111, section 7.6 Conditional Probability: Bayes' Theorem

notes by Tim Pilachowski

Recall: A **probability model** assigns probabilities to all the events in a sample space. We defined the probability of an event as "the proportion of times the event is expected to occur". Formally, for an event E

$$P(E)=\frac{\text{number of ways } E \text{ can happen}}{\text{number of possible outcomes}}=\frac{\text{number of elementary outcomes in } E}{\text{number of elementary outcomes in } S}=\frac{n(E)}{n(S)}$$

We'll also need the **complement rule**: for any event E, $P(E^c) = P(S) - P(E) = 1 - P(E)$.

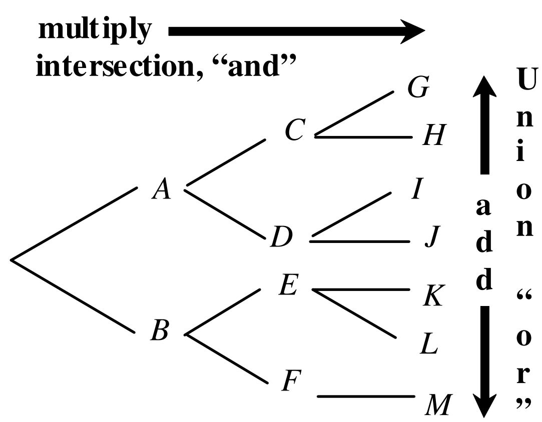
In **conditional probability** an outcome or event B is dependent upon another outcome or event A. Formally, $P(B \mid A) = P(B \text{ given } A)$ = probability that B will happen given the prior condition that A has already happened.

$$P(B \mid A)=\frac{P(A \cap B)}{P(A)}$$

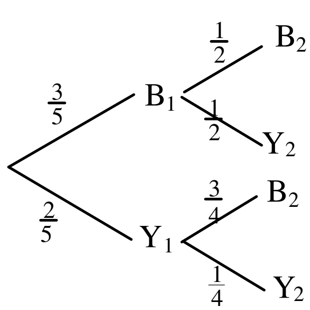
This same formula, from a different perspective, is an application of the multiplication principle.

$$P(A \cap B)=P(A) * P(B \mid A)$$

In a tree diagram, moving from root to branch, events are intersections and probabilities are multiplied.
In a tree diagram, moving up and down from branch to branch, events are unions and probabilities are added.



7.1, 7.2, 7.4, & 7.5 Example D revisited: A box contains 3 blue blocks and 2 yellow blocks.
Experiment: Pick two blocks
without replacement.



The tree diagram developed in earlier lectures provides the conditional probabilities involved in this two-stage experiment.

$P(B_1)=\frac{3}{5}$ $\quad$ $P(Y_1)=\frac{2}{5}$

$P(B_2 \mid B_1)=\frac{1}{2}$ $\quad$ $P(B_2 \mid Y_1)=\frac{3}{4}$

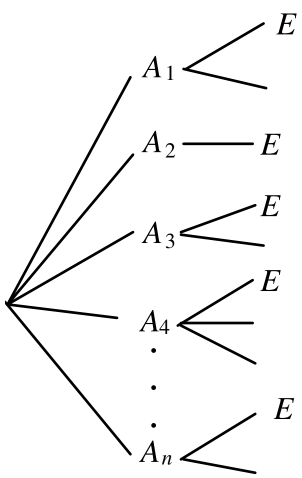
$P(Y_2 \mid B_1)=\frac{1}{2}$ $\quad$ $P(Y_2 \mid Y_1)=\frac{1}{4}$

In this scenario, the presumption is that either B_1 or Y_1 would happen first, and then B_2 or Y_2 would happen second. But what if we were interested in looking backwards? What if we asked, "We know B_2 has happened. What is the probability that the preceding event was B_1?" That is, "What is $P(B_1 \mid B_2)$?"

