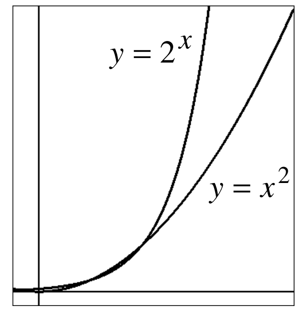
Calculus 130, section 2.1-2.2 Exponential and Logarithmic Functions

notes by Tim Pilachowski

Take a piece of paper, and fold it in half. You've doubled the number of layers—from 1 to 2. Fold it in half again, and you've once again doubled the layers—from 2 to 4. After the next fold you'd have 8 layers. After the next, 16. Then 32. Then 64, 128, 256, 512, 1024, etc. After just 10 foldings, we have the paperback edition of just the first book of *Lord of the Rings* that we're attempting to fold in half.



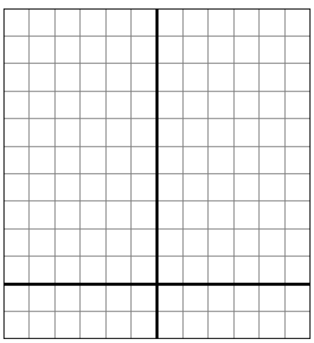
When x is the base, we have a *power function*. When x is the exponent we have an *exponential function*. The scenario above illustrates the exponential function $y = 2^x$.

If we compare the graph of $y = x^2$ to the graph of $y = 2^x$, we can see that for positive values of x the exponential function grows much more quickly than the power function. (Thus our difficulty in folding a piece of paper in half successive times.) Eventually, we'll be interested in finding a way to describe the slope of an exponential graph, i.e. a way to find its derivative.

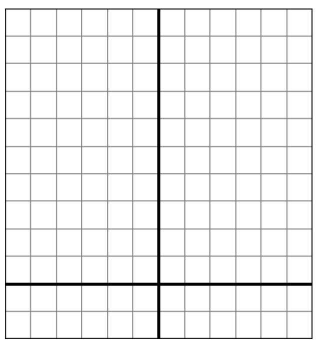
Exponential functions have many applications because they model many kinds of growth and shrinking: e.g. populations, bank deposits, radioactive decay.

Examples A: Sketch the graphs of the following functions, using translations and shifts.

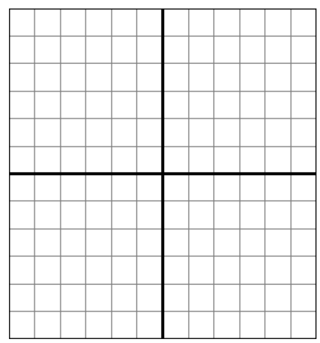
$y = 2^{x-1}$



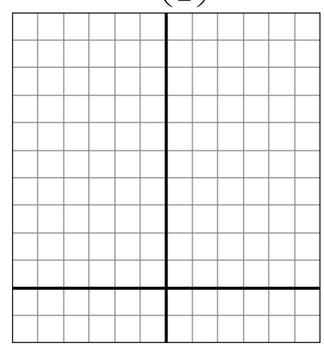
$y = 2^x - 1$



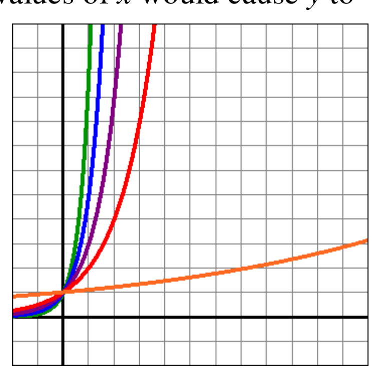
$y = -2^x$



$y = \left(\frac{1}{2}\right)^x$



Functions with the basic form $y = b^x$ are actually a family of functions. We'll consider only values for b that are positive. (Negative values of b are extremely problematic, since even and odd values of x would cause y to fluctuate between positive and negative.)



Consider the functions $y = 10^x$, $y = 5^x$, $y = 3^x$, $y = 2^x$, $y = 1.1^x$ pictured in the graph to the right. Note first the similarities: $b^0 = 1$ for all values of $b \neq 0$, so (0, 1) makes a good reference point. Each of the basic exponential functions has a horizontal asymptote $y = 0$. The graphs also have similar shape—the major difference is slope of the curve at specific values of x. Note that at $x = 0$ slope of $y = 10^x$ is steepest; slope of $y = 1.1^x$ is most shallow.

