as $g \mapsto \{(\mathrm{id} \otimes \phi_g^*)|_V\}_V$, where restriction is with respect to the inclusions a_V^*.

2.3. **Tannakian description of $\overline{G/H}$.** For $V \in \mathrm{Rep}_k G$, let $V^H \subset V$ denote the subspace of invariants under H.

Let ρ denote the right regular representation of G on $k[G]$. It is easy to see that the action comorphism $a_V^* : V \to V \otimes k[G]$ intertwines the actions a_V and $\mathrm{id} \otimes \rho$. Hence a_V^* induces an injective map

$$a_V^* : V^H \to V \otimes k[G]^H.$$

Suppose $x \in \overline{G/H}$. Then there is a unique G-equivariant morphism

$$\phi_x : G \to \overline{G/H}$$

such that $\phi_x(e) = x$. Let $\phi_x^* : k[G]^H \to k[G]$ denote its comorphism.

Lemma 2.2. *The map*

$$(\mathrm{id} \otimes \phi_x^*) \circ a_V^* : V^H \to V \otimes k[G]^H \to V \otimes k[G]$$

factors (uniquely) through $a_V^* : V \to V \otimes k[G]$. That is, there is a unique map $\alpha_V : V^H \to V$ such that the following diagram commutes

$$\begin{array}{ccc} V^H & \xrightarrow{a_V^*} & V \otimes k[G]^H \\ {\scriptstyle \alpha_V}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\phi_x^*} \\ V & \xrightarrow{a_V^*} & V \otimes k[G]. \end{array}$$

Proof. Since a_V^* is injective, it is clear that we need only prove the existence of the factoring α_V, i.e., that $(\mathrm{id} \otimes \phi_x^*)(a_V^*(V^H)) \subset a_V^*(V)$. Since $\xi : G \to \overline{G/H}$ has dense image, the proof consists in verifying the following points:

(1) If $x = \xi(g)$, for some $g \in G$, then the factoring exists.
(2) For a fixed $v \in V^H$, the set of elements $x \in \overline{G/H}$ such that

$$(\mathrm{id} \otimes \phi_x^*)(a_V^*(v)) \in a_V^*(V)$$

is a closed subvariety.

The first point (1) follows from 2.2. For point (2), fix $v \in V^H$ and write $f = a_V^*(v) \in V \otimes k[G]^H$; we will think of f as a V-valued polynomial function on G, invariant under H.

Let f_i' $(i \in I(V))$ denote a k-basis for $a_V^*(V) \subset V \otimes k[G]$, and let us extend it to a k-basis f_i' $(i \in I)$ for $V \otimes k[G]$.

By abuse we denote by

$$a^* : V \otimes k[G]^H \to V \otimes k[G]^H \otimes k[G]$$

the map $\mathrm{id}_V \otimes a^*$, where $a^* : k[G]^H \to k[G]^H \otimes k[G]$ is the comorphism for the G-action on $\overline{G/H}$. We think of $a^* f$ as a V-valued polynomial function on $G \times \overline{G/H}$.

By definition we have $[(\mathrm{id} \otimes \phi_x^*) a_V^*(v)](g) = f(g \cdot x)$. Moreover, writing

$$a^* f = \sum_{i \in I} f_i' \otimes f_i''$$

for some uniquely determined $f_i'' \in k[G]^H$, we see

$$f(g \cdot x) = a^* f(g, x) = \sum_{i \in I} f_i''(x) f_i'(g),$$

which is in $a_V^*(V)$ if and only if $f_i''(x) = 0$ for every $i \in I \backslash I(V)$. Thus

$$(\mathrm{id} \otimes \phi_x^*) a_V^*(v) \in a_V^*(V) \Leftrightarrow f_i''(x) = 0, \forall i \in I \backslash I(V),$$

which proves point (2). The lemma is proved. □

By Lemma 2.2, for each $x \in \overline{G/H}$ we may define a k-linear map $\alpha_V : V^H \to V$ by $\alpha_V = (\mathrm{id}_V \otimes \phi_x^*)|_{V^H}$, where restriction is taken with respect to the inclusions a_V^*.

The following proposition gives the Tannakian description of $\overline{G/H}$.

Proposition 2.3. *The map $\Phi : x \mapsto \{\alpha_V = (\mathrm{id}_V \otimes \phi_x^*)|_{V^H}\}_V$ defines an isomorphism between $\overline{G/H}$ and the set of collections of k-linear maps $\{\alpha_V \in \mathrm{Hom}_k(V^H, V)\}_V$ satisfying the following properties:*

(1) $\alpha_{\mathbb{I}} = \mathrm{id}$,

(2) *For every G-morphism $V \to W$, the following diagram commutes*

$$\begin{CD} V^H @>{\alpha_v}>> W \\ @VVV @VVV \\ W^H @>{\alpha_W}>> W \end{CD}$$

(3) *For any $V, W \in \mathrm{Rep}_k G$, the following diagram commutes*

$$\begin{CD} V^H \otimes W^H @>{\alpha_V \otimes \alpha_W}>> V \otimes W \\ @V{can}VV @| \\ (V \otimes W)^H @>{\alpha_{V \otimes W}}>> V \otimes W \end{CD}$$

Proof. We will define the map Ψ which is inverse to Φ. But first we must verify that Φ takes values in the right hand side. Fix $x \in \overline{G/H}$, and let $\Phi(x) = \{\alpha_V\}_V$, where $\alpha_V = (\mathrm{id}_V \otimes \phi_x^*)|_{V^H}$ as above. Since ϕ_x^* is a k-algebra homomorphism, it is clear that property (1) is satisfied.

To check (3), let $\mathrm{diag}^* : k[G] \otimes \to k[G]$ denote the comorphism of the diagonal map $\mathrm{diag} : G \to G \times G$. Then (3) follows easily using the equalities

$$a_{V \otimes W}^* = (\mathrm{id}_{V \otimes W} \otimes \mathrm{diag}^*) \circ (a_V^* \otimes a_W^*)$$
$$\phi_x^* \circ \mathrm{diag}^* = \mathrm{diag}^* \circ (\phi_x^* \otimes \phi_x^*)$$

and the fact that $a_{V \otimes W}^*$ is injective.

To check (2), let $\theta : V \to W$ be a G-morphism. Then we have $a_V^*(v)(h) = h \cdot v$, for every $h \in G$ and $v \in V$. Thus

$$(\theta \otimes \mathrm{id}) \circ a_V^* = a_W^* \circ \theta,$$

which, together with the injectivity of a_W^*, implies (2).

Definition of Ψ:

Let $\{\alpha_V\}_V$ be a collection satisfying properties (1-3) in the proposition. We note that we can easily extend the family of maps $\alpha_V : V^H \to V$ to include all *locally finite* representation of G, such that properties (1-3) continue to hold; in particular for the right regular representation ρ on $k[G]$ we may define

$$\alpha : k[G]^H \to k[G],$$

by setting $\alpha(v) = \alpha_V(v)$ whenever $v \in V$, where $V \subset k[G]$ is a finite dimensional G-stable subspace.

Claim 2.4. $\alpha : k[G]^H \to k[G]$ is a k-algebra homomorphism.

Proof. First α preserves the identity element since $\alpha_{\mathbb{I}} = \mathrm{id}$. To show α preserves multiplication, use properties (2) and (3) applied to the multiplication morphism:

$$m : k[G] \otimes k[G] \to k[G]$$

(this intertwines the G-module structures $\rho \otimes \rho$ and ρ). □

Using the claim, we get a morphism $\phi = \alpha^* : G \to \overline{G/H}$.

Lemma 2.5. *ϕ is G-equivariant.*

Proof. Denote the comultiplication by $\Delta : k[G] \to k[G] \otimes k[G]$. It is easily seen that

$$\phi \text{ is equivariant} \Leftrightarrow (\mathrm{id}_{k[G]} \otimes \alpha) \circ a^* = \Delta \circ \alpha.$$

On the other hand applying (2) to Δ (which intertwines the G-module structures ρ and $\mathrm{id} \otimes \rho$), we get the commutative diagram

$$\begin{array}{ccc} k[G]^H & \xrightarrow{a^*} & k[G] \otimes k[G]^H \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\alpha} \\ k[G] & \xrightarrow{\Delta} & k[G] \otimes k[G]. \end{array}$$

This proves the lemma. □

Finally, we define $\Psi(\{\alpha_V\}_V) = \phi(e) \in \overline{G/H}$. It is straightforward to check that Φ and Ψ are mutually inverse. This proves the proposition.

□

Remark. In case G is reductive and H is a reductive subgroup of G, then G/H is affine and so $\overline{G/H} = G/H$. Hence in this case the proposition gives a Tannakian description of the homogeneous space G/H.

2.4. $\overline{G/H}$ **over a k-scheme S.** Let S be any k-scheme. Let $\mathcal{O}_S$ denote its structure sheaf. For any $V \in \mathrm{Rep}_k G$, let $V \otimes \mathcal{O}_S$ denote the trivial vector bundle over S corresponding to V.

Using the same method as above, we get the following Tannakian description of the S-points $\overline{G/H}(S)$.

