111, section 8.5 The Normal Distribution

notes by Tim Pilachowski

Suppose we measure the heights of 25 people to the nearest inch and get the following results:

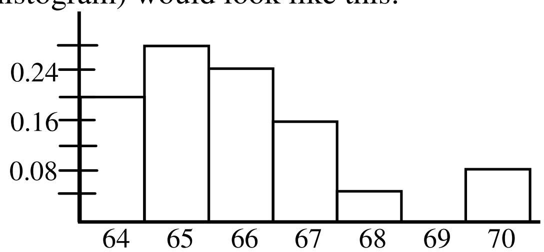
height (in.)	64	65	66	67	68	69	70
frequency	5	7	6	4	1	0	2
probability	0.2	0.28	0.24	0.16	0.04	0	0.08

We can use probabilities from the table to determine probabilities for specific intervals.

$P(64 \leq \text{height} \leq 66) =$

$P(66 \leq \text{height} \leq 69) =$

The probability distribution graph (histogram) would look like this:



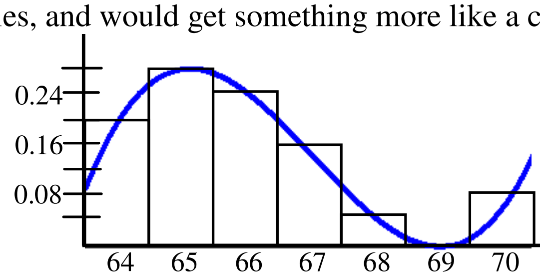
(probability as area: area of each bar = height times width = probability times 1 = percentage of people having that height = relative frequency of that height = probability of a person having that height)
Recall that the sum of the areas of the rectangles, which is the sum of the probabilities, equals 1.

We can use the areas of the rectangles to determine probabilities for intervals.

$P(64 \leq \text{height} \leq 66) =$

$P(66 \leq \text{height} \leq 69) =$

If we were to measure to the nearest half-inch, or tenth of an inch, or hundredth, or thousandth, etc., etc., etc., we'd get ever-more-narrow rectangles, and would get something more like a curve:



The area under the curve for a given interval would be the probability of people having heights within that interval. This is an example of a **continuous probability distribution** (as opposed to the discrete probability distributions we've encountered so far.

And so we come to a definition—a *probability density function* $f(x)$ for a continuous random variable has two necessary characteristics.

1. $f(x) \geq 0$ for all values of x in its domain [since all probabilities and therefore "areas under the curve" are zero or positive]
2. The area under the curve over the entire domain = 1 [since the sum of all probabilities = 1]

In Math 131 and 221 Calculus II we actually evaluate area under the curve for a probability density function and prove that $\int_A^B f(x)\,dx = 1$.

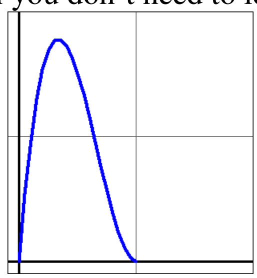
Just like the height example given earlier, the graph of a probability density function has a very useful characteristic: area under the probability density curve between a and b = $P[a \leq X \leq b]$.

Note that we are finding probabilities for *intervals* as opposed to finding probabilities for specific values x. This isn't to say that we could never have a probability equal to zero, but rather the probability for that one specific value is so small that it is negligible. Parallels in life include:

– A meteorologist will predict rain for the afternoon, not rain for 2:07 pm.
– The bullseye on a dartboard is a space, not an infinitesimally small point.
– A person growing from 65 to 67 inches will at some time be exactly 66 inches, but we have no way to measure with enough accuracy to specify exactly when it happens.

In the height example, although we might state "P(height is 66 inches) = 0.24",
we are actually referring to an interval of rounded heights, and are saying, "$P(65.5 \le \text{height} < 66.5) = 0.24$".

Lucky for you— expected value and variance for the *probability density function* $f(x)$ for a continuous random variable often requires integral calculus which you don't need to learn.

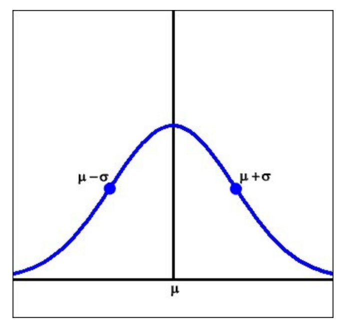


$f(x) = 12x(1-x)^2$, $0 < x < 1$

An often-useful probability density function is the normal density function, which graphs as the familiar bell-shaped curve. The generic format is

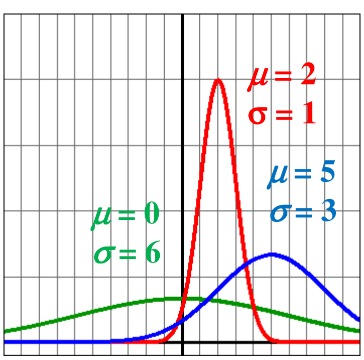
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2},$$

where $E(X) = \mu$, $\text{Var}(X) = \sigma^2$, and standard deviation $= \sqrt{\text{Var}(X)} = \sigma$.



