111, section 8.4 The Binomial Distribution

Notes by Tim Pilachowski

Definition of Bernoulli trials which make up a **binomial** experiment:
The number of trials in an experiment is fixed.
There are exactly two events/outcomes for each trial, usually labeled success and failure.
$P(\text{success}) = p$ must be the same for each trial.
Therefore, $P(\text{failure}) = q = 1 - p$.
Success and failure must be independent from one trial to the next.

Example A-1. Suppose that a box contains 3 blue blocks and 2 yellow blocks. You pick four blocks **without replacement**. Determine whether this is a binomial experiment.

number of trials is fixed?

exactly two events/outcomes for each trial?

$P(\text{success}) = p$ is the same for each trial?

$P(\text{failure}) = q = 1 - p$?

success and failure independent from one trial to the next?

Example A-2. Suppose that a box contains 3 blue blocks and 2 yellow blocks. You pick fifteen blocks ***with replacement***. Determine whether this is a binomial experiment.

number of trials is fixed?

exactly two events/outcomes for each trial?

$P(\text{success}) = p$ is the same for each trial?

$P(\text{failure}) = q = 1 - p$?

success and failure independent from one trial to the next?

Example B-1. You toss two coins. Determine whether this is a binomial experiment.

number of trials is fixed?

exactly two events/outcomes for each trial?

$P(\text{success}) = p$ is the same for each trial?

$P(\text{failure}) = q = 1 - p$?

success and failure independent from one trial to the next?

Example B-2. You toss ten coins. Determine whether this is a binomial experiment.

number of trials is fixed?

exactly two events/outcomes for each trial?

$P(\text{success}) = p$ is the same for each trial?

$P(\text{failure}) = q = 1 - p$?

success and failure independent from one trial to the next?

Example C-1. You toss two standard six-sided dice a hundred times. Define a random variable X = sum of the two dice. Determine whether this is a binomial experiment.

number of trials is fixed?

exactly two events/outcomes for each trial?

P(success) = p is the same for each trial?

P(failure) = $q = 1 - p$?

success and failure independent from one trial to the next?

Example C-2. You toss two standard six-sided dice a hundred times. Define success as "sum of the two dice is an even number". Determine whether this is a binomial experiment.

number of trials is fixed?

exactly two events/outcomes for each trial?

P(success) = p is the same for each trial?

x	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$	$\frac{5}{36}$	$\frac{1}{9}$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$

P(failure) = $q = 1 - p$?

success and failure independent from one trial to the next?

Example D. You deal five cards from a standard deck of 52. Define a random variable X = number of Aces. Determine whether this is a binomial experiment.

number of trials is fixed?

exactly two events/outcomes for each trial?

P(success) = p is the same for each trial?

P(failure) = $q = 1 - p$?

success and failure independent from one trial to the next?

Example E. Let X = the number of days each ICU patient stays in intensive care. Determine whether this is a binomial experiment.

number of trials is fixed?

exactly two events/outcomes for each trial?

P(success) = p is the same for each trial?

P(failure) = $q = 1 - p$?

success and failure independent from one trial to the next?

Example F. Suppose we measure the heights of 25 people to the nearest inch. Determine whether this is a binomial experiment.

number of trials is fixed?

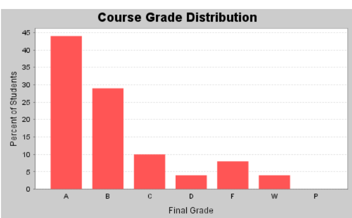
exactly two events/outcomes for each trial?

P(success) = p is the same for each trial?

P(failure) = $q = 1 - p$?

success and failure independent from one trial to the next?

Example G-1. A Math 220 class, taught in the Fall of 2010 at UMCP, had the following grade distribution. Determine whether this is a binomial experiment.



number of trials is fixed?

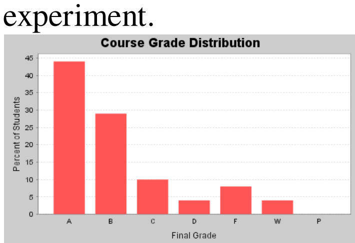
exactly two events/outcomes for each trial?

P(success) = p is the same for each trial?

P(failure) = $q = 1 - p$?

success and failure independent from one trial to the next?

Example G-2. A Math 220 class, taught in the Fall of 2010 at UMCP, had the following grade distribution. Define success as a grade of "C" or better. Determine whether this is a binomial experiment.



number of trials is fixed?

exactly two events/outcomes for each trial?