Proposition 2.6. *There is a bijection (functorial in S) between the set $\overline{G/H}(S)$ and the set of all collections*

$$\{\alpha_V : V^H \otimes \mathcal{O}_S \to V \otimes \mathcal{O}_S\}_{V \in \mathrm{Rep}_k G}$$

where each α_V is $\mathcal{O}_S$-linear, and the collection satisfies the following properties:

(1) $\alpha_{\mathbb{I}} = \mathrm{id}_{\mathcal{O}_S}$,

(2) *For any G-morphism $V \to W$, the following diagram commutes*

$$\begin{array}{ccc} V^H \otimes \mathcal{O}_S & \xrightarrow{\alpha_V} & V \otimes \mathcal{O}_S \\ \downarrow & & \downarrow \\ W^H \otimes \mathcal{O}_S & \xrightarrow{\alpha_W} & W \otimes \mathcal{O}_S. \end{array}$$

(3) *For $V, W \in \mathrm{Rep}_k G$, the following diagram commutes*

$$\begin{array}{ccc} V^H \otimes W^H \otimes \mathcal{O}_S & \xrightarrow{\alpha_V \otimes \alpha_W} & V \otimes W \otimes \mathcal{O}_S \\ {\scriptstyle can}\downarrow & & \| \\ (V \otimes W)^H \otimes \mathcal{O}_S & \xrightarrow{\alpha_{V\otimes W}} & V \otimes W \otimes \mathcal{O}_S. \end{array}$$

Proof. We define the maps Φ and Ψ as in Proposition 2.3; the proof then proceeds exactly as before.

Definition of Ψ:

Given $\{\alpha_V\}_V$ satisfying the properties (1-3), we first extend the collection to include all locally finite representations V of G, such that (1-3) continue to hold. Taking V to be the right regular representation $k[G]$, we get

$$\alpha : k[G]^H \otimes \mathcal{O}_S \to k[G] \otimes \mathcal{O}_S,$$

which by the same method as Proposition 2.3 is seen to be a morphism of quasi-coherent $\mathcal{O}_S$-algebras. Using the anti-equivalence of categories

$$\{\text{q.c sheaves of } \mathcal{O}_S\text{-algebras}\} \longleftrightarrow \{\text{schemes } X, \text{ affine over } S\},$$

α determines a map of schemes

$$\phi = \alpha^* : G \times S \to \overline{G/H} \times S.$$

By the same argument as in Lemma 2.5, this is $G \times S$-equivariant. Now we set $\Psi(\{\alpha_V\}) = \phi(e) \in \overline{G/H}(S)$, where $e \in (G \times S)(S)$ is the identity element of the S-points of the S-group scheme $G \times S$.

Definition of Φ:

A point $x \in \overline{G/H}(S)$ determines for every S-scheme $S' \to S$ an element $x(S') \in \overline{G/H}(S')$ and hence a $(G \times S)(S')$-equivariant map

$$\phi_{x(S')} : (G \times S)(S') \to (\overline{G/H} \times S)(S')$$

taking $e(S')$ to $x(S')$. The family $\{\phi_{x(S')}\}$ determines $G \times S$-equivariant morphism of S-schemes

$$\phi_x : G \times S \to \overline{G/H} \times S$$

taking $''e''$ to $''x''$. By the anti-equivalence of categories above this determines a morphism of quasi-coherent $\mathcal{O}_S$-algebras

$$\alpha_x = \phi_x : k[G]^H \otimes \mathcal{O}_S \to k[G] \otimes \mathcal{O}_S.$$

We would like to define $\Phi(x) = \{\alpha_V\}_V$ by $\alpha_V = (\mathrm{id}_V \otimes \alpha_x)|_{V^H \otimes \mathcal{O}_S}$ where restriction is taken relative to the inclusion $a_V^* : V^H \otimes \mathcal{O}_S \to V \otimes k[G]^H \otimes \mathcal{O}_S$. But first we must prove the analogue of Lemma 2.2.

Lemma 2.7. *The map*

$$(\mathrm{id}_V \otimes \alpha_x) \circ a_V^* : V^H \otimes \mathcal{O}_S \to V \otimes k[G] \otimes \mathcal{O}_S$$

factors uniquely through $a_V^* : V \otimes \mathcal{O}_S \to V \otimes k[G] \otimes \mathcal{O}_S$*.*

Proof. We first observe that the case $H = e$ is straightforward: this case is simply the S-scheme analogue of §2.2. We will use this case below to prove the general case.

By passing to an open cover of S, it is enough to check this for $S = \mathrm{Spec}(R)$, an affine k-scheme.

Let us denote

$$X(R) = \{\text{collections } \{\alpha_V : V^H \otimes R \to V \otimes R\}_V \text{ over } R, \text{ satisfying (1-3)}\}$$

and

$$X'(R) = \{\text{collections } \{\alpha_V\}_V \in X(R) \text{ such that the desired factoring exists}\}.$$

These make sense for any R-algebra R' in place of R; we denote by X and X' the resulting R-schemes. Thus X' is a subscheme of X. By §2.2, there is an obvious map $\Phi' : \mathrm{Spec}(k[G]) \to X'$ defined by the composition of Tannaka's map with restriction map which takes a collection $\{\alpha_V : V \otimes R \to V \otimes R\}$ to $\{\alpha_V : V^H \otimes R \to V \otimes R\}$. Write $A = k[G]^H \hookrightarrow k[G] = B$. Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(B) & \xrightarrow{\Phi'} & X' \\ {\scriptstyle\xi}\downarrow & & \downarrow \\ \mathrm{Spec}(A) & \xrightarrow{\Phi} & X. \end{array}$$

By the same argument as in Lemma 2.2, we see

$$\Phi^{-1}(X') \;\; \textit{is a closed subscheme of } \mathrm{Spec}(A).$$

Therefore we can write

$$\Phi^{-1}(X') = \mathrm{Spec}(A/I)$$

for some ideal $I \subset A$. Since the image of $\mathrm{Spec}(B)$ under ξ lies in $\mathrm{Spec}(A/I)$, the homomorphism $\xi^* : A \to B$ factors through A/I. But since ξ^* is inclusion, this means that $I = 0$, and hence Φ takes all of $\mathrm{Spec}(A)$ into X'. This proves that the desired factoring exists, whence the lemma. □

Now for $x \in \overline{G/H}(S)$ we define $\Phi(x) = \{(\mathrm{id}_V \otimes \alpha_x)|_{V^H \otimes \mathcal{O}_S}\}_V$ as above.

It is straightforward to check that Ψ and Φ are mutually inverse. This proves the proposition. □

3. Frobenius reciprocity

Let G be a connected reductive group over k. Suppose P is a parabolic subgroup and $H \triangleleft P$ is a normal subgroup. For $\mathcal{U} \in \mathrm{Rep}_k P/H$, we define the associated induced module:

Definition 3.1.

$$\mathrm{ind}_P^G \mathcal{U} = \{f : G \to \mathcal{U} \mid f(gp) = p^{-1} f(g), \ \forall g \in G, \, p \in P\}.$$

This is an object of $\mathrm{Rep}_k G$ where G acts by $(g \cdot f)(g') = f(g^{-1}g')$. Note that we have $\dim_k(\mathrm{ind}_P^G \mathcal{U}) < \infty$, because there is a finite-rank locally free $\mathcal{O}_{G/P}$-module $\mathcal{L}(\mathcal{U})$ associated to $\mathcal{U}$, for which we have

$$\mathrm{ind}_P^G \mathcal{U} = H^0(G/P, \mathcal{L}(\mathcal{U})).$$

For $\mathcal{V} \in \mathrm{Rep}_k G$, $\mathcal{U} \in \mathrm{Rep}_k P$, we have the usual Frobenius reciprocity:

$$\mathrm{Hom}_P(\mathcal{V}, \mathcal{U}) = \mathrm{Hom}_G(\mathcal{V}, \mathrm{ind}_P^G \mathcal{U}),$$

where $\mathcal{V}$ is regarded as an object of $\mathrm{Rep}_k P$ on the left hand side.

For $\mathcal{V} \in \mathrm{Rep}_k G$, let $\mathcal{V}^*$ denote the contragredient representation.

Definition 3.2. For $\mathcal{U} \in \mathrm{Rep}_k P/H$, define the associated *Weyl module* by

$$V(\mathcal{U}) = (\mathrm{ind}_P^G \mathcal{U}^*)^*.$$

The usual Frobenius reciprocity implies (by dualizing) the following version which we shall use.

Lemma 3.3. *For $\mathcal{V} \in \mathrm{Rep}_k G$ and $\mathcal{U} \in \mathrm{Rep}_k P/H$, we have a natural isomorphism*

$$\mathrm{Hom}_G(V(\mathcal{U}), \mathcal{V}) = \mathrm{Hom}_{P/H}(\mathcal{U}, \mathcal{V}^H).$$

Consequently, we have the adjuction maps

$$\epsilon_{\mathcal{U}} : \mathcal{U} \to V(\mathcal{U})^H \quad (P/H - \text{linear})$$
$$\delta_{\mathcal{V}} : V(\mathcal{V}^H) \to \mathcal{V} \quad (G - \text{linear}).$$

4. Describing $\overline{G/H}$ with $\mathrm{Rep}_k P/H$

Let G continue to denote a connected reductive group over k.

