Math 111, section 6.4 Permutations and Combinations

notes by Tim Pilachowski

The Multiplication Principle was introduced in section 6.3:

Suppose a task T_1 can be peformed in N_1 ways, a task T_2 can be peformed in N_2 ways, … and finally, a task T_m can be peformed in N_m ways. Then the number of ways of performing the tasks T_1, T_2, … , T_m is given by the product N_1 times N_2 times … times N_m.

The Examples used in Lecture 6.3 were carefully chosen so that successive tasks were independent, i.e., so that the first event had no influence on the next. The first die rolled did not affect the second. In the club elections, no one person was a candidate for more than one office. Farnsworth's choice of screen size did not affect his choice of CPU speed. Letters and numbers in the license plates could be used more than once.

But what if the result of the first event does have an effect on the outcome of the second?

Example A-1. Eddington has three blocks to play with: red, yellow, and blue. If he lays them out into a line one at a time, how many designs can he create?

Example B-1. Eddington has six blocks to play with: purple, red, orange, yellow, green and blue. If he lays them out into a line one at a time, how many designs can he create?

Example C-1. Eddington has ten blocks to play with: purple, red, orange, yellow, green, blue, white, black, gray, and brown. If he lays them out into a line one at a time, how many designs can he create?

This type of arrangement, in which the order of objects or events makes a difference, e.g. RYB ≠ RBY, is called a ***permutation***. For our text and for this class, we will assume that there is no repetition in a permutation, e.g. Edd could not and would not have two red blocks side by side because he picks a block and does not put it back.

The Multiplication Principle applied to a permutation involves what is called a ***factorial***.
For any positive integer n, "n factorial" is written and defined as

$$n! = n(n-1)(n-2)\ldots(3)(2)(1)$$

and by definition 0! = 1.

Notes on factorials:

Example A-1 revisited. Eddington has three blocks to play with: red, yellow, and blue. If he lays two of them out into a line one at a time, how many designs can he create?

Example B-1 revisited. Eddington has six blocks to play with: purple, red, orange, yellow, green and blue. If he lays three of them out into a line one at a time, how many designs can he create?

Example C-1 revisited. Eddington has ten blocks to play with: purple, red, orange, yellow, green, blue, white, black, gray, and brown. If he lays four of them out into a line one at a time, how many designs can he create?

The process developed in Examples A-1 through C-1 can be generalized into the following formula for the number of permutations of n distinct objects taken r at a time:

$$P(n, r) = \frac{n!}{(n-r)!}.$$

Other notations that are used and you may encounter include ${}^{n}P_{r}$ and ${}_{n}P_{r}$.

Example D-1. Eddington has fifteen blocks to play with, each one a different color. How many ways can he pick up just one? How many ways can he line up all fifteen?

Example A-2. Eddington has three blocks to play with: red, yellow, and blue. If he picks up two of them at the same time, how many ways can he choose blocks?

Example B-2. Eddington has six blocks to play with: purple, red, orange, yellow, green and blue. If he picks up three of them at the same time, how many ways can he choose blocks?

Example C-2. Eddington has ten blocks to play with: purple, red, orange, yellow, green, blue, white, black, gray, and brown. If he picks up four of them at the same time, how many ways can he choose blocks?

The process developed in Examples A-2 through C-2 can be generalized into the following formula for the number of combinations of n distinct objects taken r at a time:

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}.$$

Other notations that are used and you may encounter include $\binom{n}{r}$ which your text has also, ${}^{n}C_{r}$ and ${}_{n}C_{r}$.

Theory:

Example D-2. Eddington has fifteen blocks to play with, each one a different color. How many combinations of 1 block are there? How many combinations of 15 blocks are there?

Your textbook exercises for day 1 of section 6.4 ask you to calculate values for various versions of $P(n, r)$ and $C(n, r)$.

Note that for something like $P(20, 24)$ and $C(17, 18)$ the correct answer is “not possible”. Why?

Example E-1. How many arrangements of 3 digits (repeated digits allowed) are there, chosen from the ten digits 0 to 9 inclusive? How many permutations of 3 different digits (i.e., no repeats) are there, chosen from the ten digits 0 to 9 inclusive? How many combinations of 3 different digits are there, chosen from the ten digits 0 to 9 inclusive?

Example E-2. How many arrangements of 4 letters (repeated letters allowed) are there, chosen from the twenty six letters of the alphabet? How many permutations of 4 different letters (i.e., no repeats) are there, chosen from the twenty six letters of the alphabet? How many combinations of 4 different letters are there, chosen from the twenty six letters of the alphabet?

Examples E-1 and E-2 identified whether the situation was a permutation or combination, with or without repeated elements. In word problems it's not always so easy.