$P(B_1 \mid B_2) =$

interpretation:

We can take the ideas used in Example D and expand to develop a general-use formula, which is Bayes' Theorem.



$$P(B_1 \mid B_2) = \frac{P(B_1 \cap B_2)}{P(B_2)} \qquad = \frac{P(B_1 \cap B_2)}{P(B_1 \cap B_2) + P(Y_1 \cap B_2)}$$

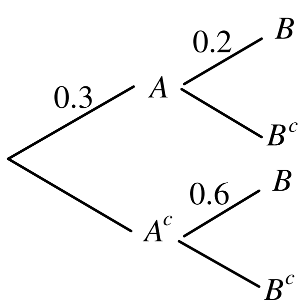
$$= \frac{P(B_1) * P(B_2 \mid B_1)}{P(B_1) * P(B_2 \mid B_1) + P(Y_1) * P(B_2 \mid Y_1)}$$

$$P(A_i \mid E) = \frac{P(E \cap A_i)}{P(E)} \qquad = \frac{P(E \cap A_i)}{P(E \cap A_1) + P(E \cap A_2) + \ \dots \ + P(E \cap A_n)}$$

$$= \frac{P(A_i) * P(E \mid A_i)}{P(A_1) * P(E \mid A_1) + P(A_2) * P(E \mid A_2) + \ \dots \ + P(A_n) * P(E \mid A_n)}$$

You can memorize this formula if you wish, however my recommendation is to draw the tree diagram and work out the probabilities for the intersections from there. This is the approach I'll use in the examples below.

Example A: The tree diagram below provides the conditional probabilities involved in a two-stage experiment.



$P(A^c) =$

$P(B^c \mid A) =$

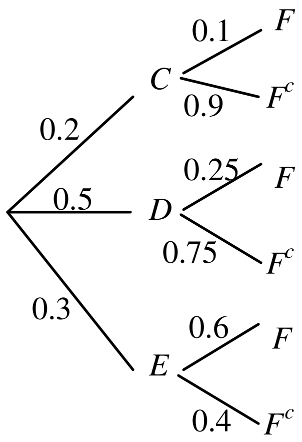
$P(B^c \mid A^c) =$

$P(A \mid B) =$

$P(A \mid B^c) =$

$P(A^c \mid B) =$

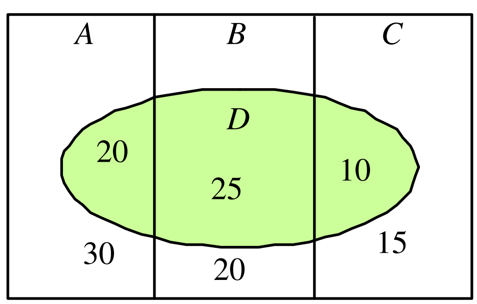
Example B: The tree diagram below provides the conditional probabilities involved in a two-stage experiment.



$P(C \mid F) =$

$P(D \mid F^c) =$

Example C: The Venn diagram below depicts an experiment in which three mutually exclusive events form a partition of a uniform sample space S. a) Draw a tree diagram. b) Determine $P(A \mid D)$. c) Determine $P(D)$. d) Determine $P(D^c)$. e) Determine $P(C \mid D^c)$.



Example E. Silver Springs, Florida, has a snack bar and a gift shop. The management observes 100 visitors, and counts 65 who eat in the snack bar. Among those who ate in the snack bar, 40 also make a purchase in the gift shop. Of the patrons who did not eat in the snack bar, 15 bought something in the gift shop. What is the probability that a person who made a purchase in the gift shop ate in the snack bar?

Example F. At Matriarch University (U. Mama), 35% of the students have an academic scholarships. Of the students who have an academic scholarship, 80% live on campus. Among the students without an academic scholarship, 70% live on campus. What is the probability that a person who lives on campus has an academic scholarship?

Example G. Two cards are dealt from a standard deck of 52. The first is placed face down. The second one, dealt face-up, is an Ace. What is the probability that the first card is a King, Queen, Jack or 10?

Example H. The Shockingly Good Company has factories in Assateague, Betterton and Chestertown. The percentage of production and percentage of defective spark plugs made at each factory is given in the table below. What is the probability that a defective spark plug came from the Assateague factory? Betterton? Chestertown?

	Percent of Production	Probability of Defective
Assateague	45%	0.08%
Betterton	20%	0.4%
Chestertown	35%	0.1%

interpretation:

If the company is making decisions about where to focus efforts to improve quality, which factory should they choose? Why?

Example I: The Gallup organization conducted 10 separate surveys conducted from January through May 2009. At the time of the report, Gallup had found an average of 35% of Americans considering themselves Democratic, 28% Republican, and 37% independent. Within those affiliations, the following percentages identified themselves as Conservative, Moderate or Liberal.