All of the usual properties of exponents apply to exponential functions:

$$b^x * b^y = b^{x+y} \qquad \frac{b^x}{b^y} = b^{x-y} \qquad \frac{1}{b^y} = b^{0-y} = b^{-y} \qquad \left(b^x\right)^y = b^{xy} \qquad a^x * b^x = (ab)^x \qquad \frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x.$$

Examples B: We can use these properties to simplify expressions and solve equations.

Simplify $2^{x-1} * 8^{x+3}$. *Answer*: $2^{4x+8} = 2^8 * 2^{4x}$

Simplify $\frac{2^{x-1}}{8^{x+3}}$. *Answer*: $2^{-2x-10} = \frac{1}{2^{10}} * 2^{-2x}$

Solve $2^{x^2+3} = 16$. *Answer*: $x = \pm 1$

Solve $\left(\sqrt{3}\right)^x = \frac{1}{27}$. *Answer*: $x = -6$

Solve $\frac{1}{2} = 1 - 2^{x+1}$. *Answer*: $x = -2$

Example C: You deposit \$100 into a certificate of deposit which pays 5% each year on the balance current at the time. Find an equation to describe the growth of your money.
Define $A(t)$ = amount of money accumulated after t years. The table below summarizes the growth over 5 years.

T	interest earned	A = accumulated amount
	percent * [current balance]	{current balance} + {percent * [current balance] } = [current balance] * (1 + percent)
0	0	{100}
1	0.05 * [100]	{100} +{ 0.05[100] } = 100(1 + 0.05)
2	0.05 * [100(1 + 0.05)]	{100(1 + 0.05) } + {0.05[100(1 + 0.05)] } = [100(1 + 0.05)] * (1 + 0.05) = $100(1 + 0.05)^2$
3	0.05 * [$100(1 + 0.05)^2$]	{$100(1 + 0.05)^2$} + {0.05[$100(1 + 0.05)^2$] } = [$100(1 + 0.05)^2$] * (1 + 0.05) = $100(1 + 0.05)^3$
4	0.05 * [$100(1 + 0.05)^3$]	{$100(1 + 0.05)^3$} + {0.05[$100(1 + 0.05)^3$] } = [$100(1 + 0.05)^3$] * (1 + 0.05) = $100(1 + 0.05)^4$
5	0.05 * [$100(1 + 0.05)^4$]	{$100(1 + 0.05)^4$} + {0.05[$100(1 + 0.05)^4$] } = [$100(1 + 0.05)^4$] * (1 + 0.05) = $100(1 + 0.05)^5$

We can use the pattern to state a general formula for interest added annually for n years:

N	0.05 * [$100(1 + 0.05)^{n-1}$]	{$100(1 + 0.05)^{n-1}$} + {0.05[$100(1 + 0.05)^{n-1}$] } = [$100(1 + 0.05)^{n-1}$] * (1 + 0.05) = $100(1 + 0.05)^n$

So, If we left our money there for 10 years (t = 10) we'd have $A(10) = 100(1+0.05)^{10} \cong \$162.89$.
If the interest was compounded quarterly, the 5% annual rate would be divided up among the four quarters, and the number of interest calculations would be n = 4(10): $A(10) = 100\left(1+\frac{0.05}{4}\right)^{4(10)} \cong \$164.36$.

For interest compounded monthly, we'd have: $A(10) = 100\left(1+\frac{0.05}{12}\right)^{12(10)} \cong \$164.70$.

For interest compounded daily, we'd have: $A(10) = 100\left(1+\frac{0.05}{365}\right)^{365(10)} \cong \$164.87$.

For different principals, P, rates of interest, r, compounding periods, m, and numbers of years, t, we can generalize: $A(P, r, m, t) = P\left(1+\frac{r}{m}\right)^{mt}$.

Using this formula we could recalculate our balance compounding every hour, second, or fraction of a second. What happens if we increase the number of times interest is calculated and approach infinity?

An "alternate" way of defining e is $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$.

If we now take our generalized formula, and replace $\frac{r}{m} = \frac{1}{n} \Rightarrow nr = m$, and take $\lim_{n\to\infty}$ we have

$$A(P, r, t) = \lim_{n\to\infty} P\left(1+\frac{1}{n}\right)^{nrt} = P\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]^{rt} = Pe^{rt}.$$

This "continuous compounding" formula for money also models some types of biological growth. We'll investigate applications in section 2.3.