P(success) = p is the same for each trial?

P(failure) = $q = 1 - p$?

success and failure independent from one trial to the next?

Example A-2 revisited. Suppose that a box contains 3 blue blocks and 2 yellow blocks. You pick four blocks with replacement. Define success as "picking a blue block". Develop the probability distribution table.

$n =$ $p =$ $q =$

possible outcomes:

Let random variable X = number of successes, i.e. number of blues picked.

x					
$P(X = x)$					

For a binomial distribution – n trials, $P(\text{success}) = p$, and x = number of successes = 0, 1, 2, … , n

$$P(X = x) = C(n, x)\, p^x q^{n-x} = \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

Example A-2 revisited. Suppose that a box contains 3 blue blocks and 2 yellow blocks. You pick four blocks with replacement. Define success as "picking a blue block". X = number of successes

$n =$ $p =$ $q =$

$P(X = 0) =$

$P(X = 1) =$

$P(X = 2) =$

$P(X = 3) =$

$P(X = 4) =$

Example A-2 yet again. Suppose that a box contains 3 blue blocks and 2 yellow blocks. You pick four blocks with replacement. Define success as "picking a blue block". X = number of successes

x	0	1	2	3	4
$P(X = x)$ decimal					

Appendix D, Table 1, in your text has binomial distribution tables which provide approximate probabilities for various values of n, p and x. If the numbers you need happen to be there, you can use the tables instead of doing the calculations by hand.

If we were to go to the tables, which one would we need?

From the binomial probability distribution table:

Example A-2 variation. Suppose that a box contains 3 blue blocks and 2 yellow blocks. You pick fifteen blocks with replacement. Define success as "picking a blue block". X = number of successes
What is the probability of picking at least 10 blue blocks?

$n =$ $p =$ $q =$

The calculation would look like this:

$$P(X \geq 10) = P(X = 10) + P(X = 11) + P(X = 12) + P(X = 13) + P(X = 14) + P(X = 15)$$
$$= \frac{15!}{10!5!}0.6^{10}0.4^{5} + \frac{15!}{11!4!}0.6^{11}0.4^{4} + \frac{15!}{12!3!}0.6^{12}0.4^{3} + \frac{15!}{13!2!}0.6^{13}0.4^{2} + \frac{15!}{14!1!}0.6^{14}0.4^{1} + \frac{15!}{15!0!}0.6^{15}0.4^{0}$$
$$\approx 0.40321555041484$$

Which binomial probability table would we use?

From the binomial probability distribution table:

Example B-2 revisited. You toss a coin ten times. Define success as "coin is heads".

$n =$ $p =$ $q =$

a) What is the probability of tossing all heads?

calculation:

from the binomial probability distribution table:

b) What is the probability of tossing exactly three heads?

calculation:

from the binomial probability distribution table:

c) What is the probability of tossing at most three heads?

calculation:

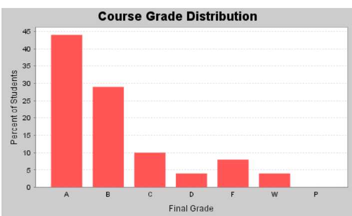
from the binomial probability distribution table:

Example C-2 revisited. You toss two standard six-sided dice a hundred times. Define success as "sum of the two dice is an even number".

$n =$ $p =$ $q =$

Why can't we use the tables from the text?

What is the probability that at most 95 of the tosses result in even sums?



Example G-2 revisited. A Math 220 class, taught in the Fall of 2010 at UMCP, had the following grade distribution. Define success as a grade of "C" or better. Pick 15 students at random (*with replacement*). What is the probability that between 11 and 14 have a grade of C or better?

$n =$ $p =$

$q =$

calculations:

final notes for lecture 8.4, part 2:

The expected value (mean) of a binomial probability distribution is a simple formula:

$$E(X) = np.$$

It is reasonable to expect that a previously-observed proportion p will still hold for any sample of size n.

Using some extended algebra we can derive a formula for variance of a binomial probability distribution:

$$\text{Var}(X) = npq = np(1 - p).$$

Then standard deviation is, as before, the square root of variance:

$$\sigma_x = \sqrt{npq} = \sqrt{np(1-p)}.$$

Example A-2 yet again. Suppose that a box contains 3 blue blocks and 2 yellow blocks. You pick four blocks with replacement. Define success as "picking a blue block". X = number of successes

What is the expected number of blue blocks (i.e. the mean)?

What are the variance and standard deviation?

