111, section 8.2 Expected Value

notes prepared by Tim Pilachowski

Do you remember how to calculate an average? The word "average", however, has connotations outside of a strict mathematical definition, so mathematicians and statisticians have a different name: the ***mean***.

8.1 Example F revisited: Suppose we measure the heights of 25 people to the nearest inch and get the following results:

height (in.)	64	65	66	67	68	69	70
frequency	5	7	6	4	1	0	2

What is the mean height? *answer*: 65.88 in.

first method:

second method:

third method:

In the mathematics of probability, the formula "sum of (value times probability)" is called "expected value" as well as "mean".

For a discrete random variable X we will calculate the expected value or mean using the following formula:

$$E(X) = x_1 * p_1 + x_2 * p_2 + x_3 * p_3 + \ldots + x_n * p_n = \sum_{i=1}^{n} x_i * p_i$$

where x_i is an amount and $f(x_i)$ is its probability.

Example A *fourth method*: We measure the heights of 25 people to the nearest inch with the following results:

height (in.)	64	65	66	67	68	69	70
frequency	5	7	6	4	1	0	2
probability							

What is the expected value, $E(X)$, for height?

8.1 Example A-2 revisited: You toss a coin ten times. What is the expected value for random variable X = number of heads in ten tosses of a coin?

x	0	1	2	3	4	5	6	7	8	9	10	
$P(X = x)$												total = 1

answer: 5

8.1 Example B revisited: You roll two dice. What is the expected value for random variable X = the sum of the two dice ?

x	2	3	4	5	6	7	8	9	10	11	12	
$P(X = x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$	total = 1

answer: 7

8.1 Example C revisited: You deal five cards from a standard deck of 52. What is the expected value for random variable X = number of Aces?

x	0	1	2	3	4	
$P(X = x)$						total = 1

answer: 0.31463

8.1 Example D revisited: Suppose that you pick three blocks without replacement from a box that contains 3 blue blocks and 2 yellow blocks. What is the expected value for X = number of blue blocks drawn?

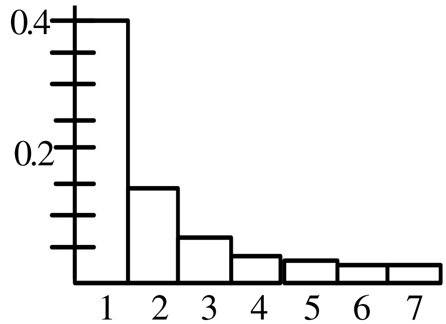
x	1	2	3	
$P(X = x)$	$\frac{3}{10}$	$\frac{3}{5}$	$\frac{1}{10}$	total = 1

answer: 1.8

8.1 Example E revisited. Let X = the number of days each ICU patient stays in intensive care.

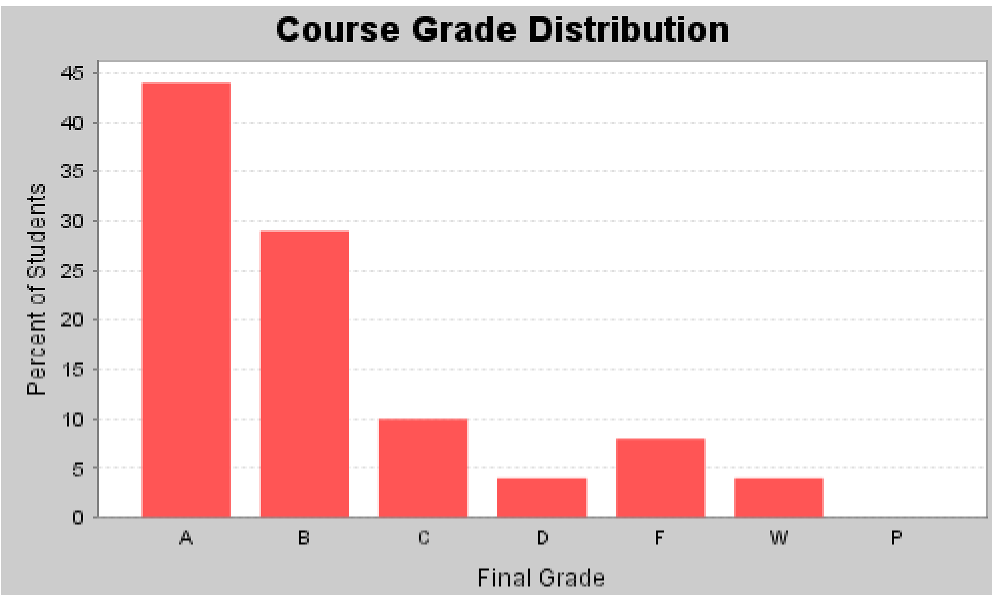
$$X = 1, 2, 3, 4, 5, \ldots$$

The probabilities would be developed based on relative frequencies—observations made from hospital and patient records. The histogram might look something like this.



For your general knowledge, this probability distribution is approximately exponential, with formula $f(x) = 0.4e^{-0.4x}$. Calculus would be needed to find the value of this expected value sum.

8.1 Example G revisited: A Math 220 class, taught in the Fall of 2010 at UMCP, had the following grade distribution. Define X = grade points (GPs), where an A is 4 GPs, a B is 3 GPs, a C is 2 GPs, a D is 4 GPs, and an F or W is 0 GPs. What is the grade point average (expected value) for this class?



x	4	3	2	1	0	
$P(X = x)$	0.443	0.293	0.100	0.042	0.122	total = 1

answer: 2.893

Example H: Insurance companies use actuarial data to set rates for policies. Collected data indicate that, on a \$1000 policy, an average of 1 in every 100 policy holders will file a \$20,000 claim. An average of 1 in every 200 policy holders will file a \$50,000 claim. An average of 1 in every 500 policy holders will file a \$100,000 claim. What is the expected value of a policy to the company?

X = value of a policy					
$P(X = x)$					total = 1

answer: \$350

Example I: In 1953, French economist Maurice Allais studied how people assess risk by giving them two decisions to make.

1) Choose between A = { 100% chance of getting \$1 million } and B = { 10% chance of getting \$2.5 million, 89% chance of getting \$1 million, 1% chance of getting nothing }.
2) Choose between A = { 11% chance of getting \$1 million, 89% chance of getting nothing } and B = {10% chance of getting \$2.5 million, 90% chance of getting nothing}.

Allais found that most people chose A for decision 1) and B for decision 2). Use expected value to determine whether these choices are supported by the numbers.
answers: 1) \$1 million, \$1.14 million; 2) \$110,000, \$250,000

Example J: An airline prices 150 seats according to the following schema. If the aircraft is sold out, what is the expected value to the airline of a ticket?

type	first class	unrestricted coach	restricted coach	frequent flyer
number	20	45	81	4
price	$1200	$750	$320	$0

answer: $557.80

Example K: Five coins are tossed. If 0, 1, or 2 heads come up, the player wins nothing. If 3 heads come up the player wins $2. If 4 heads come up the player wins $5, and if 5 heads come up the player wins $10.
a) What fee charged to the player would mean the house breaks even? b) If the house charges $2 to play and 1000 people play, estimate the house profit.

no. of heads	0	1	2	3	4	5	
X = winnings							
$P(X = x)$							total = 1

answers: ≈ $1.72, $281.50