Example F. a) In Arithmetic, addition is both commutative and associative, that is numbers can be rearranged and regrouped in any way and the answer will still be the same. Is addition more like a combination or permutation? b) In Arithmetic, division is neither commutative nor associative. It makes a difference whether one is calculating "*p* divided by *q*" or "*q* divided by *p*". Is division more like a combination or permutation?

Example G. a) The cook at Pappy's Restaurant makes a pizza by first rolling out and stretching the dough for the crust, then spreading out the sauce, then adding cheese and toppings. Is making a pizza more like a combination or permutation? b) The cook at Pine Tree Café D' makes sausage stew by throwing all of the ingredients into a pot and letting it simmer for four hours. Is making stew more like a combination or permutation?

Example H. a) Harold Hill puts his knick knacks on a shelf according to whatever whim hits him at the moment, and is likely to move them around periodically. Is Harold's method an example of a combination or permutation? b) Marian puts books on the shelf in the town's library, arranging them by using their call numbers. Is Marian's method an example of a combination or permutation?

Example I. In the Maryland Lottery's Pick 4 game, players have a choice of the type of bet. "Straight – Your number must match the winning number in exact order. Box – Your number can match the winning number in any order." (source: mdlottery.com) a) Is playing "Straight" an example of a combination or permutation? b) Is playing "Box" an example of a combination or permutation?

Example J. a) Twenty runners stand on the starting line at the beginning of a marathon. Is this an example of a combination or a permutation? b) Ribbons are awarded to first-, second-, and third-place finishers. Is this an example of a combination or a permutation?

Example K. A combination lock requires the user to know a set of three one-digit numbers which must be entered in a particular way. For example, if the correct combination is "1-3-5", then entering "5-1-3" or "3-5-1" won't open the lock. Is a "combination lock" an example of a combination or a permutation?

Example L. In a game of bridge, using a standard deck of fifty-two playing cards, each of four players is dealt thirteen cards. Is this an example of a combination or permutation? How many possible bridge hands are there?

Example M. a) Odysseus can name every player on his baseball team's roster. Is this an example of a combination or permutation? b) As the club's manager, Persephone decides which player will bat when in the lineup for each game. Is this an example of a combination or permutation? c) If the team roster has 25 members, how many ways can Persephone set up the starting lineup of nine batters?

Example N-a. One hundred names are put into a hat. Is this an example of a combination or permutation?

Example N-b. One hundred names are put into a hat. Names will be drawn for prizes. The top prize is $500, the next prize is $300, and the third prize is $100. Is this an example of a combination or permutation? How many ways can the prizes be awarded?

Example N-c. One hundred names are put into a hat. Names will be drawn for three prizes of $300 each. Is this an example of a combination or permutation? How many ways can the prizes be awarded?

Example O. For the Powerball lottery, "Every Wednesday and Saturday night, five white balls from 1 to 59 and one red Powerball from 1 to 39 will be drawn. You win a prize by matching some or all of the numbers drawn." (source: mdlottery.com). Are the choices for a Powerball ticket a combination or a permutation? How many ways can a player pick the numbers for a Powerball ticket?

Example P-a. A club is electing its three top officers. Is this an example of a combination or permutation? If the club has twenty members, how many different slates of officers can be elected?

Example P-b. A club is forming a committee of five members to plan its annual Solstice Dance. Is this an example of a combination or permutation? If the club has twenty members, how many different ways can the Solstice Dance Committee be formed?

Example P-c. A club is forming a committee to plan its annual Equinox Dance. One person will serve as Chair of the committee, and there will be four other members. Is this an example of a combination or permutation? If the club has twenty members, how many different ways can the Equinox Dance Committee be formed?

Example P-d. A club is forming a committee to plan its annual Holiday Dance. One person will serve as Chair of the committee, another will be Vice-Chair and there will be three other members. Is this an example of a combination or permutation? If the club has twenty members, how many different ways can the Holiday Dance Committee be formed?

Example P-e. A club is forming a committee to plan its annual Anniversary Dance. Two people will serve as Co-Chairs of the committee, and there will be three other members. Is this an example of a combination or permutation? If the club has twenty members, how many different ways can the Anniversary Dance Committee be formed?

Example Q-a. Find the number of distinguishable permutations that can be formed from the letters of the word INCLUDE.

Example Q-b. Find the number of distinguishable permutations that can be formed from the letters of the word PRELUDE.

Example Q-c. Find the number of distinguishable permutations that can be formed from the letters of the word SUCCEEDED.

On day 3 of Lecture 6.4, if we have time we can go over some of the already-assigned homework questions from the textbook, which includes numbers 1 through 56.

If no one has specific requests, I'll choose some good ones from among the unassigned questions.