For $\mathcal{U}_1, \mathcal{U}_2 \in \mathrm{Rep}_k P/H$, there is a canonical G-morphism

$$V(\mathcal{U}_1 \otimes \mathcal{U}_2) \to V(\mathcal{U}_1) \otimes V(\mathcal{U}_2),$$

defined to be the dual of the natural morphism

$$\mathrm{ind}_P^G \mathcal{U}_1^* \otimes \mathrm{ind}_P^G \mathcal{U}_2^* \to \mathrm{ind}_P^G (\mathcal{U}_1^* \otimes \mathcal{U}_2^*).$$

Note that since G/P is connected and complete, we have a canonical isomorphism $V(\mathbb{I}) = \mathbb{I}$ (we denote the trivial representations of both G and P/H by $\mathbb{I}$).

In this section we will consider collections

$$\{\beta_{\mathcal{U}}\}_{\mathcal{U} \in \mathrm{Rep}_k P/H}$$

consisting of k-linear maps $\beta_{\mathcal{U}} : \mathcal{U} \to V(\mathcal{U})$ satisfying the following conditions:

(1) $\beta_{\mathbb{I}} = \mathrm{id}$,
(2) For a P/H-morphism $\mathcal{U}_1 \to \mathcal{U}_2$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{U}_1 & \xrightarrow{\beta_{\mathcal{U}_1}} & V(\mathcal{U}_1) \\ \downarrow & & \downarrow \\ \mathcal{U}_2 & \xrightarrow{\beta_{\mathcal{U}_2}} & V(\mathcal{U}_2). \end{array}$$

(3) For $\mathcal{U}_1, \mathcal{U}_2 \in \mathrm{Rep}_k P/H$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{U}_1 \otimes \mathcal{U}_2 & \xrightarrow{\beta_{\mathcal{U}_1 \otimes \mathcal{U}_2}} & V(\mathcal{U}_1 \otimes \mathcal{U}_2) \\ \Big\downarrow {\scriptstyle \beta_{\mathcal{U}_1} \otimes \beta_{\mathcal{U}_2}} & & \Big\downarrow {\scriptstyle can} \\ V(\mathcal{U}_1) \otimes V(\mathcal{U}_2) & = & V(\mathcal{U}_1) \otimes V(\mathcal{U}_2). \end{array}$$

We call these conditions (1-3) *tensor conditions*. (We will use the same terminology for the previous conditions (1-3) imposed on a collection $\{\alpha_{\mathcal{V}}\}_{\mathcal{V}}$.)

We will say an arbitrary collection of k-linear maps $\{\beta_{\mathcal{U}}\}_{\mathcal{U} \in \mathrm{Rep}_k P/H}$ is *compatible with P/H-morphisms* provided it satisfies condition (2) above. Similarly, we say an arbitrary collection $\{\alpha_{\mathcal{V}}\}_{\mathcal{V} \in \mathrm{Rep}_k G}$ is *compatible with G-morphisms* if it satisfies the corresponding property (2) from Proposition 2.3.

Now suppose $\{\alpha_{\mathcal{V}}\}_{\mathcal{V} \in \mathrm{Rep}_k G}$ is any collection of k-linear maps $\alpha_{\mathcal{V}} : \mathcal{V}^H \to \mathcal{V}$. We construct a collection $\{\beta_{\mathcal{U}}\}_{\mathcal{U} \in \mathrm{Rep}_k P/H}$ of k-linear maps $\beta_{\mathcal{U}} : \mathcal{U} \to V(\mathcal{U})$ by setting

$$\beta_{\mathcal{U}} = \alpha_{V(\mathcal{U})} \circ \epsilon_{\mathcal{U}}.$$

Conversely, given $\{\beta_{\mathcal{U}}\}_{\mathcal{U} \in \mathrm{Rep}_k P/H}$, we construct $\{\alpha_{\mathcal{V}}\}_{\mathcal{V} \in \mathrm{Rep}_k G}$ by setting

$$\alpha_{\mathcal{V}} = \delta_{\mathcal{V}} \circ \beta_{\mathcal{V}^H}.$$

We have the following lemma.

Lemma 4.1. (a) *The association*

$$\{\alpha_{\mathcal{V}}\}_{\mathcal{V} \in \mathrm{Rep}_k G} \longleftrightarrow \{\beta_{\mathcal{U}}\}_{\mathcal{U} \in \mathrm{Rep}_k P/H}.$$

described above gives a bijection between the set of collections compatible with G-morphisms and the set of collections compatible with P/H-morphisms.

(b) *Moreover, $\{\alpha_{\mathcal{V}}\}_{\mathcal{V} \in \mathrm{Rep}_k G}$ satisfies the tensor conditions if and only if the corresponding collection $\{\beta_{\mathcal{U}}\}_{\mathcal{U} \in \mathrm{Rep}_k P/H}$ does.*

Proof. Part (a) follows easily using the following identities, which follow by adjunction:

$$\delta_{\mathcal{V}}|_{V(\mathcal{V}^H)^H} \circ \epsilon_{\mathcal{V}^H} = \mathrm{id}_{\mathcal{V}^H}$$
$$\delta_{V(\mathcal{U})} \circ V(\epsilon_{\mathcal{U}}) = \mathrm{id}_{V(\mathcal{U})}.$$

Part (b) is a straightforward diagram-chase, which we omit. □

It is clear that this discussion goes over without change over an arbitrary k-scheme S. Therefore we have the following analogue of Proposition 2.6.

Proposition 4.2. *There is a bijection (functorial in S) between the set $\overline{G/H}(S)$ and the set of all collections*

$$\{\beta_{\mathcal{U}} : \mathcal{U} \otimes \mathcal{O}_S \to V(\mathcal{U}) \otimes \mathcal{O}_S\}_{\mathcal{U} \in \mathrm{Rep}_k P/H}$$

consisting of $\mathcal{O}_S$-linear morphisms satisfying the following tensor conditions:

(1) $\beta_{\mathbb{I}} = \mathrm{id}_{\mathcal{O}_S}$,

(2) *For a P/H-morphism $\mathcal{U}_1 \to \mathcal{U}_2$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{U}_1 \otimes \mathcal{O}_S & \xrightarrow{\beta_{\mathcal{U}_1}} & V(\mathcal{U}_1) \otimes \mathcal{O}_S \\ \downarrow & & \downarrow \\ \mathcal{U}_2 \otimes \mathcal{O}_S & \xrightarrow{\beta_{\mathcal{U}_2}} & V(\mathcal{U}_2) \otimes \mathcal{O}_S. \end{array}$$

(3) *For $\mathcal{U}_1, \mathcal{U}_2 \in \mathrm{Rep}_k P/H$, the following diagram commutes*

$$\begin{array}{ccc} \mathcal{U}_1 \otimes \mathcal{U}_2 \otimes \mathcal{O}_S & \xrightarrow{\beta_{\mathcal{U}_1 \otimes \mathcal{U}_2}} & V(\mathcal{U}_1 \otimes \mathcal{U}_2) \otimes \mathcal{O}_S \\ {\scriptstyle \beta_{\mathcal{U}_1} \otimes \beta_{\mathcal{U}_2}} \downarrow & & \downarrow {\scriptstyle can} \\ V(\mathcal{U}_1) \otimes V(\mathcal{U}_2) \otimes \mathcal{O}_S & = & V(\mathcal{U}_1) \otimes V(\mathcal{U}_2) \otimes \mathcal{O}_S. \end{array}$$

5. Plücker description of G/U

5.1. **Strongly quasi-affine quotients.** We say the quotient G/H is *strongly quasi-affine* if the canonical map

$$G/H \to \overline{G/H}$$

is an open immersion.

We will make use of the following theorem of F. Grosshans [6], Thm. 4.3.

Theorem 5.1 (Grosshans). *If $U(P)$ is the unipotent radical of a parabolic subgroup $P \subset G$, then $G/U(P)$ is strongly quasi-affine.*

5.2. **Tannakian description of $G/U(P)$.** It is easy to see which collections $\{\alpha_{\mathcal{V}}\}$ correspond to elements in the open subset $G/U(P)$ of $\overline{G/U(P)}$.

