111, section 8.1 Distributions of Random Variables

notes by Tim Pilachowski

For a given situation, or **experiment**, observations are made and data is recorded.

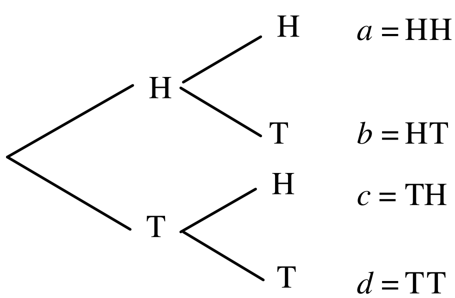
A **sample space** S must contain *all possible* outcomes for an experiment.

An **event** (designated with a capital letter A, B, C, etc.) is a subset of the sample space, and will incorporate one or more of the outcomes.

Example A-1. You toss two coins. Describe an appropriate sample space.

We did this back in 7.1 and constructed the following tree diagram.

Experiment: toss two coins
outcomes):



The sample space is S = { HH, HT, TH, TT } and has 4 simple events or outcomes.

We could take a different perspective and consider these simple events in terms of “number of heads”.

We have defined a variable X = number of heads in two tosses.

Note carefully: The probabilities for the values of X are not equal!

Example A-2. You toss a coin ten times. What are the possible outcomes?

Define X = number of heads in ten tosses of a coin. What are the values that X may assume?

Note carefully: The probabilities for the values of X are not equal!

Example A-3. You toss a coin until a head comes up. What are the possible outcomes?

Define X = number of tosses until a head comes up. What are the values that X may assume?

Note carefully: The probabilities for the values of X are not equal!

A **random variable** is a rule/formula that assigns a number value to each outcome of a chance experiment.

“Random” reminds us that we cannot predict specific outcomes, only discuss the probabilities.

Each value x_i assigned to a discrete random variable X will have an associated probability, $P(X = x_i)$.

In Example A-1 we defined our random variable as X = number of heads in two tosses of a coin.
In Example A-2 we defined our random variable as X = number of heads in ten tosses of a coin.
In Example A-2 we defined our random variable as X = number of tosses until a head comes up.

Example B. You toss two standard six-sided dice. Define a random variable X = sum of the two dice.
Think one red die and one white die. The two dice are independent. There are thirty-six possible tosses.

(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

What are the values that X may assume?

Note carefully: The probabilities for the values of X are not equal!

Example C. You deal five cards from a standard deck of 52. Define a random variable X = number of Aces.
What are the values that X may assume?

Note carefully: The probabilities for the values of X are not equal!

Example D. Suppose that a box contains 3 blue blocks and 2 yellow blocks. You pick three blocks without replacement. Define X = number of blue blocks drawn.

What are the values that X may assume?

x			
$P(X = x)$			

Each value x_i assigned to a discrete random variable X will have an associated probability, $P(X = x_i)$.

The x-values and their probabilities can be displayed in a probability distribution table and/or histogram.

A function $f(x_i) = P(X = x_i)$ is called (what a surprise!) a **probability distribution function**.

A probability distribution function must meet the basic criteria for all probabilities.

$$0 \leq P(X = x_i) \leq 1 \text{ for each value } x_i \text{, and } \sum_{\text{all } x_i \text{ in } S} P(X = x_i) = 1$$

Why can't the table below give the probability distribution for a random variable X ?

x	0	3	5
$P(X = x)$	$-\frac{1}{5}$	$\frac{3}{5}$	$\frac{3}{5}$

Why can't the table below give the probability distribution for a random variable X ?

X	-3	2	4
$P(X = x)$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{4}{7}$

Example B revisited. You roll two dice. Define X = the sum of the two dice.

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$											

We can use this probability distribution table to answer questions about various probabilities.

$P(X = 5) =$

$P(X < 5) =$

$P(5 \le X \le 8) =$

Most of the examples done so far illustrate **finite random variables**. The values X may assume are limited in scope. (The exception is Example A-3.)

Variables that are unlimited are **infinite random variables**. (For Example A-3, $X = 1, 2, 3, 4, 5, 6, \ldots$)

Example E. Let X = the number of days each ICU patient stays in intensive care. What are the values that X can assume?

All of the examples done so far illustrate **discrete random variables**. The values X may assume are distinct.

Measurements such as length or distance would be **continuous random variables**.

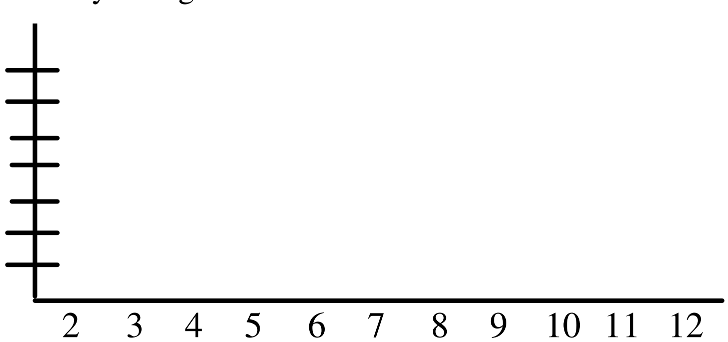
Example F. Suppose we measure the heights of 25 people and we define X = height in inches. What are the values that X may assume?