The graph of a normal curve is symmetric with respect to the line $x = \mu$, and has tails on both the left and the right.

Side note: More than 68% of a normally distributed population will fall within $\pm 1\sigma$ of the mean, more than 95% will fall within $\pm 2\sigma$, and more than 99.5% within $\pm 3\sigma$.



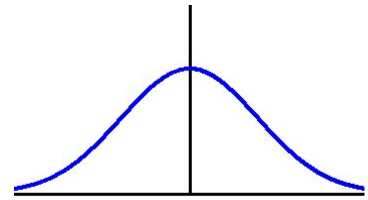
Differences in the means result in shifts left and right. A smaller standard deviation will result in a taller, more narrow "bell". Each curve is symmetric about its own mean. Note that in all three cases, probabilities beyond $\mu \pm 3\sigma$ become so small as to usually be considered insignificant.

If we consider the special case where $E(X) = \mu = 0$ and standard deviation $= \sigma = 1$, we get what is called the standard normal distribution, $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$, with its graph called the standard normal curve. In the science of statistics, where things such as sampling distributions are known to be normally distributed, a random variable X will be converted to a standard variable Z using the formula $Z = \frac{X-\mu}{\sigma}$.

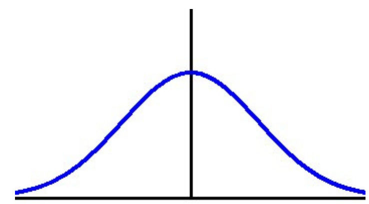
There is no method for integrating to find area under the curve (and thus probabilities) of this standard normal probability density function, but a calculation using a Taylor polynomial (covered in Math 221) has been done to construct tables of values such as the one found in the Appendix of your text (Table 3). For a standardized random variable Z, this text's normal distribution table gives us (area under the curve from "forever left" to z = $P(-\infty < Z \le$ specified value.

In an Excel spreadsheet, the function =NORMSDIST(z) also gives area under the standard normal curve = probability for the interval from $-\infty$ to z. Some, but not all handheld calculators work the same way.

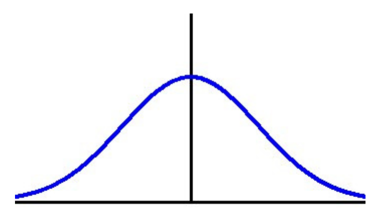
Example A: Given a standard normal random variable Z, find $P(Z \le 1.23)$.
answer: 0.8907



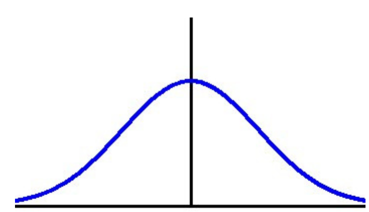
Example B: Given a standard normal random variable Z, find $P(0 \le Z \le 1.23)$.
answer: 0.3907



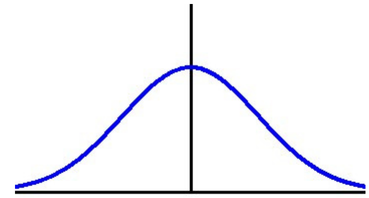
Example C: Given a standard normal random variable Z, find $P(-1.23 \le Z \le 0)$.
answer: 0.3907



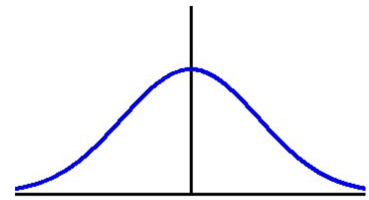
Example D: Given a standard normal random variable Z, find $P(-2.14 \le Z \le 1.23)$.
answer: 0.8745



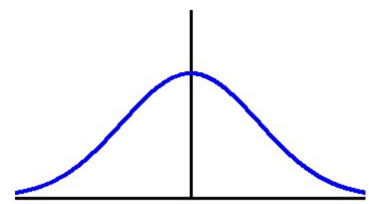
Example E: Given a standard normal random variable Z, find $P(Z \le 1.70)$.
answer: 0.9554



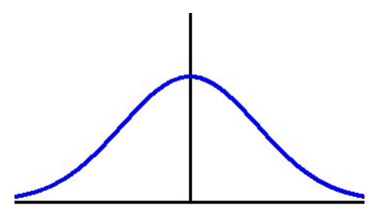
Example F: Given a standard normal random variable Z, find $P(0.14 \le Z \le 2.28)$.
answer: 0.4330

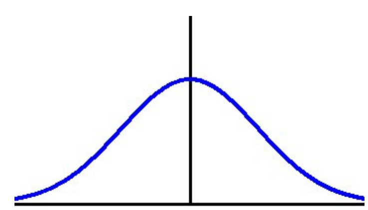


Example G: Given a standard normal random variable Z, find $P(Z \ge 2.33)$.
answer: 0.0099

