Calculus 130, section 2.2-2.3 Logarithmic Properties & Exponential Growth

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Recall the properties of exponential functions:

$$b^x * b^y = b^{x+y} \qquad \frac{b^x}{b^y} = b^{x-y} \qquad \frac{1}{b^y} = b^{0-y} = b^{-y} \qquad \left(b^x\right)^y = b^{xy} \qquad a^x * b^x = (ab)^x \qquad \frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x.$$

Logarithm functions, and in particular the *natural logarithm* function, $y = \ln x$, have properties of their own, related to but not exactly the same as the exponential function properties. While the logarithm properties apply to logarithms in any base, most often we will be rewriting all bases in terms of the natural logarithm. So the logarithm properties are given below as applied to $\ln(x)$. Of the properties listed in the text, two are vital. The others can be easily derived from these or from the definition of logarithms.

$$\ln(xy) = \ln x + \ln y \qquad \ln\left(x^a\right) = a \ln x$$

To prove the first: Note that $e^{\ln(xy)} = xy$. But since $x = e^{\ln x}$ and $y = e^{\ln y}$, we can use the properties of exponents to write $e^{\ln(xy)} = x * y = e^{\ln x} * e^{\ln y} = e^{\ln x + \ln y}$. Since the exponents must be equal, we're done.

Proof of the second is along similar lines: $e^{\ln\left(x^a\right)} = x^a = \left(e^{\ln x}\right)^a = e^{a * \ln x}$.

Knowing that $\frac{1}{x} = x^{-1}$, property b. given by the text becomes fairly obvious:

$$\ln\frac{1}{x} = \ln\left(x^{-1}\right) = -\ln x \qquad \ln\frac{x}{y} = \ln\left(xy^{-1}\right) = \ln x + \ln\left(y^{-1}\right) = \ln x - \ln y.$$

Example A: Simplify $\ln 60 - \ln 4 - \ln 5$. *Answer*: $\ln(3)$

Example B: Simplify $e^{3\ln x + 2\ln y}$. *Answer*: x^3y^2 , $x > 0$ and $y > 0$

Restrictions on the original expression restrict the domains of the simplified version: $x > 0$, $y > 0$.

Example C: Using properties of logarithms, expand and simplify $\ln\left(x^2 e^x\right)$. *Answer*: $2\ln(|x|) + x$

Absolute value is necessary to retain the domain of the original: $-\infty < x < 0$ and $0 < x < \infty$.

Example D: Using properties of logarithms, expand $\ln\left[\dfrac{x+1}{(x^2-5)(3x+7)^5}\right]$.

Answer: $\ln(x+1)-\ln(x^2-5)-5\ln(3x+7)$

Be careful when using the logarithm properties! They *do not* allow us to expand this example any more—the properties apply when a product or quotient is inside the logarithm, but *not* when there is a sum or difference.

Example E: Solve $\log_2 x+\log_2(x-4)=5$. *Answer*: 8

Example F: The intensity, I, of a floogle-blast is given by the formula $I=\log_3(t+7)-\log_3(t+1)$ where t represents the duration of the blast in seconds. How long is a blast of intensity 1? *Answer*: 2 seconds

One more useful set of exponential/logarithm properties: the change of base theorem, which allows us to rewrite a logarithm in any base as a natural logarithm (ln), and rewrite an exponential in any base as a natural exponential (base e).

$$y=\log_b x \quad\Rightarrow\quad b^y=x \quad\Rightarrow\quad \ln(b^y)=y\ln b=\ln x \quad\Rightarrow\quad y=\log_b x=\frac{\ln x}{\ln b}$$

$$e^{\ln b}=b \quad\Rightarrow\quad b^x=\left(e^{\ln b}\right)^x=e^{\ln b*x}$$

Example G: Given $f(x)=\log_3(x+2)-1$, estimate the coordinates of the y-intercept to three decimal places.
Answer: (0, –0.369)

(Here begins section 2.3)

In Lecture 2.1, Example C, we derived an equation for continuous compounding of interest: $A = Pe^{rt}$. This is one application of the exponential growth and decay function: $y = y_0 e^{kt}$. If we take this basic form, and define t as representing time, then it is a simple process to note that when time = t = 0, $y = y_0 e^{k(0)} = y_0$. In other words, y_0 is an initial value, the amount we start out with at time = 0.

When $k > 0$, we have exponential growth, and k is called the *growth constant*.
When $k < 0$, we have exponential decay, and k is called the *decay constant*.

Notice that the only real difference is that the coefficient of t is positive for exponential growth and negative for exponential decay.

Example H: A fictional country's population grows according to the model $y = 72e^{0.025t}$ where $t = 0$ represents the year 1980 and y = population in millions. What was the population in 1990 (rounded to the nearest 10,000)?
Answer: 92,450,000 people

Example I: Yeast in a culture increases from 4 grams to 10 grams after 7 hours. Find the growth constant k.
Answer: $\frac{\ln(2.5)}{7}$

Note that $k = \frac{\ln(2.5)}{7}$ is the *exact* answer; 0.13 is a *decimal approximation*. Translated into words, it means that the yeast culture is growing at a rate of about 13% per hour.

Example I extended: How long will it take for the original yeast culture to triple in size? Give both an exact answer and an approximation to the nearest hundredth of an hour.

The exact answer is $\frac{7\ln(3)}{\ln(2.5)}$ hours. (Note that there is no logarithm property that will allow us to combine the quotient of two logarithms into a simpler form!) The approximate answer is (8.392… rounded to) 8.39 hours.

Example J: The half-life of krypton-92 (^{92}Kr) is 3 seconds. If you begin with 100 g, how much is left after 3 seconds? 6 seconds? 9 seconds? 12 seconds?
Using the definition of half-life, each three seconds that passes reduces the amount to half of what it was.
After 3 seconds you'd have half of 100 = 50 g.
After 6 seconds (3 more) you'd have half of 50 = 25 g.
After 9 seconds (3 more) you'd have half of 25 = 12.5 g.
After 12 seconds (3 more) you'd have half of 12.5 = 6.25 g.

Example J extended: After how much time would you have 2 g of the original 100 g? Give both an exact answer and an approximation to the nearest thousandth of a second. *Answers*: $\frac{3\ln(0.02)}{\ln(0.5)} \cong$ 16.932 seconds

Limited Growth Functions Exponential growth functions grow without bound. In reality, populations will have limitations of space and resources. As time goes on, the population growth rate will slow until the size of the population reaches an equilibrium. Limited growth functions incorporate a maximum population size that is sustainable in a given environment. Limited growth models can sometimes be useful in describing a population in its later stages, as the population size is approaching its maximum.

Example K: A small lake has limited size and resources to support its population of bluegill fish. (Young bluegill eat microscopic animals, while adults prey on insects, fish eggs, small crayfish, and occasionally small fish.) Let y equal the number of bluegill, and t be time measured in days. In this particular lake, researchers have determined that, following a restocking in the spring, the population of bluegill over the summer and into the fall is modeled by the limited growth equation $y = 1000 - 750e^{-0.25t}$. a) At time = 0, when the lake is freshly restocked, what is the size of the bluegill population? b) What is the maximum sustainable bluegill population? c) How long will it take for the bluegill population to reach half of its maximum?

Answers: 250; 1000; $-4\ln\left(\frac{2}{3}\right)$ days