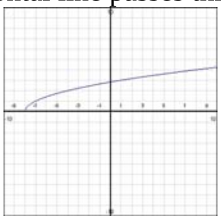
Math 113 Chapter 2.7 Examples and Extra Notes

Inverse of a function

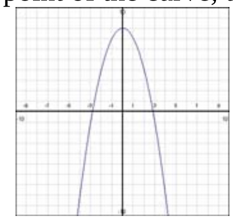
You can tell whether on not a function has an inverse by applying the "horizontal line test" to the graph: If any horizontal line passes through only one point of the curve, then the function has an inverse.



$y = \sqrt{x}$

passes the horizontal line test

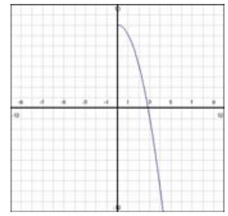
has an inverse



$y = -x^2 + 8$

doesn't pass the horizontal line test

doesn't have an inverse



$y = -x^2 + 8, x \geq 0$

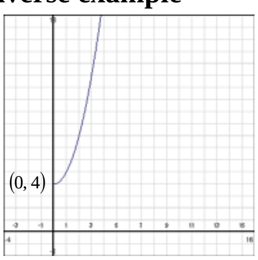
passes the horizontal line test only because domain is restricted

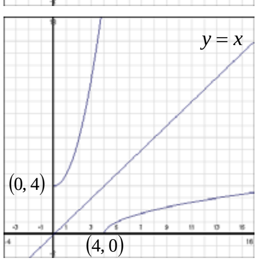
has an inverse

Given a function f and its inverse, f^{-1}, the following will always be true:

1. If $f(a) = b$, then $f^{-1}(b) = a$. (This fact and the statement in point #2 below is actually the same information.)
2. If (a, b) is a point on the graph of f, then (b, a) will be on the graph of f^{-1}. You can use this fact to use a graph to sketch its inverse: locate two or three points on the graph of *f*, swap coordinates, plot the new points, connect the dots, and you have the graph of f^{-1}. (See example below.)
3. The graph of *f* and the graph of f^{-1} are symmetric with respect to the line $y = x$.
4. The domain of *f* = the range of f^{-1}, and the range of *f* = the domain of f^{-1}.
5. $f \circ f^{-1} = x$ and $f^{-1} \circ f = x$. To show that two functions are inverses, you must do *both* compositions.

Inverse example





$f(x) = x^2 + 4,\ x \geq 0$, passes the horizontal line test (but only because the domain is restricted) so it has an inverse. Using #1 and #2 from above, since the points $(0, 4)$ and $(2, 8)$ are on the graph of *f*, then $(4, 0)$ and $(8, 2)$ will be on the graph of $f^{-1}(x)$.

The graph of f^{-1} has been added in the second picture to the left. Item #3 above is verified, since the two graphs are symmetric across the line $y = x$.

The domain of *f* has been restricted to $[0, \infty)$. From the graph we can tell that its range is $[4, \infty)$. Likewise, using the graph of f^{-1} we can see that its domain is $[4, \infty)$, while its range is $[0, \infty)$. That is, the domain of *f* = the range of f^{-1}, and the range of *f* = the domain of f^{-1} (#4 above).

Using what we know about transformations of graphs, it appears that the equation of f^{-1} is $f^{-1}(x) = \sqrt{x-4}$. We can verify this using item #5:

$$f \circ f^{-1} = f\left(\sqrt{x-4}\right) = \left(\sqrt{x-4}\right)^2 + 4 = x - 4 + 4 = x \qquad f^{-1} \circ f = f^{-1}\left(x^2+4\right) = \sqrt{\left(x^2+4\right) - 4} = \sqrt{x^2} = |x| = x$$

Important note: Because the domain of *f*, and therefore $f^{-1} \circ f$, is restricted to positive numbers, and absolute value of a positive number is the number itself, for *these two functions* the composition $f \circ f^{-1}$ will *always* equal *x*.