Example A-2 variation. Suppose that a box contains 3 blue blocks and 2 yellow blocks. You pick fifteen blocks with replacement. Define success as "picking a blue block". X = number of successes

What is the expected number of blue blocks (i.e. the mean)?

What are the variance and standard deviation?

Example B-2 revisited. You toss a coin ten times. Define success as "coin is heads".

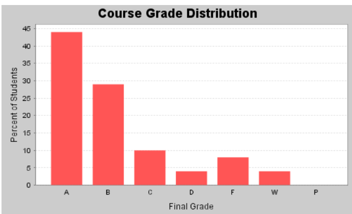
What is the expected number of heads (i.e. the mean)?

What are the variance and standard deviation?

Example C-2 revisited. You toss two standard six-sided dice a hundred times. Define success as "sum of the two dice is an even number".

What is the expected number of even sums (i.e. the mean)?

What are the variance and standard deviation?



Example G-2 revisited. A Math 220 class, taught in the Fall of 2010 at UMCP, had the following grade distribution. Define success as a grade of "C" or better. Pick 15 students at random (*with replacement*).

What is the expected number who have a C or better?

What are the variance and standard deviation?

In most of the Examples above the probability p was theoretical. In Example G-2 we used an empirical observation based on prior experience.

Example H. From prior experience and testing, Shockingly Good, Inc. has determined that 2 out of every 90 spark plugs produced is defective. The company picks 20 spark plugs at random (*with replacement*) from the production line. Define random variable X = number of good spark plugs.

$p =$ $n =$

a) What is the probability that exactly 1 is defective?

b) What is the probability that at most 1 is defective?

c) What is the probability that at least 2 are defective?

d) When 1800 spark plugs are produced, what is the expected number of number of good spark plugs?

e) When 1800 spark plugs are produced, what are the variance and standard deviation for X = number of good spark plugs?