	Democrat (event *D*)	Independent (event *I*)	Republican (event *R*)
Conservative (event *C*)	22%	35%	73%
Moderate (event *M*)	40%	45%	24%
Liberal (event *L*)	38%	20%	3%

http://www.gallup.com/poll/120857/conservatives-single-largest-ideological-group.aspx

Results are based on aggregated Gallup Poll surveys of approximately 1,000 national adults, aged 18 and older, interviewed by telephone. Sample sizes for the annual compilations range from approximately 10,000 to approximately 40,000. For these results, one can say with 95% confidence that the maximum margin of sampling error is ±1 percentage point.

This table gives conditional probabilities. $P(\text{Conservative} \mid \text{Independent}) = 35\%$. $P(\text{Liberal} \mid \text{Republican}) = 3\%$.

a) Draw a tree diagram to illustrate the events and probabilities for this two-stage experiment.

b) Calculate $P(I \mid C)$. Write a verbal description of what $P(I \mid C)$ means. Determine whether I and C are independent events and state how you know.

c) Calculate $P(D \mid L^c)$. Write a verbal description of what $P(D \mid L^c)$ means. Determine whether D and L^c are independent events and state how you know.

7.5 Example K revisited: In early 2010, households were surveyed about health insurance coverage.

	Under 65 years	18-64 years	Under 18 years
Uninsured %	17.5%	21.5%	7.4%
No. Uninsured (millions)	46.4	40.9	5.5
Public %	21.2%	14.4%	38.4%
Private %	62.7%	65.5%	55.5%

Data from Table 1, Table 2 and Table 3, *Health Insurance Coverage: Early Release of Estimates From the National Health Interview Survey, January–March 2010* by Robin A. Cohen, Ph.D., and Michael E. Martinez, M.P.H., M.H.S.A., Division of Health Interview Statistics, National Center for Health Statistics. **IMPORTANT:** A footnote states, "A small number of persons were covered by both public and private plans and were included in both categories." As a result, note that the percents in each column add up to greater than 100%.

This table gives conditional probabilities! $P(\text{Uninsured} \mid 18\text{-}64 \text{ years}) = 21.5\%$. $P(\text{Public} \mid \text{Under } 65) = 21.2\%$.
S = persons in the U.S. under 65 years of age.
Let F = age under 18 years, so F^c = age 18-64 years.
Let E = lacking health insurance coverage, so E^c = having public or private health coverage (union).

a) Draw a tree diagram to illustrate the relationship between events E and F.

b) Determine $P(E \mid F)$ and $P(E^c \mid F)$ and place them on the tree diagram.

c) Determine $P(E \mid F^c)$ and $P(E^c \mid F^c)$ and place them on the tree diagram.

d) Calculate $P(F)$ and $P(F^c)$ then place them on the tree diagram.

e) Calculate $P(F \mid E)$.

f) Write a verbal description of what $P(E \mid F)$ means.

Write a verbal description of what $P(F \mid E)$ means.

7.5 Example L revisited: U.S. Department of Labor employment reports place jobs into the five following categories:

Management, professional, and related occupations	(Mgt)
Service occupations	(Svc)
Sales and office occupations	(SO)
Natural resources, construction, and maintenance occupations	(NCM)
Production, transportation, and material moving occupations	(PTM)

In 2010, 39.3% of jobs were classified as Mgt, 14.5% were Svc, 23.2% were SO, 9.9% were NCM, and 13.1% were PTM. Among Mgt, 48.6% were held by male workers, and 51.4% were held by female workers. The other categories were: Svc, 50.6% male, 49.4% female; SO, 38.3% male, 61.7% female; NCM, 95.9% male, 4.1% female; PTM, 80.2% male, 19.8% female.

source: United States Department of Labor, Bureau of Labor Statistics report “Median weekly earnings of full-time wage and salary workers by detailed occupation and sex” covering the year 2010.
ftp://ftp.bls.gov/pub/special.requests/lf/aat39.txt

a) Put this data into a tree diagram.

b) Determine *P*(being female | employed in a management/professional job). Determine *P*(being employed in a management/professional job | female).Are “being female” and “employed in a management/professional job” independent events? How do you know?

c) Among the male members of the U.S. workforce, what proportion were in production/transportation jobs? Are “being male” and “working in a production/transportation job” independent events? How do you know?