Example B revisited. You roll two dice. Define X = the sum of the two dice.

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$											

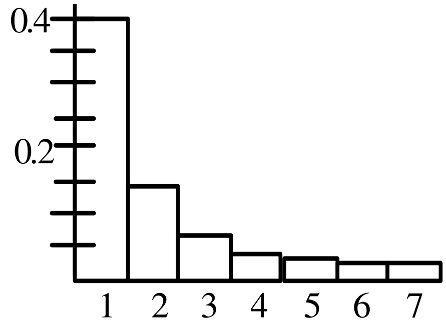
Earlier, we developed the probability distribution table for this random variable. We could also take these numbers and display them in a bar graph called a **histogram.**

The probability histogram would look like this:



Example E revisited. Let X = the number of days each ICU patient stays in intensive care. X = 1, 2, 3, 4, 5, …

The probabilities would be developed based on relative frequencies—observations made from hospital and patient records. The histogram might look something like this.



This histogram is extrapolated from research articles. In the literature, researchers have tried correlating length of stay with diagnosis. For your general knowledge, this probability distribution is approximately exponential, with formula $f(x) = 0.4e^{-0.4x}$.

Note that, for each probability, $0 \leq P(X = x) \leq 1$. There are Calculus methods which prove that S $P(X = x) = 1$. This probability distribution is ***not*** symmetric.

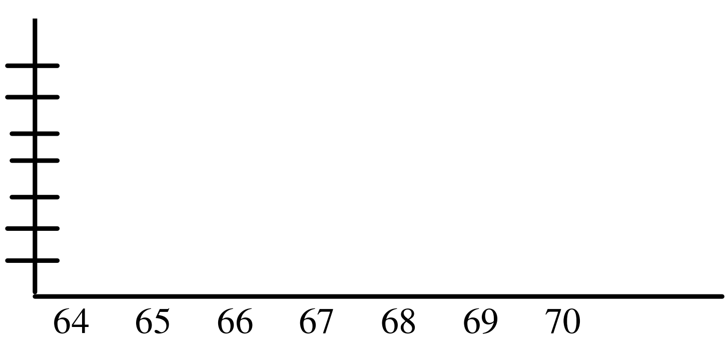
Example F revisited. Suppose we measure the heights of 25 people and we define X = height in inches. $X \in \{h \mid h \geq 0\}$

Although height is a continuous random variable, we can (and in practice, do) treat it as a discrete random variable by rounding off to a specified accuracy.

Suppose we measure the heights of 25 people to the nearest inch and get the following results:

height (in.)	64	65	66	67	68	69	70
frequency	5	7	6	4	1	0	2
probability							

The probability histogram would look like this:

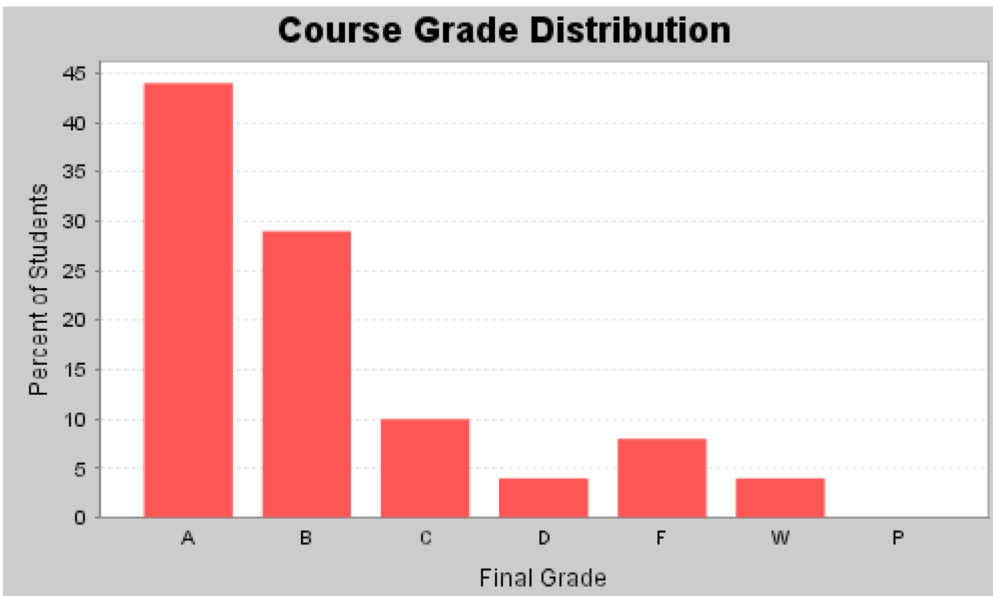


We can calculate probabilities for various values of X.

a) $P(X < 67) =$

b) The probability that height is at least 68 inches is

Example G. A Math 220 class, taught in the Fall of 2010 at UMCP, had the following grade distribution. Define X = number of As.



a) What is the probability that the first three students on the roster all received an A?

b) What is the probability that exactly two of the first three students on the roster received an A?

c) What is the probability that at least two of the first three students on the roster received an A?

d) What is the probability that at most two of the first three students on the roster received an A?