Appendix D, text pages 588-590 Table 1 Binomial Probabilities

n	*x*	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
3	0	0.857	0.729	0.512	0.343	0.216	0.125	0.064	0.027	0.008	0.001	
	1	0.135	0.243	0.384	0.441	0.432	0.375	0.288	0.189	0.096	0.027	0.007
	2	0.007	0.027	0.096	0.189	0.288	0.375	0.432	0.441	0.384	0.243	0.135
	3		0.001	0.008	0.027	0.064	0.125	0.216	0.343	0.512	0.729	0.857
4	0	0.815	0.656	0.410	0.240	0.130	0.062	0.026	0.008	0.002		
	1	0.171	0.292	0.410	0.412	0.346	0.250	0.154	0.076	0.026	0.004	
	2	0.014	0.049	0.154	0.265	0.346	0.375	0.346	0.265	0.154	0.049	0.014
	3		0.004	0.026	0.076	0.154	0.250	0.346	0.412	0.410	0.292	0.171
	4			0.002	0.008	0.026	0.062	0.130	0.240	0.410	0.656	0.815
5	0	0.774	0.590	0.328	0.168	0.078	0.031	0.010	0.002			
	1	0.204	0.328	0.410	0.360	0.259	0.156	0.077	0.028	0.006		
	2	0.021	0.073	0.205	0.309	0.346	0.312	0.230	0.132	0.051	0.008	0.001
	3	0.001	0.008	0.051	0.132	0.230	0.312	0.346	0.309	0.205	0.073	0.021
	4			0.006	0.028	0.077	0.156	0.259	0.360	0.410	0.328	0.204
	5				0.002	0.010	0.031	0.078	0.168	0.328	0.590	0.774
6	0	0.735	0.531	0.262	0.118	0.047	0.016	0.004	0.001			
	1	0.232	0.354	0.393	0.303	0.187	0.094	0.037	0.010	0.002		
	2	0.031	0.098	0.246	0.324	0.311	0.234	0.138	0.060	0.015	0.001	
	3	0.002	0.015	0.082	0.185	0.276	0.312	0.276	0.185	0.082	0.015	0.002
	4		0.001	0.015	0.060	0.138	0.234	0.311	0.324	0.246	0.098	0.031
	5			0.002	0.010	0.037	0.094	0.187	0.303	0.393	0.354	0.232
	6			0.000	0.001	0.004	0.016	0.047	0.118	0.262	0.531	0.735
7	0	0.698	0.478	0.210	0.082	0.028	0.008	0.002				
	1	0.257	0.372	0.367	0.247	0.131	0.055	0.017	0.004			
	2	0.041	0.124	0.275	0.318	0.261	0.164	0.077	0.025	0.004		
	3	0.004	0.023	0.115	0.227	0.290	0.273	0.194	0.097	0.029	0.003	
	4		0.003	0.029	0.097	0.194	0.273	0.290	0.227	0.115	0.023	0.004
	5			0.004	0.025	0.077	0.164	0.261	0.318	0.275	0.124	0.041
	6				0.004	0.017	0.055	0.131	0.247	0.367	0.372	0.257
	7					0.002	0.008	0.028	0.082	0.210	0.478	0.698
8	0	0.663	0.430	0.168	0.058	0.017	0.004	0.001				
	1	0.279	0.383	0.336	0.198	0.090	0.031	0.008	0.001			
	2	0.051	0.149	0.294	0.296	0.209	0.109	0.041	0.010	0.001		
	3	0.005	0.033	0.147	0.254	0.279	0.219	0.124	0.047	0.009		
	4		0.005	0.046	0.136	0.232	0.273	0.232	0.136	0.046	0.005	
	5			0.009	0.047	0.124	0.219	0.279	0.254	0.147	0.033	0.005
	6			0.001	0.010	0.041	0.109	0.209	0.296	0.294	0.149	0.051
	7				0.001	0.008	0.031	0.090	0.198	0.336	0.383	0.279
	8					0.001	0.004	0.017	0.058	0.168	0.430	0.663
9	0	0.630	0.387	0.134	0.040	0.010	0.002					
	1	0.299	0.387	0.302	0.156	0.060	0.018	0.004				
	2	0.063	0.172	0.302	0.267	0.161	0.070	0.021	0.004			
	3	0.008	0.045	0.176	0.267	0.251	0.164	0.074	0.021	0.003		
	4	0.001	0.007	0.066	0.172	0.251	0.246	0.167	0.074	0.017	0.001	
	5		0.001	0.017	0.074	0.167	0.246	0.251	0.172	0.066	0.007	0.001
	6			0.003	0.021	0.074	0.164	0.251	0.267	0.176	0.045	0.008
	7				0.004	0.021	0.070	0.161	0.267	0.302	0.172	0.063
	8					0.004	0.018	0.060	0.156	0.302	0.387	0.299
	9						0.002	0.010	0.040	0.134	0.387	0.630
10	0	0.599	0.349	0.107	0.028	0.006	0.001					
	1	0.315	0.387	0.268	0.121	0.040	0.010	0.002				
	2	0.075	0.194	0.302	0.233	0.121	0.044	0.011	0.001			
	3	0.010	0.057	0.201	0.267	0.215	0.117	0.042	0.009	0.001		
	4	0.001	0.011	0.088	0.200	0.251	0.205	0.111	0.037	0.006		
	5		0.001	0.026	0.103	0.201	0.246	0.201	0.103	0.026	0.001	
	6			0.006	0.037	0.111	0.205	0.251	0.200	0.088	0.011	0.001
	7			0.001	0.009	0.042	0.117	0.215	0.267	0.201	0.057	0.010
	8				0.001	0.011	0.044	0.121	0.233	0.302	0.194	0.075
	9					0.002	0.010	0.040	0.121	0.268	0.387	0.315
	10						0.001	0.006	0.028	0.107	0.349	0.599

Appendix D, text pages 588-590 Table 1 Binomial Probabilities (continued)