Lemma 5.2. *In the Tannakian description for $\overline{G/U(P)}$, a collection $\{\alpha_{\mathcal{V}}\}$ corresponds to an element in $G/U(P)$ if and only if $\alpha_{\mathcal{V}}$ is injective for every $\mathcal{V} \in \mathrm{Rep}_k G$.*

Proof. Suppose $x \in \overline{G/U(P)}$ corresponds to the collection $\{\alpha_{\mathcal{V}}\}$. Taking into account that each comorphism $a^*_{\mathcal{V}} : \mathcal{V} \to \mathcal{V} \otimes k[G]$ is injective, Theorem 5.1 yields the following equivalences:

$$\begin{aligned} \alpha_{\mathcal{V}} \ \text{ is injective } \ \forall \mathcal{V} \in \mathrm{Rep}_k G &\Leftrightarrow \alpha : k[G]^{U(P)} \to k[G] \ \text{ is injective} \\ &\Leftrightarrow \phi_x = \alpha^* : G \to \overline{G/U(P)} \ \text{ is dominant} \\ &\Leftrightarrow \mathrm{im}(\phi_x) = G/U(P) \\ &\Leftrightarrow x \in G/U(P). \end{aligned}$$

□

5.3. **The Weyl modules $V(\lambda)$.** Let $B = TU$ denote a Borel subgroup of G, where T is a maximal torus, and $U = U(B)$. We denote by $X^*(T)$ the lattice of characters of T, and by X^+ the cone of B-dominant characters in $X^*(T)$. We will use the following notation: For each $\lambda \in X^*(T)$, we let k_λ denote the 1-dimensional representation of $B/U = T$ where T acts via by the character λ. When confusion is not possible, we denote k_λ simply by λ. We write

$$V(\lambda) = (\mathrm{ind}_B^G(-\lambda))^*;$$

this representation is called the *Weyl module* for λ. It is known that

$$V(\lambda) \neq 0 \Leftrightarrow \lambda \in X^+.$$

We suppose now that $\lambda \in X^+$. Then $V(\lambda) \in \mathrm{Rep}_k G$ enjoys the following properties (cf. [9], I.2.13).

- $V(\lambda)$ possesses a B-highest weight vector v_λ, which generates a B-stable line $\mathbb{L}(\lambda) \subset V(\lambda)$.
- The G-module $V(\lambda)$ is generated by $\mathbb{L}(\lambda)$.
- Any G-module generated by a B-stable line is a quotient of $V(\lambda)$.

In general $V(\lambda)$ is not irreducible (although this holds if $\mathrm{char}(k) = 0$). Let V_λ denote the irreducible G-module with highest weight λ; by the above remarks, this is a quotient of $V(\lambda)$. We have an exact sequence

$$0 \to \mathrm{rad}_G V(\lambda) \to V(\lambda) \to V_\lambda \to 0,$$

where $\mathrm{rad}_G V(\lambda)$ is the intersection of all maximal proper G-submodules in $V(\lambda)$.

We note also the following fact.

Lemma 5.3. *For any weights $\lambda, \mu \in X^+$, the canonical map*

$$V(\lambda + \mu) \to V(\lambda) \otimes V(\mu)$$

is injective.

Proof. The dual map

$$H^0(G/B, \mathcal{L}(-\lambda)) \otimes H^0(G/B, \mathcal{L}(-\mu)) \to H^0(G/B, \mathcal{L}(-\lambda - \mu))$$

(induced by cup-product) is surjective (cf. [9], II.14.20). □

5.4. **Plücker description for G/U.** Using Frobenius reciprocity as in Lemma 4.1, it is not hard to translate a collection $\{\alpha_{\mathcal{V}}\}_{\mathcal{V} \in \mathrm{Rep}_k G}$ giving a point in G/U into a corresponding collection $\{\beta_{\mathcal{U}}\}_{\mathcal{U} \in \mathrm{Rep}_k B/U}$. When we do so, the condition

$$\alpha_{\mathcal{V}} \ \text{ is injective } \ \forall \mathcal{V} \in \mathrm{Rep}_k G$$

is converted into the condition

$$\beta^{-1}_{\mathcal{V}^U}(\ker(\delta_{\mathcal{V}})) = 0 \ \ \forall \mathcal{V} \in \mathrm{Rep}_k G.$$

Next we want to translate the collection $\{\beta_{\mathcal{U}}\}_{\mathcal{U} \in \mathrm{Rep}_k B/U}$ into an even simpler collection $\{\beta_\lambda\}_{\lambda \in X^+}$.

The category $\mathrm{Rep}_k B/U$ is semi-simple, with simple objects k_λ ($\lambda \in X^*(T)$). Furthermore the functor $V(\cdot) : \mathrm{Rep}_k B/U \to \mathrm{Rep}_k G$ commutes with direct sums. This means that any collection of k-linear maps

$$\beta_{\mathcal{U}} : \mathcal{U} \to V(\mathcal{U}),$$

($\mathcal{U} \in \mathrm{Rep}_k B/U$), is uniquely determined by the sub-collection

$$\beta_\lambda : k_\lambda \to V(\lambda),$$

(for $\lambda \in X^+$) – here we have used that $V(\lambda) = 0$ unless $\lambda \in X^+$. In fact it is not hard to translate the tensor conditions on the collection $\{\beta_{\mathcal{U}}\}$ into similar tensor conditions on $\{\beta_\lambda\}$.

In the following statement, we note that if $\lambda = 0$, we have $k_\lambda = \mathbb{I}$ and $V(\lambda) = V(0) = \mathbb{I}$. Also, by abuse, we write $\lambda + \mu$ for the representation $k_{\lambda+\mu} = k_\lambda \otimes k_\mu$. We get the following Plücker description for G/U.

Proposition 5.4. *There is a bijection (functorial in S) between the set $(G/U)(S)$ and the set of all collections*

$$\{\beta_\lambda : \lambda \otimes \mathcal{O}_S \to V(\lambda) \otimes \mathcal{O}_S\}_{\lambda \in X^+}$$

of $\mathcal{O}_S$-linear maps β_λ satisfying the following tensor conditions:

(1) $\beta_{\mathbb{I}} = \mathrm{id}_{\mathcal{O}_S}$,

(2) *For each $\lambda, \mu \in X^+$ the following diagram commutes:*

$$\begin{array}{ccc} (\lambda+\mu) \otimes \mathcal{O}_S & \xrightarrow{\beta_{\lambda+\mu}} & V(\lambda+\mu) \otimes \mathcal{O}_S \\ \downarrow{\scriptstyle \beta_\lambda \otimes \beta_\mu} & & \downarrow{\scriptstyle can} \\ V(\lambda) \otimes V(\mu) \otimes \mathcal{O}_S & = & V(\lambda) \otimes V(\mu) \otimes \mathcal{O}_S. \end{array}$$

(3) *For each $\lambda \in X^+$, we have $\beta_\lambda(k_\lambda \otimes \mathcal{O}_S) \not\subset \mathrm{rad}_G V(\lambda) \otimes \mathcal{O}_S$.*

Proof. Given a collection $\{\beta_{\mathcal{U}}\}$ we let $\{\beta_\lambda\}$ denote the subcollection indexed by the B/U-modules k_λ, for $\lambda \in X^+$. Conversely, suppose we are given $\{\beta_\lambda\}_{\lambda \in X^+}$. We have a canonical decomposition

$$\mathcal{U} = \oplus_{\lambda \in X^*(T)} \mathrm{Hom}_{B/U}(\lambda, \mathcal{U}) \otimes \lambda,$$

and thus

$$V(\mathcal{U}) = \oplus_{\lambda \in X^*(T)} \mathrm{Hom}_{B/U}(\lambda, \mathcal{U}) \otimes V(\lambda).$$

We recall that $V(\lambda) = 0$ unless $\lambda \in X^+$, so we may define

$$\beta_{\mathcal{U}} = \oplus_{\lambda \in X^*(T)} \mathrm{id} \otimes \beta_\lambda,$$

where we understand $\beta_\lambda = 0$ unless $\lambda \in X^+$.

It is straightforward to verify that this determines a bijection $\{\beta_{\mathcal{U}}\}_{\mathcal{U} \in \mathrm{Rep}_k B/U} \leftrightarrow \{\beta_\lambda\}_{\lambda \in X^+}$ which preserves the tensor conditions. Finally, one can check that

$$\beta^{-1}_{\mathcal{V}^U}(\ker(\delta_{\mathcal{V}})) = 0 \ \ \forall \mathcal{V} \in \mathrm{Rep}_k G \Leftrightarrow \beta_\lambda(k_\lambda) \not\subset \mathrm{rad}_G V(\lambda) \ \ \forall \lambda \in X^+.$$

□

5.5. **Plücker description for G/B.** To simplify notation, in this section we will consider only the case $S = \mathrm{Spec}(k)$ (this is all we need for our intended applications).

The torus T acts freely on G/U by $(t, gU) \mapsto gtU$, and the quotient of this action is the finite flag variety G/B. This transports to the following action of T on a collection $\{\beta_\lambda\}_{\lambda \in X^+}$:

$$(t, \{\beta_\lambda\}_\lambda) \mapsto \{\lambda(t)\beta_\lambda\}_\lambda.$$

Given a collection $\{\beta_\lambda\}_{\lambda \in X^+}$, we define a collection of lines $\{L(\lambda) \subset V(\lambda)\}_{\lambda \in X^+}$ by setting

$$L(\lambda) = \beta_\lambda(k_\lambda)$$

for every $\lambda \in X^+$. Recall the "standard" line $\mathbb{L}(\lambda) \subset V(\lambda)$ as described in §5.3. We have the following Plücker description for G/B:

Proposition 5.5. *The map $g \mapsto \{g\mathbb{L}(\lambda)\}_{\lambda \in x^+}$ determines a natural bijection between the finite flag variety G/B and the set of all collections*

$$\{L(\lambda)\}_{\lambda \in X^+}$$

consisting of lines $L(\lambda) \subset V(\lambda)$ satisfying the following tensor conditions:

(1) *For* $\lambda, \mu \in X^+$, *let* $i_{\lambda,\mu} : V(\lambda+\mu) \hookrightarrow V(\lambda) \otimes V(\mu)$ *denote the canonical inclusion. Then* $i_{\lambda,\mu}(L(\lambda+\mu)) = L(\lambda) \otimes L(\mu)$.
(2) *For* $\lambda \in X^+$, *we have* $L(\lambda) \not\subset \mathrm{rad}_G V(\lambda)$.