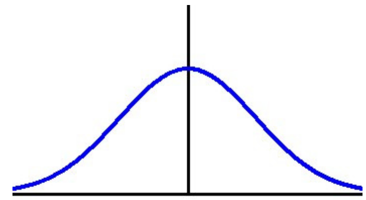


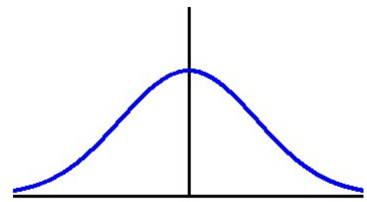
Example H: Let Z be the standard normal variable. Find the values of z that satisfy a) $P(Z < z) = 0.1685$
b) $P(Z < z) = 0.9838$. *answers*: -0.96, 2.14



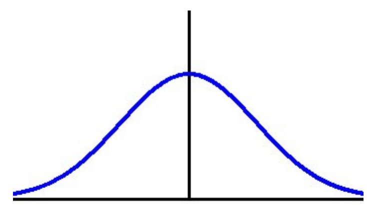


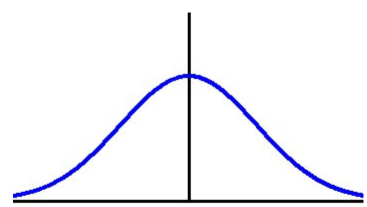
Example I: Let Z be the standard normal variable. Find the values of z that satisfy a) $P(Z > z) = 0.1515$ b) $P(Z > -z) = 0.7157$.
answers: 1.03, 0.57





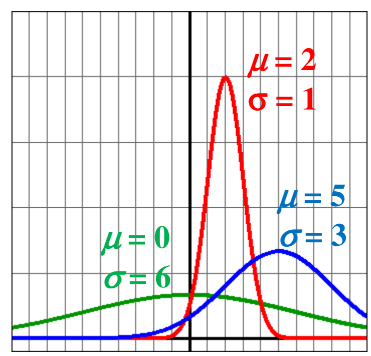
Example J: Given a standard normal random variable Z, find a) the value z that marks the lowest 10%, b) the value z that marks the top 5%.
answers: a) -1.28, b) 1.645





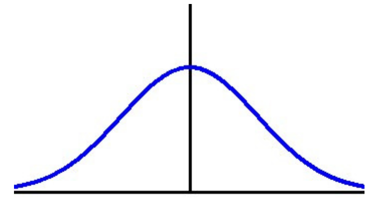
In all the the previous Examples, we worked with various values for the standardized normal random variable Z and were able to get our answers by going directly to the normal distribution table.

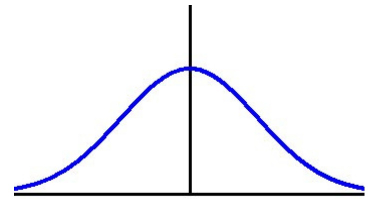
Rather than approximate values for *every possible* normal density function, the common practice is to convert everything to a standard normal distribution and use the same normal distribution table over and over.

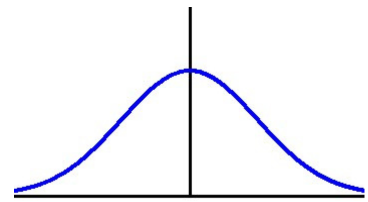


The z-score formula for converting a normal random variable X into the standardized normal random variable Z is $Z = \frac{X - \mu}{\sigma}$.

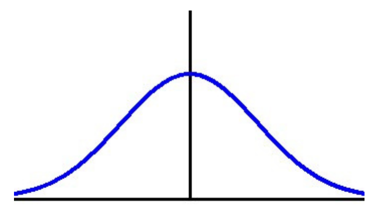
Example K: If X has a normal distribution with $\mu = 120$ and $\sigma = 35$, find a) $P(X < 100)$, b) $P(100 < X < 157.63)$, c) $P(X > 157.63)$
answers: 0.2843, 0.5756, 0.1401

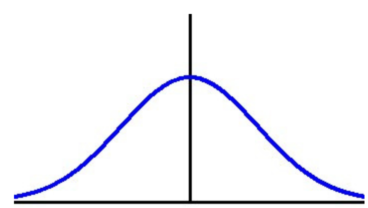






Example L: If X has a normal distribution with $\mu = 120$ and $\sigma = 35$, find b such that a) $P(X < b) = 0.3015$, b) $P(X > b) = 0.0322$
answers: 101.8, 184.75





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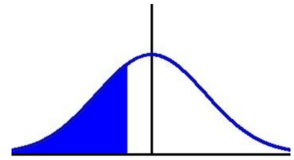

Table 2 Area Under a Standard Normal Curve to the Left of z ($z < 0$)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
-3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
-3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
-3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
-3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
-2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
-2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
-2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
-2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
-2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
-2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
-2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
-1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
-0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
-0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
-0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641

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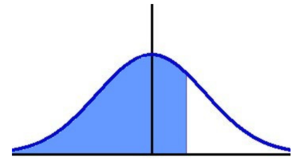
Table 2 Area Under a Standard Normal Curve to the Left of z ($z > 0$)



z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998