n	***x***	**0.05**	**0.1**	**0.2**	**0.3**	**0.4**	**0.5**	**0.6**	**0.7**	**0.8**	**0.9**	**0.95**
11	0	0.569	0.314	0.086	0.020	0.004						
	1	0.329	0.384	0.236	0.093	0.027	0.005	0.001				
	2	0.087	0.213	0.295	0.200	0.089	0.027	0.005	0.001			
	3	0.014	0.071	0.221	0.257	0.177	0.081	0.023	0.004			
	4	0.001	0.016	0.111	0.220	0.236	0.161	0.070	0.017	0.002		
	5		0.002	0.039	0.132	0.221	0.226	0.147	0.057	0.010		
	6			0.010	0.057	0.147	0.226	0.221	0.132	0.039	0.002	
	7			0.002	0.017	0.070	0.161	0.236	0.220	0.111	0.016	0.001
	8				0.004	0.023	0.081	0.177	0.257	0.221	0.071	0.014
	9				0.001	0.005	0.027	0.089	0.200	0.295	0.213	0.087
	10					0.001	0.005	0.027	0.093	0.236	0.384	0.329
	11							0.004	0.020	0.086	0.314	0.569
12	0	0.540	0.282	0.069	0.014	0.002						
	1	0.341	0.377	0.206	0.071	0.017	0.003					
	2	0.099	0.230	0.283	0.168	0.064	0.016	0.002				
	3	0.017	0.085	0.236	0.240	0.142	0.054	0.012	0.001			
	4	0.002	0.021	0.133	0.231	0.213	0.121	0.042	0.008	0.001		
	5		0.004	0.053	0.158	0.227	0.193	0.101	0.029	0.003		
	6			0.016	0.079	0.177	0.226	0.177	0.079	0.016		
	7			0.003	0.029	0.101	0.193	0.227	0.158	0.053	0.004	
	8			0.001	0.008	0.042	0.121	0.213	0.231	0.133	0.021	0.002
	9				0.001	0.012	0.054	0.142	0.240	0.236	0.085	0.017
	10					0.002	0.016	0.064	0.168	0.283	0.230	0.099
	11						0.003	0.017	0.071	0.206	0.377	0.341
	12							0.002	0.014	0.069	0.282	0.540
13	0	0.513	0.254	0.055	0.010	0.001						
	1	0.351	0.367	0.179	0.054	0.011	0.002					
	2	0.111	0.245	0.268	0.139	0.045	0.010	0.001				
	3	0.021	0.100	0.246	0.218	0.111	0.035	0.006	0.001			
	4	0.003	0.028	0.154	0.234	0.184	0.087	0.024	0.003			
	5		0.006	0.069	0.180	0.221	0.157	0.066	0.014	0.001		
	6		0.001	0.023	0.103	0.197	0.209	0.131	0.044	0.006		
	7			0.006	0.044	0.131	0.209	0.197	0.103	0.023	0.001	
	8			0.001	0.014	0.066	0.157	0.221	0.180	0.069	0.006	
	9				0.003	0.024	0.087	0.184	0.234	0.154	0.028	0.003
	10				0.001	0.006	0.035	0.111	0.218	0.246	0.100	0.021
	11					0.001	0.010	0.045	0.139	0.268	0.245	0.111
	12						0.002	0.011	0.054	0.179	0.367	0.351
	13							0.001	0.010	0.055	0.254	0.513
14	0	0.488	0.229	0.044	0.007	0.001						
	1	0.359	0.356	0.154	0.041	0.007	0.001					
	2	0.123	0.257	0.250	0.113	0.032	0.006	0.001				
	3	0.026	0.114	0.250	0.194	0.085	0.022	0.003				
	4	0.004	0.035	0.172	0.229	0.155	0.061	0.014	0.001			
	5		0.008	0.086	0.196	0.207	0.122	0.041	0.007			
	6		0.001	0.032	0.126	0.207	0.183	0.092	0.023	0.002		
	7			0.009	0.062	0.157	0.209	0.157	0.062	0.009		
	8			0.002	0.023	0.092	0.183	0.207	0.126	0.032	0.001	
	9				0.007	0.041	0.122	0.207	0.196	0.086	0.008	
	10				0.001	0.014	0.061	0.155	0.229	0.172	0.035	0.004
	11					0.003	0.022	0.085	0.194	0.250	0.114	0.026
	12					0.001	0.006	0.032	0.113	0.250	0.257	0.123
	13						0.001	0.007	0.041	0.154	0.356	0.359
	14							0.001	0.007	0.044	0.229	0.488
15	0	0.463	0.206	0.035	0.005							
	1	0.366	0.343	0.132	0.031	0.005						
	2	0.135	0.267	0.231	0.092	0.022	0.003					
	3	0.031	0.129	0.250	0.170	0.063	0.014	0.002				
	4	0.005	0.043	0.188	0.219	0.127	0.042	0.007	0.001			
	5	0.001	0.010	0.103	0.206	0.186	0.092	0.024	0.003			
	6		0.002	0.043	0.147	0.207	0.153	0.061	0.012	0.001		
	7			0.014	0.081	0.177	0.196	0.118	0.035	0.003		
	8			0.003	0.035	0.118	0.196	0.177	0.081	0.014		
	9			0.001	0.012	0.061	0.153	0.207	0.147	0.043	0.002	
	10				0.003	0.024	0.092	0.186	0.206	0.103	0.010	0.001
	11				0.001	0.007	0.042	0.127	0.219	0.188	0.043	0.005
	12					0.002	0.014	0.063	0.170	0.250	0.129	0.031
	13						0.003	0.022	0.092	0.231	0.267	0.135
	14							0.005	0.031	0.132	0.343	0.366
	15								0.005	0.035	0.206	0.463