Proof. The map $F : \{\beta_\lambda\} \mapsto \{L(\lambda)\}$ defined by $L(\lambda) = \beta_\lambda(k_\lambda)$ clearly takes two collections in the same T-orbit to the same collection of lines. To construct an inverse map, note first that a collection of lines $\{L(\lambda)\}$ satisfying the given tensor conditions is uniquely determined by the finite collection $\{L(\omega_i)\}_i$, where ω_i ranges over the fundamental weights of X^+. For each i, choose a generator $v_i \in L(\omega_i)$. We can define $\beta_{\omega_i} : k_{\omega_i} \to V(\omega_i)$ by letting $\beta_{\omega_i}(\gamma) = \gamma v_i,\ \forall \gamma \in k_{\omega_i}$. (Here we identify each k_λ with the underlying space k.)

Claim 5.6. If we have defined β_λ and β_μ such that $\beta_\lambda(k) = L(\lambda)$ and $\beta_\mu(k) = L(\mu)$, then there is a unique way to define $\beta_{\lambda+\mu}$ such that $i_{\lambda,\mu} \circ \beta_{\lambda+\mu} = \beta_\lambda \otimes \beta_\mu$ and $\beta_{\lambda+\mu}(k) = L(\lambda+\mu)$.

Proof. This follows easily from the injectivity of $i_{\lambda,\mu}$ and the relation

$$i_{\lambda,\mu}(L(\lambda+\mu)) = L(\lambda) \otimes L(\mu).$$

□

By the claim, the choice of $\{\beta_{\omega_i}\}_i$ uniquely determines a collection $\{\beta_\lambda\}_{\lambda \in X^+}$ which satisfies the tensor conditions and satisfies as well $\beta_\lambda(k) = L(\lambda)$ for every λ. The T-orbit of $\{\beta_\lambda\}$ depends only on $\{L(\lambda)\}$ (i.e., it is independent of the choice of v_i), and this defines an inverse map to F. □

Remark. If $\mathrm{char}(k) = 0$, then the second condition in Proposition 5.5 can be omitted, since $V(\lambda) = V_\lambda$ is irreducible. The Drinfeld-Plücker relations for G/B announced by Kuznetsov [11] differ from those appearing in Proposition 5.5, in that the tensor conditions here are expressed using the canonical inclusion $V_{\lambda+\mu} \hookrightarrow V_\lambda \otimes V_\mu$ and those in [11] use all G-projections $V_\lambda \otimes V_\mu \to V_\nu$, for any weight $\nu \preceq \lambda + \mu$. The tensor conditions are in fact equivalent, so the above arguments provide a proof for the statement in [11], and also a generalization to characteristic $p > 0$. For the sequel we will find it useful to slightly reformulate the tensor conditions in Proposition 5.5 in a way that looks more like [11]. This yields the proposition below.

Proposition 5.7. *Assume* $\mathrm{char}(k) = 0$. *Then* $g \mapsto \{g\mathbb{L}(\lambda)\}_{\lambda \in X^+}$ *gives a bijection between the flag variety* G/B *and the set of collections*

$$\{L(\lambda)\}_{\lambda \in X^+}$$

of lines $L(\lambda) \subset V(\lambda)$ *which satisfy the following tensor condition:*

Let $\lambda, \mu, \nu \in X^+$. *Then for every* G-*inclusion* $\phi : V(\nu) \hookrightarrow V(\lambda) \otimes V(\mu)$ *we have*

$$\phi^{-1}(L(\lambda) \otimes L(\mu)) = \begin{cases} 0, & \text{if } \nu \prec \lambda + \mu, \\ L(\lambda+\mu) & \text{if } \nu = \lambda + \mu. \end{cases}$$

6. Affine Grassmannians

6.1. **Notation.** For this section and the rest of this paper, we assume $\mathrm{char}(k) = 0$. (The case of characteristic $p > 0$ will be considered in a subsequent version of this work.)

We assume G is a connected reductive group over k. Suppose its based root system is $(X^*, X_*R, \check{R}, \Pi)$. Choose a maximal torus T and a Borel subgroup $B = TU$ and identify X^* with $X^*(T)$ and Π with the B-positive simple roots in R. Let $\check{Q}$ denote the subgroup of X_* generated by the coroots $\check{R}$. Let

$$\langle\, ,\, \rangle : X^* \times X_* \to \mathbb{Z}$$

denote the canonical pairing.

At some points below, we shall assume the derived group G^{der} is simply-connected.

We will use the following notation for the remainder of this paper:

- $\check{\mu}$ will always denote a dominant coweight for T, and λ will always denote a dominant weight for T.
- Given $\check{\mu}$ (resp. λ), we will denote by $\Omega(\check{\mu})$ (resp. $\Omega(\lambda)$) the set of coweights (resp. weights) in the Weyl module associated to $\check{\mu}$ (resp. λ).
- $\check{\lambda}$ (resp. λ') will always denote an element of $\Omega(\check{\mu})$ (resp. $\Omega(\lambda)$).
- for $x, y \in X_*$ (resp. X^*) the relation $x \preceq y$ will mean that $y - x$ is a sum of positive coroots (resp. roots).
- We will denote the extended affine Weyl group by $\widetilde{W} = X_* \rtimes W$, where W denotes the finite Weyl group. We let w_0 denote the longest element in W.
- We fix a parameter t and let $\mathcal{O}_t = k[[t]]$ and $\mathcal{K}_t = k((t))$.
- We denote translation elements in $\widetilde{W}$ by t_x ($x \in X_*$). We identify t_x with an element of $T(\mathcal{K}_t) \subset G(\mathcal{K}_t)$ by the rule:
  $$t_x = x(t^{-1})$$
  where x is viewed as a cocharacter $x \in X_*(T)$. Thus if $v_{\lambda'}$ is vector with weight λ' in some G-module, then
  $$t_x \cdot v_{\lambda'} = t^{-\langle \lambda', x\rangle} v_{\lambda'}.$$
- For any $\mathcal{V} \in \mathrm{Rep}_k G$ denote $\mathcal{V}^0 = \mathcal{V} \otimes \mathcal{O}_t$. Thus $\mathcal{V}^0$ is the "standard" $\mathcal{O}_t$-lattice in $\mathcal{V} \otimes \mathcal{K}_t$.

6.2. **Plücker description for affine Grassmannians.** We denote the affine Grassmannian over $\mathrm{Spec}(\mathcal{O}_t)$ by $\mathcal{Q}$. In the forgoing it is possible to describe $\mathcal{Q}$ as an ind-scheme, giving the points $\mathcal{Q}(R)$ for every k-algebra R. To illustrate just the main ideas, we will only describe the k-points

$$\mathcal{Q}(k) = G(\mathcal{K}_t)/G(\mathcal{O}_t).$$

Proposition 6.1. *The map $g \mapsto \{gV(\lambda)^0\}_{\lambda \in X^+}$ defines a bijection between $G(\mathcal{K}_t)/G(\mathcal{O}_t)$ and the set of collections*

$$\{\mathcal{L}(\lambda)\}_{\lambda \in X^+}$$

consisting of $\mathcal{O}_t$-lattices $\mathcal{L}(\lambda) \subset V(\lambda) \otimes \mathcal{K}_t$ satsifying the following tensor condition:

Let $\lambda, \mu, \nu \in X^+$. Then for every G-inclusion $\phi : V(\nu) \hookrightarrow V(\lambda) \otimes V(\mu)$ we have

$$\phi^{-1}(\mathcal{L}(\lambda) \otimes \mathcal{L}(\mu)) = \mathcal{L}(\nu),$$

where ϕ is also used to denote the induced map

$$\phi \otimes \mathrm{id} : V(\nu) \otimes \mathcal{K}_t \to V(\lambda) \otimes V(\mu) \otimes \mathcal{K}_t.$$

Proof. The injectivity follows from

$$\cap_\lambda \mathrm{Stab}_{G(\mathcal{K}_t)} V(\lambda)^0 = G(\mathcal{O}_t),$$

which is a consequence of the Cartan decomposition

$$G(\mathcal{K}_t) = \coprod_{\check{\mu}} G(\mathcal{O}_t) t_{\check{\mu}} G(\mathcal{O}_t).$$

It remains to prove the surjectivity. Since the category $\mathrm{Rep}_k G$ is semi-simple, every G-module $\mathcal{V}$ has a canonical decomposition

$$\mathcal{V} = \oplus_\lambda \mathrm{Hom}_G(V(\lambda), \mathcal{V}) \otimes V(\lambda),$$

where G acts trivially on the factor $\mathrm{Hom}_G(V(\lambda), \mathcal{V})$. Given the data $\{\mathcal{L}(\lambda)\}_{\lambda \in X^+}$, we define for each $\mathcal{V} \in \mathrm{Rep}_k G$ an $\mathcal{O}_t$-lattice in $\mathcal{V} \otimes \mathcal{K}_t$ by

$$\mathcal{L}(\mathcal{V}) = \oplus_\lambda \mathrm{Hom}_G(V(\lambda), \mathcal{V}) \otimes \mathcal{L}(\lambda).$$