Example C again: For an initial balance of \$100 and an annual interest rate of 5% compounded continuously over 10 years, calculate the closing balance rounded to the nearest penny. *Answer*: \$164.87

If you are calculating interest on a hand calculator, the continuous compounding formula is much easier to use. If you are working at a bank or investment firm, doing a massive number of this type of calculation, the continuous compounding formula uses a lot less computer time and memory.

The number e is Euler's number. Like π or $\sqrt{2}$, e is an irrational number. The value of e is *approximately* 2.717. The corresponding function, $y = e^x$, is called the natural exponential function.

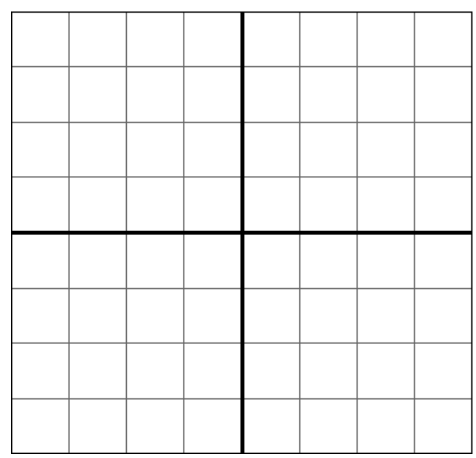
As a side note, this is only one way to define e and approximate its value. There are other definitions, including

$$e = \sum_{n=0}^{\infty} \frac{1}{n!}.$$

(Here begins section 2.2)

Given a function f and its inverse, f^{-1}, the following will always be true:

1. If $f(a) = b$, then $f^{-1}(b) = a$ (This fact and the statement in point #2 below is actually the same information.)
2. If (a, b) is a point on the graph of f, then (b, a) will be on the graph of f^{-1}.
3. The domain of f = the range of f^{-1}, and the range of f = the domain of f^{-1}.
4. $f \circ f^{-1} = x$ and $f^{-1} \circ f = x$. To show that two functions are inverses you must do *both* compositions.
5. The graph of f and the graph of f^{-1} are symmetric with respect to the line $y = x$.



A *logarithm* function is a constructed inverse for an exponential function. The *natural logarithm* function, $y = \ln(x)$, is the inverse of the natural exponential function, $f(x) = e^x$. Applying the above:

1. For one example, $f(0) = e^0 = 1$ and $f^{-1}(1) = \ln(1) = 0$.
2. For one example, $(0, 1)$ is on the graph of $f(x) = e^x$ and $(1, 0)$ is on the graph of $f^{-1}(x) = \ln(x)$.
3. The domain of $f(x) = e^x$ is the range of $f^{-1}(x) = \ln(x)$: $-\infty < x < \infty$.

   The range of $f(x) = e^x$ is the domain of $f^{-1}(x) = \ln(x)$: $0 < x < \infty$.

   Note also that while the graph of $f(x) = e^x$ has a horizontal asymptote at $y = 0$,

   the graph of $f^{-1}(x) = \ln(x)$ has a vertical asymptote at $x = 0$.
4. $f \circ f^{-1} = e^{\ln x} = x$ and $f^{-1} \circ f = \ln\left(e^x\right) = x$.
5. The graphs of $f(x) = e^x$ and $f^{-1}(x) = \ln(x)$ are symmetric with respect to the line $y = x$.

Use the reference point $\log_b(1) = 0$ and knowledge of the basic shape to graph simple logarithm functions using shifts and translations.

Examples D. Rewrite the following exponentials in logarithm form.

a. $5^x = 125$

b. $13 = e^x$

Examples E. Rewrite the following logarithms in exponential form.

a. $\log_b 36 = 2$

b. $\ln x = 7$

Example F: Simplify $\ln\left(\sqrt[5]{e^3}\right)$. *Answer*: $\frac{3}{5}$

Example G: Simplify $e^{\ln(x+2)}$. *Answer*: $x + 2$

Example H: Simplify $e^{\ln(x)+2}$. *Answer*: xe^2

Example I: Solve $5\ln(x-1)+4=0$. *Answer*: $e^{-4/5}+1$

Example J: Solve $\ln\left(x^2-1\right)=4$. *Answer*: $\pm\sqrt{e^4+1}$

Example K: Solve $5-2e^{-2x}=0$. *Answer*: $-\frac{1}{2}\ln\frac{5}{2}$