Calculus 111, section 7.2 Definition of Probability

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In chapter 6, the question was, "How many ways...", and we counted the number of possible outcomes.

In section 7.1, we constructed sample spaces by asking, "What could happen?"

Now, in section 7.2, we begin asking and answering the question, "How likely are the sample points and events from our sample space?"

Concept review: A **sample space** S contains *all possible* outcomes for an experiment.
The elements in a sample space are the outcomes of the experiment, and are called **sample points**.
The sample points in a sample space must be **mutually exclusive** outcomes.
An **event** (designated with a capital letter A, B, C, etc.) is a subset of the sample space, and will incorporate one or more of the outcomes.

Example A: You toss two coins. The sample space is S = { }.
a) What are the probabilities for the simple events described in the sample space?

A **probability model** assigns probabilities to all the events in a sample space. In essence, we will define the probability of an event as "the proportion of times the event is expected to occur". For simple events that are equally likely to occur, we can use a "uniform probability model", as we did in Example A-a.

Formally, for an event E

The **probability** of an event $E = P(E) = \dfrac{\text{number of ways } E \text{ can happen}}{\text{number of possible outcomes}} = \dfrac{\text{number of simple events in } E}{\text{number of simple events in } S}$.

That is, if the outcomes in sample space S are equally likely, then $P(E) = \dfrac{n(E)}{n(S)}$.

Example A continued: You toss two coins. The sample space is S = { HH, HT, TH, TT }, 4 simple events that are equally likely.

A = both coins are heads =

probability that both coins are heads = $P(A)$ =

B = one coin is heads and the other is tails =

P(one coin is heads and the other is tails) = $P(B)$ =

Note that $P(S)$ =

Each event (simple or compound) in a sample space will have a **probability** associated with it. These probabilities will have the following properties.

1. For any event E, $0 \le P(E) \le 1$.
2. For any event $F = s_1 \cup s_2 \cup \ldots \cup s_n$, $P(F) = P(s_1) + P(s_2) + \ldots + P(s_n)$.
3. For any sample space S, $P(S) = \sum_{\text{all } s \text{ in } S} P(s) = 1$.

Properties 2 and 3 rely on all simple events s being mutually exclusive!!

Example B-1: You toss a standard six-sided die. The sample space is $S = \{ 1, 2, 3, 4, 5, 6 \}$, 6 simple events that are equally likely.

A = the number rolled is even =

$P(A) =$

B = the number rolled is at least 2 =

$P(B) =$

C = the number rolled is more than 3 =

$P(C) =$

D = the number rolled is at most 4 =

$P(D) =$

Note that $P(S) =$

Example B-2: You toss two standard six-sided dice.

$$
\begin{aligned}
S = \{ \quad & (1, 1), (2, 1), (3, 1), (4, 1), (5, 1), (6, 1), \\
& (1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2), \\
& (1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (6, 3), \\
& (1, 4), (2, 4), (3, 4), (4, 4), (5, 4), (6, 4), \\
& (1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (6, 5), \\
& (1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6) \}
\end{aligned}
$$

A = at least one of the dice is a 1 =

$P(A) =$

B = the sum of the two dice is 9 =

$P(B)$ =

C = at least one of the dice is a 1 and the sum of the two dice is 9 =

$P(C)$ =

Events A and B are **mutually exclusive.**

Note that $P(S)$ =

Example C: You pick a card from a standard deck of 52 cards. [4 suits: spades (S), hearts (H), diamonds (D), clubs (C); 13 cards in each suit: ace (A), king (K), queen (Q), jack (J), 10, 9, 8, 7, 6, 5, 4, 3, 2]

S = { A-S, A-H, A-D, A-C, K-S, K-H, K-D, K-C, Q-S, Q-H, Q-D, Q-C, … , 2-S, 2-H, 2-D, 2-C }. $n(S) = 52$. The 52 outcomes are equally likely.

A = the card is an Ace =

$P(A)$ =

B = the card is a Spade =

$P(B)$ =

C = the card is the Ace of Spades =

$P(C)$ =

D = the card is an Ace or a Spade =

$P(D)$ =

E = the card is neither an Ace nor a Spade =

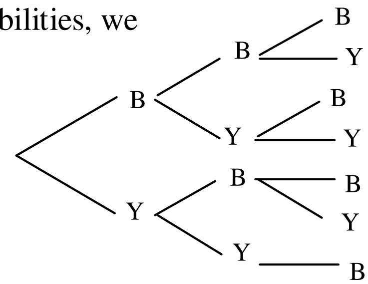
$P(E)$ =

Note that $P(S)$ =

Example D: Suppose that a box contains 3 blue blocks and 2 yellow blocks. You pick three blocks without replacement.

Experiment: Pick three blocks without replacement.

Since "blue" and "yellow" do not have uniform probabilities, we cannot simply assign each a probability of 0.5.



A = the first block is blue

$P(A)$ =

B = the first block is yellow

$P(B)$ =

C = if the first block is blue, the second is also blue

$P(C)$ =

D = if the first block is yellow, the second is blue

$P(D)$ =

So far, the probabilities encountered have been theoretical. Probabilities can also be determined in **empirical** situations, through observations made about actual phenomena. The proportion of times a given event occurs in the long run is its **relative frequency**. In this type of situation, the probability of an event E is $P(E)$ = value to which the relative frequency stabilizes with an increasing number of trials.

Example E: A hospital records the number of days each ICU patient stays in intensive care. $S = \{ 1, 2, 3, \dots \}$. Out of 1247 ICU patients in the last 15 years, 536 stayed in ICU two weeks or less. If a patient is selected at random, what is the probability that she or he will stay in ICU two weeks or less?

Example F: Frodo has kept track of the number of times each of his friends has dropped in for a visit during the past month. Find the probability distribution associated with these data.

friend	Sam	Merry	Pippin	Gandalf
frequency of visits	30	15	21	2

Example G: In 2009, households were surveyed about health insurance coverage, with the following results.

	Age 18-64	Age < 18
Public Plan	10286	10771
Private Plan	47000	15914
Uninsured	14144	1885

DATA SOURCE: CDC/NCHS, National Health Interview Survey, 1997-2009, Family Core component. Data are based on household interviews of a sample of the civilian noninstitutionalized population. Numbers given above were extrapolated from summary tables.

E_1 = individual is under 18; $P(E_1) =$

E_2 = individual has health insurance; $P(E_2) =$

E_3 = individual is under 18 ***and*** has health insurance; $P(E_3) =$

E_4 = individual is under 18 ***or*** has health insurance; $P(E_4) =$