Let $\mathrm{ProjMod}(\mathcal{O}_t)$ denote the category of finite-type projective $\mathcal{O}_t$-modules. It is straightforward to check that

$$\mathcal{L} : \mathrm{Rep}_k G \to \mathrm{ProjMod}(\mathcal{O}_t)$$

is a faithful, exact, $\otimes$-functor (i.e., it is a fiber functor). Therefore $\mathcal{L}$ comes from a unique G-torsor $\mathbb{L}$ over $\mathrm{Spec}(\mathcal{O}_t)$, see [3]. Moreover, the isomorphisms $\mathcal{L}(\lambda) \otimes \mathcal{K}_t \cong V(\lambda) \otimes \mathcal{K}_t$ induced by the inclusions $\mathcal{L}(\lambda) \hookrightarrow V(\lambda) \otimes \mathcal{K}_t$ determine a trivialization of $\mathbb{L}$ over the generic fiber $\mathrm{Spec}(\mathcal{K}_t)$. Now by the theorem of Beauville-Laszlo [1] there is a natural correspondence

$$G(\mathcal{K}_t)/G(\mathcal{O}_t) \longleftrightarrow \{G\text{-torsors over } \mathrm{Spec}(\mathcal{O}_t) \text{ with trivialization on } \mathrm{Spec}(\mathcal{K}_t)\}.$$

Thus we see $\mathbb{L}$ and hence the collection $\{\mathcal{L}(\lambda)\}_\lambda$ comes from a well-defined coset $gG(\mathcal{O}_t) \in G(\mathcal{K}_t)/G(\mathcal{O}_t)$. This completes the proof.

□

6.3. **Plücker description for $\overline{\mathcal{Q}}_{\check{\mu}}$.** By the Cartan decomposition, the $G(\mathcal{O}_t)$-orbits on $\mathcal{Q}$ are indexed by the dominant coweights $\check{\mu}$. Let $\overline{\mathcal{Q}}_{\check{\mu}}$ denote the closure of the orbit

$$\mathcal{Q}_{\check{\mu}} = G(\mathcal{O}_t) t_{\check{\mu}} G(\mathcal{O}_t)/G(\mathcal{O}_t)$$

corresponding to $\check{\mu}$. Our description of $\overline{\mathcal{Q}}_{\check{\mu}}$ is based on the following lemmas, whose proofs are straightforward.

Lemma 6.2. *Suppose $\check{\lambda}$ and $\check{\mu}$ are two dominant coweights, and $\check{\mu} - \check{\lambda} \in \check{Q}$. Then the following conditions are equivalent:*

(a) $\check{\lambda} \preceq \check{\mu}$.
(b) $\langle \lambda, \check{\lambda} \rangle \leq \langle \lambda, \check{\mu} \rangle$, *for every* $\lambda \in X^+$.
(c) $t_{\check{\lambda}} V(\lambda)^0 \subset t^{-\langle \lambda, \check{\mu} \rangle} V(\lambda)^0$, *for every* $\lambda \in X^+$.
(d) $t^{-\langle w_0\lambda, \check{\mu} \rangle} V(\lambda)^0 \subset t_{\check{\lambda}} V(\lambda)^0 \subset t^{-\langle \lambda, \check{\mu} \rangle} V(\lambda)^0$, *for every* $\lambda \in X^+$.

For the next lemma, let us denote $d_\lambda = \dim_k V(\lambda)$, for $\lambda \in X^+$.

Lemma 6.3. *Suppose G^{der} is simply connected (i.e. $X_*/\check{Q}$ is torsion-free). Then the following statements are equivalent for any dominant coweights $\check{\lambda}, \check{\mu} \in X_*$:*

(a) $\check{\mu} - \check{\lambda} \in \check{Q}$.

(b) $\wedge^{d_\lambda} t_{\check{\lambda}} V(\lambda)^0 = \wedge^{d_\lambda} t_{\check{\mu}} V(\lambda)^0$, *for every* $\lambda \in X^+$.

The closure relations in the affine Grassmannian are given by the partial order on X_*:

$$\mathcal{Q}_{\check{\lambda}} \subset \overline{\mathcal{Q}}_{\check{\mu}} \Longleftrightarrow \check{\lambda} \preceq \check{\mu}.$$

Hence from Lemmas 6.2, 6.3 and Proposition 6.1 we derive the following Plücker description for $\overline{\mathcal{Q}}_{\check{\mu}}$:

Proposition 6.4. *Suppose* $G^{der} = G^{sc}$. *Then there is a natural bijection between* $\overline{\mathcal{Q}}_{\check{\mu}}$ *and the set of collections*

$$\{\mathcal{L}(\lambda)\}_{\lambda \in X^+}$$

consisting of $\mathcal{O}_t$*-lattices* $\mathcal{L}(\lambda) \subset V(\lambda) \otimes \mathcal{K}_t$ *which satisfy the following properties:*

(1) *Let* $\nu, \lambda, \mu \in X^+$. *For every* G*-injection* $\phi : V(\nu) \to V(\lambda) \otimes V(\mu)$, *we have* $\phi^{-1}(\mathcal{L}(\lambda) \otimes \mathcal{L}(\mu)) = \mathcal{L}(\nu)$.
(2) $\wedge^{d_\lambda} \mathcal{L}(\lambda) = \wedge^{d_\lambda} t_{\check{\mu}} V(\lambda)^0$, *for every* $\lambda \in X^+$.
(3) $t^{-\langle w_0\lambda, \check{\mu}\rangle} V(\lambda)^0 \subset \mathcal{L}(\lambda) \subset t^{-\langle \lambda, \check{\mu}\rangle} V(\lambda)^0$, *for every* $\lambda \in X^+$.

6.4. **Ind-scheme structure on $\mathcal{Q}$.**
Assume $G^{der} = G^{sc}$.

As mentioned above, we can easily define $\overline{Q}_{\check{\mu}}$ as a k-scheme, using a similar definition as that given above. Let ω_i denote the fundamental weights for G. It is easy to see that a collection $\{\mathcal{L}(\lambda)\}_{\lambda \in X^+}$ is uniquely determined by the finitely many lattices in the subcollection $\{\mathcal{L}(\omega_i)\}_i$. In fact, the forgetful functor

$$\{\mathcal{L}(\lambda)\}_\lambda \mapsto \{\mathcal{L}(\omega_i)\}_i$$

defines a closed immersion of $\overline{\mathcal{Q}}_{\check{\mu}}$ into a finite-dimensional projective k-scheme.

By bounding the relative positions of the finitely-many lattices $\mathcal{L}(\omega_i)$, we see as well that every element in $\mathcal{Q}$ is contained in some $\overline{\mathcal{Q}}_{\check{\mu}}$. Hence we have the following ind-scheme structure on $\mathcal{Q}$:

Proposition 6.5. *The equality*

$$\mathcal{Q} = \cup_{\check{\mu}} \overline{\mathcal{Q}}_{\check{\mu}}$$

realizes the affine Grassmannian as an inductive limit of finite-dimensional projective schemes over k.

7. Affine flag varieties

7.1. **Notation.** Let $\pi : G(\mathcal{O}_t) \to G(k)$ be the homomorphism determined by $t \mapsto 0$. Let $I = I_k \subset G(\mathcal{O}_t)$ be the "standard" Iwahori subgroup:

$$I = \pi^{-1}(B).$$

Recall that each Weyl module $V(\lambda)$ is generated by its "standard" highest-weight line $\mathbb{L}(\lambda) \subset V(\lambda)$. By definition, this line is the image of the adjunction map $\epsilon_\lambda : \lambda \to V(\lambda)^U$ (cf. Lemma 3.3).

Choose a generator $v_\lambda \in \mathbb{L}(\lambda)$. We define the standard two-step lattice chain

$$V(\lambda)^\bullet = (V(\lambda)^0 \subset V(\lambda)^1 \subset t^{-1}V(\lambda)^0)$$

by defining $V(\lambda)^0 \subset V(\lambda) \otimes \mathcal{K}_t$ as before, and by defining

$$V(\lambda)^1 = t^{-1}\mathcal{O}_t v_\lambda \oplus (\oplus_{\lambda' \prec \lambda} \mathcal{O}_t v_{\lambda'}).$$

Here the summand on the right denotes the $\mathcal{O}_t$-span of all weight vectors in $V(\lambda)$ with weight less than λ.

There is a natural isomorphism

$$t^{-1}V(\lambda)^0/V(\lambda)^0 = V(\lambda)$$

under which we have the identification

$$V(\lambda)^1/V(\lambda)^0 = \mathbb{L}(\lambda).$$

Furthermore, if $\lambda, \mu \in X^+$, then there is a natural isomorphism

$$t^{-1}(V(\lambda)^0 \otimes V(\mu)^0)/V(\lambda)^0 \otimes V(\mu)^0 = V(\lambda) \otimes V(\mu)$$

under which we have the identification

$$[(V(\lambda)^0 \otimes V(\mu)^0) + t(V(\lambda)^1 \otimes V(\mu)^1)]/V(\lambda)^0 \otimes V(\mu)^0 = \mathbb{L}(\lambda) \otimes \mathbb{L}(\mu).$$

7.2. **Plücker description for affine flag varieties.** We denote the affine flag variety by $\mathcal{F}l$. As for affine Grassmannians, we can describe it as an ind-scheme, but for simplicity we describe here only its k-points

$$\mathcal{F}l(k) = G(k((t)))/I.$$

Proposition 7.1. *The map $g \mapsto \{gV(\lambda)^\bullet\}_\lambda$ determines a natural bijection between $G(k((t)))/I$ and the set of collections $\{\mathcal{L}(\lambda)^\bullet\}_{\lambda \in X^+}$ consisting of two-step lattice chains*

$$\mathcal{L}(\lambda)^\bullet = (\mathcal{L}(\lambda)^0 \subset \mathcal{L}(\lambda)^1 \subset t^{-1}\mathcal{L}(\lambda)^0)$$

in $V(\lambda) \otimes \mathcal{K}_t$ which satisfy the following properties:

(1) $\dim_k \mathcal{L}(\lambda)^1/\mathcal{L}(\lambda)^0 = 1$.
(2) *Let $\nu, \lambda, \mu \in X^+$. For every G-inclusion $\phi : V(\nu) \to V(\lambda) \otimes V(\mu)$, we have:*

$$\phi^{-1}(\mathcal{L}(\lambda)^0 \otimes \mathcal{L}(\mu)^0) = \mathcal{L}(\nu)^0.$$

(3) *For ϕ as above, we have*

$$\phi^{-1}[\mathcal{L}(\lambda)^0 \otimes \mathcal{L}(\mu)^0 + t(\mathcal{L}(\lambda)^1 \otimes \mathcal{L}(\mu)^1)] = \begin{cases} \mathcal{L}(\nu)^0, & \text{if } \nu \prec \lambda + \mu, \\ \mathcal{L}(\nu)^1, & \text{if } \nu = \lambda + \mu. \end{cases}$$

Proof. The map is injective, because

$$I = \cap_\lambda \mathrm{Stab}_{G(\mathcal{K}_t)}(V(\lambda)^\bullet).$$

To see the surjectivity, it suffices to prove that $G(\mathcal{K}_t)$ acts transitively on the set of collections of two-step lattice chains. This follows from Proposition 6.1 and the fact that the simultaneous stabilizer $G(\mathcal{O}_t)$ of the standard lattices $V(\lambda)^0$ acts transitively on the collections of lines in $V(\lambda)$ which satisfy the conditions in Proposition 5.7. □

7.3. **Plücker definition of the subvariety $\mathcal{F}l_{\leq\check{\mu}}$.**

For the rest of this section, we assume $G^{der} = G^{sc}$.

Given a dominant coweight $\check{\mu}$, we will define here a finite-dimensional projective subscheme $\mathcal{F}l_{\leq\check{\mu}} \subset \mathcal{F}l$. This will be a natural analogue of the subscheme $\overline{\mathcal{Q}}_{\check{\mu}} \subset \mathcal{Q}$.

Definition 7.2. We denote by $\mathcal{F}l_{\leq\check{\mu}}$ the set of collections $\{\mathcal{L}(\lambda)^\bullet\}_\lambda \in \mathcal{F}l$ which satisfy the following properties:

(1) $\wedge^{d_\lambda}\mathcal{L}(\lambda)^0 = \wedge^{d_\lambda} t_{\check{\mu}} V(\lambda)^0$, for every $\lambda \in X^+$.
(2) $t^{-\langle w_0\lambda,\check{\mu}\rangle} V(\lambda)^i \subset \mathcal{L}(\lambda)^i \subset t^{-\langle\lambda,\check{\mu}\rangle} V(\lambda)^i$, $(i \in \{0,1\})$, for every $\lambda \in X^+$.

7.4. **Ind-scheme structure on $\mathcal{F}l$.**

By the same argument as in §6.4, the forgetful functor

$$\{\mathcal{L}(\lambda)^\bullet\}_{\lambda\in X^+} \mapsto \{\mathcal{L}(\omega_i)^\bullet\}_i$$

defines a closed immersion of $\mathcal{F}l_{\leq\check{\mu}}$ into a finite-dimensional projective k-scheme.

Moreoever, by bounding the relative positions of the finitely-many two-step lattice chains $\mathcal{L}(\omega_i)^\bullet$, we see that any element of $\mathcal{F}l$ is contained in some $\mathcal{F}l_{\leq\check{\mu}}$. Hence we have the following ind-scheme structure on $\mathcal{F}l$:

Proposition 7.3. *The equality*

$$\mathcal{F}l = \cup_{\check{\mu}} \mathcal{F}l_{\leq\check{\mu}}$$

realizes the affine flag variety as an inductive limit of finite-dimensional projective schemes over k.

7.5. **The stratification of $\mathcal{F}l_{\leq\check{\mu}}$.** For $w \in \widetilde{W}$, let $Y(w) = IwI/I \subset \mathcal{F}l$ denote the corresponding Bruhat cell, and let $X(w) = \overline{Y(w)}$ denote the corresponding affine Schubert variety. Recall the the closure relation between Schubert varieties is given by the Bruhat order on $\widetilde{W}$:

$$X(w) \subset X(w') \Longleftrightarrow w \leq w'.$$

The ind-scheme $\mathcal{F}l$ carries an obvious action of the Iwahori subgroup I, and this action preserves the subscheme $\mathcal{F}l_{\leq\check{\mu}}$. Since the latter is finite-dimensional, the I-orbits in $\mathcal{F}l_{\leq\check{\mu}}$ are indexed by a finite set

$$\mathrm{Strat}(\check{\mu}) \subset \widetilde{W}.$$

The following result gives a combinatorial description of this set.

Proposition 7.4. *Let $x \in \widetilde{W}$, and write $x = t_{\check{\lambda}} w$, with $\check{\lambda} \in X_*$ and $w \in W$. Then $x \in \mathrm{Strat}(\check{\mu})$ if and only if the following properties are satisfied:*

(1) *$\check{\lambda} \in \Omega(\check{\mu})$; equivalently, $\langle w_0\lambda, \check{\mu}\rangle \leq \langle \lambda', \check{\lambda}\rangle \leq \langle\lambda, \check{\mu}\rangle$, for every $\lambda \in X^+$ and every $\lambda' \in \Omega(\lambda)$.*
(2) *For every $\lambda \in X^+$, the following condition holds: if $w\lambda \neq \lambda$ then*
  - *$\langle \lambda, w^{-1}\check{\lambda}\rangle < \langle\lambda, \check{\mu}\rangle$; and*
  - *$\langle w_0\lambda, \check{\mu}\rangle < \langle\lambda, \check{\lambda}\rangle$.*

The set $\mathrm{Strat}(\check{\mu})$ shares certain properties with the sets $\mathrm{Perm}(\check{\mu})$ and $\mathrm{Adm}(\check{\mu})$ defined by Kottwitz-Rapoport [10]. Some of these are listed below. It is to be hoped that $\mathrm{Strat}(\check{\mu})$ is a better approximation to $\mathrm{Adm}(\check{\mu})$ than was the case for $\mathrm{Perm}(\check{\mu})$ (cf. [8]). In some sense, this hope and the corollary below were the motivation for this approach to local models.

Corollary 7.5. *For every dominant coweight $\check{\mu}$, we have the following properties of* $\mathrm{Strat}(\check{\mu})$*:*

(1) $\mathrm{Adm}(\check{\mu}) \subset \mathrm{Strat}(\check{\mu})$.
(2) $\mathrm{Adm}(\check{\mu}) \cap X_* = \mathrm{Strat}(\check{\mu}) \cap X_* = \mathrm{Perm}(\check{\mu}) \cap X_* = \Omega(\check{\mu})$.

Proof. It is clear from Proposition 7.4 that

$$\mathrm{Strat}(\check{\mu}) \cap X_* = \Omega(\check{\mu}).$$

It is already known that this set coincides with both $\mathrm{Perm}(\check{\mu}) \cap X_*$ and $\mathrm{Adm}(\check{\mu}) \cap X_*$. Thus (2) is proved.

Since $\mathcal{F}l_{\leq \check{\mu}}$ is a projective scheme, it is a closed subscheme in $\mathcal{F}l$. Therefore the set $\mathrm{Strat}(\check{\mu})$ is closed with respect to the Bruhat order. But by (2), the translation elements $t_{w\check{\mu}}$ ($w \in W$) lie in $\mathrm{Strat}(\check{\mu})$. Hence all of $\mathrm{Adm}(\check{\mu})$ lies in $\mathrm{Strat}(\check{\mu})$, and (1) is proved. □

Unfortunately, as was the case for $\mathrm{Perm}(\check{\mu})$ [8], the set $\mathrm{Strat}(\check{\mu})$ does not coincide with $\mathrm{Adm}(\check{\mu})$ in general:

Proposition 7.6. *Suppose G is not of type A_n and is of rank ≥ 4. Then for every* regular *dominant coweight $\check{\mu}$, we have*

$$\mathrm{Adm}(\check{\mu}) \neq \mathrm{Strat}(\check{\mu}).$$

Proof. We use Deodhar's examples, as in [8]. For groups of the given type, there exist $w, w' \in W$, such that

- $w \neq w'$,
- $l(w) = l(w')$,
- $w\lambda - w'\lambda$ is a sum of positive roots, for every $\lambda \in X^+$.

Now one can see by Proposition 7.4 and by similar reasoning as in [8], that the element

$$x = t_{w^{-1}\check{\mu}} w^{-1} w'$$

belongs to $\mathrm{Strat}(\check{\mu})$ but not $\mathrm{Adm}(\check{\mu})$. □

It is still possible that $\mathrm{Adm}(\check{\mu}) = \mathrm{Strat}(\check{\mu})$ whenever $\check{\mu}$ is a minuscule coweight, or more generally, a sum of minuscule coweights.

8. Rapoport-Zink models

8.1. **Notation.** Let ϖ be a parameter. Write $\mathcal{O}_F = k[[\varpi]]$ and $F = k((\varpi))$. Let R be any $\mathcal{O}_F$-algebra. We are going to consider two-step chains of $R[t]$-modules contained in the module $V(\lambda) \otimes R[t, t^{-1}, (t+\varpi)^{-1}]$.

We define the "standard" two-step chain of $R[t]$-modules in $V(\lambda) \otimes R[t, t^{-1}, (t+\varpi)^{-1}]$

$$V(\lambda)^\bullet_R = (V(\lambda)^0_R \subset V(\lambda)^1_R \subset (t+\varpi)^{-1} V(\lambda)^0_R)$$

as follows: we define $V(\lambda)^0_R = V(\lambda) \otimes R[t]$ and

$$V(\lambda)^1_R = (t+\varpi)^{-1} R[t] v_\lambda \oplus (\oplus_{\lambda' \prec \lambda} R[t] v_{\lambda'}).$$

Here the summand on the right denotes the $R[t]$-span of all weight vectors in $V(\lambda)$ having weight less than λ.

If $\phi : V \to W$ is any G-morphism in $\mathrm{Rep}_k G$, then we denote by

$$\tilde{\phi} : V \otimes R[t, t^{-1}, (t+\varpi)^{-1}] \to W \otimes R[t, t^{-1}, (t+\varpi)^{-1}]$$

the induced map on the tensor product with $R[t, t^{-1}, (t+\varpi)^{-1}]$.

8.2. **Plücker definition for Rapoport-Zink models.** In this section we propose a general Plücker definition of "Rapoport-Zink" models $M_{\check{\mu}}$. The models defined here are the "Plücker analogues" of the ones for GL_n and GSp_{2n}, used in [7], but here are defined only in the case of equi-characteristic 0 function fields.

Definition 8.1. For any $\mathcal{O}_F$-algebra R, we define the R-points $M_{\check{\mu}}(R)$ to be the set of all collections $\{\mathcal{L}(\lambda)^\bullet\}_{\lambda \in X^+}$ of two-step chains of $R[t]$-submodules in $V(\lambda) \otimes R[t, t^{-1}, (t+\varpi)^{-1}]$

$$\mathcal{L}(\lambda)^\bullet = (\mathcal{L}(\lambda)^0 \subset \mathcal{L}(\lambda)^1 \subset (t+\varpi)^{-1}\mathcal{L}(\lambda)^0)$$

which satisfy the following properties:

(1) $\wedge^{d_\lambda}_{R[t]} \mathcal{L}(\lambda)^i = \wedge^{d_\lambda}_{R[t]} t_{\check{\mu}} V(\lambda)^i_R$, $(i \in \{0,1\})$, for every $\lambda \in X^+$.
(2) $t^{-\langle w_0\lambda, \check{\mu}\rangle} V(\lambda)^i_R \subset \mathcal{L}(\lambda)^i \subset t^{-\langle \lambda, \check{\mu}\rangle} V(\lambda)^i_R$ $(i \in \{0,1\})$, for every $\lambda \in X^+$.
(3) $t^{-\langle \lambda, \check{\mu}\rangle} V(\lambda)^i_R / \mathcal{L}(\lambda)^i$ $(i \in \{0,1\})$ is R-locally free, for every $\lambda \in X^+$.
(4) Let $\nu, \lambda, \mu \in X^+$. Then for every G-inclusion $\phi : V(\nu) \to V(\lambda) \otimes V(\mu)$, we have $\tilde{\phi}^{-1}(\mathcal{L}(\lambda)^0 \otimes \mathcal{L}(\mu)^0) = \mathcal{L}(\nu)^0$.
(5) For ϕ as above, we have

$$\tilde{\phi}^{-1}[\mathcal{L}(\lambda)^0 \otimes \mathcal{L}(\mu)^0 + (t+\varpi)(\mathcal{L}(\lambda)^1 \otimes \mathcal{L}(\mu)^1)] = \begin{cases} \mathcal{L}(\nu)^0, & \text{if } \nu \prec \lambda + \mu, \\ \mathcal{L}(\nu)^1, & \text{if } \nu = \lambda + \mu. \end{cases}$$

By arguing as in [7], we can easily describe the generic and special fibers for these models. One finds that

$$M_{\check{\mu}}(k((\varpi))) = \overline{\mathcal{Q}}_{\check{\mu}}(k((\varpi)))$$
$$M_{\check{\mu}}(k) = \mathcal{F}l_{\leq \check{\mu}}(k).$$

The functor $R \mapsto M_{\check{\mu}}(R)$ is represented by a finite-dimensional projective $\mathcal{O}_F$-scheme; this is seen as in §6.4,7.4, by considering the forgetful functor

$$\{\mathcal{L}(\lambda)^\bullet\}_{\lambda \in X^+} \mapsto \{\mathcal{L}(\omega_i)^\bullet\}_i.$$

We remark that the union $\cup_{\check{\mu}} M_{\check{\mu}}$ gives an ind-scheme structure to Beilinson's deformation of the affine Grassmannian to the affine flag variety, as mentioned in the introduction.

Finally, we point out one of the main reasons that Weyl modules look promising for use in giving the analogue of Definition 8.1 over $\mathbb{Z}_p$ instead of $k[[\varpi]]$: all that is needed to transfer the definition, nearly word-for-word, to one over $\mathbb{Z}_p$ are a suitable integral structure $V(\lambda)^0_{\mathbb{Z}_p}$ for $V(\lambda)$ over $\mathbb{Q}_p$, and also a good notion for $V(\lambda)^1_{\mathbb{Z}_p}$. But if G is a split reductive group over $\mathbb{Z}_p$, and if $\lambda \in X^+$, then the choice of a highest weight vector $v_\lambda \in V(\lambda)_{\mathbb{Q}_p}$ (the irreducible Weyl module over $\mathbb{Q}_p$) determines an integral structure

$$V(\lambda)^0_{\mathbb{Z}_p} = \mathrm{Dist}(G_{\mathbb{Z}_p}) \cdot v_\lambda$$

for $V(\lambda)_{\mathbb{Q}_p}$, where $\mathrm{Dist}(G_{\mathbb{Z}_p})$ denotes the algebra of distributions (cf. [9], II.8.2).

Moreover, this integral structure has the property

$$V(\lambda)_k = V(\lambda)^0_{\mathbb{Z}_p} \otimes k,$$

where the left hand side is the Weyl module attached to λ over k, for every field k which is a $\mathbb{Z}_p$-algebra. We can therefore set

$$V(\lambda)^1_{\mathbb{Z}_p} = p^{-1}\mathbb{Z}_p v_\lambda \oplus (\oplus_{\lambda' \prec \lambda} \mathbb{Z}_p v_{\lambda'}),$$

and get the standard two-step chain as above. Then one can use this to define $M_{\check{\mu}}$ over $\mathrm{Spec}(\mathbb{Z}_p)$. The problem is that the special fiber could be too large: one cannot necessarily describe the affine flag variety or the affine Grassmannian over a characteristic $p > 0$ field using only Weyl modules (at least not for general groups: it can be done for GL_n however).

Thus, to define the models so that the special fiber is correct, it should then be necessary to use more representations than just the Weyl modules in the definition of $M_{\check{\mu}}$.

Questions that will be considered in the continuation of this work:

- Is $\mathrm{Adm}(\check{\mu}) = \mathrm{Strat}(\check{\mu})$ for every minuscule coweight $\check{\mu}$? [1]
- Let $M^{RZ}_{\check{\mu}}$ denote a Rapoport-Zink local model. There is (at least in examples I have considered) a natural map

  $$M^{RZ}_{\check{\mu}} \to M_{\check{\mu}}.$$

  Is this an isomorphism?
- Define the "Tannakian" models $M_{\check{\mu}}$ over a p-adic number ring.
- Are these models flat?
- Use the models $M_{\check{\mu}}$ to prove Kottwitz' conjecture for semi-simple trace of Frobenius on nearby cycles for these models, following [7].

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[1] Added in August 2003: I have found that in fact this equality is almost never true. Therefore the set $\mathrm{Strat}(\mu)$ and the model giving rise to it are not the correct ones to consider...

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