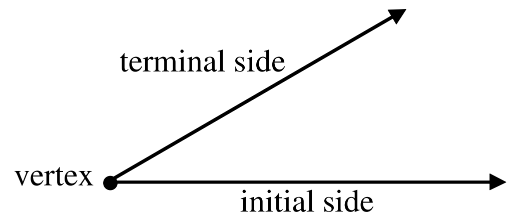
Calculus 130, section 2.4 Trigonometric Functions

notes prepared by Tim Pilachowski

We begin with an *angle*, a geometric and trigonometric figure consisting of a *vertex* and two sides.
One of the sides will be designated the *initial side*, and by convention will be drawn horizontally to the right of the vertex.
The other side will be called the *terminal side*.

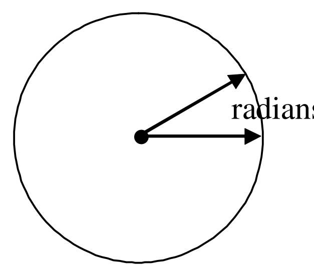


In elementary and high school Geometry, you would likely have measured angles with degrees, minutes and seconds—a method devised in ancient Babylon. Angle measures such as 0º, 30º, 45º, 60º, 90º (right angle), 180º (straight angle), and 360º (full circle) would have been often-used.

For purposes of Trigonometry and Calculus, we'll find it much more convenient to measure angles in terms of *radians.* To define radians, we put our angle into a circle so that the vertex is located at the center, and designate the radius of the circle equal to 1. On a Cartesian grid we'll place the (vertex of the angle) = (center of the circle) at the origin. This *unit circle* has a circumference (formula $C = 2\pi r$) equal to 2π.

Some quick notes on the number π. It is an irrational number, i.e. it cannot be written as an $\frac{\text{integer}}{\text{integer}}$ fraction, and as a decimal it is non-terminating and non-repeating. (See Pilachowski's Rules of Mathematics 3.14 and 3.14159.) It is an *exact* value when written as π — both 3.14 and $\frac{22}{7}$ are *decimal approximations*. In this class it will be standard procedure to give the exact value as your answer.

Back to the unit circle. Its circumference is equal to 2π. Radian measure is defined as the distance around the circumference one must travel to get from the initial side of the angle to the terminal side. If one travels all the way around the unit circle, it would be an angle of 360º and 2π radians. This relationship gives us the conversion factor to convert degrees to radians and back again:



$$\text{degrees} * \frac{2\pi}{360} = \text{radians} \quad \text{and} \quad \text{radians} * \frac{360}{2\pi} = \text{degrees}.$$

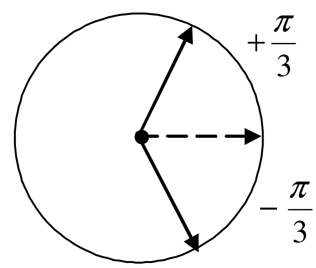
Note that the text uses the simplest form fractions $\frac{\pi}{180}$ and $\frac{180}{\pi}$.

Examples A: Convert 30º, 45º, 90º and 180º to radians. *answers:* $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{2}$, π; *Note that* $\frac{\pi}{2}$ *and* π *radians, when placed on the unit circle, match what we know about right and straight angles.*

Examples B: Convert $\frac{7\pi}{6}$ and $\frac{3\pi}{2}$ radians to degrees. *answers:* 210º, 270º

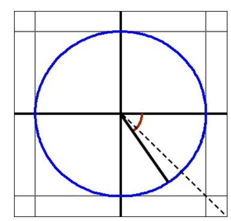
Note that in the two answers to examples B, we came up with angles not seen in Geometry, i.e angles larger than 180º. (This is just one of the reasons why radians will be more useful—and convenient—than degrees.)

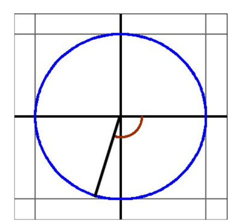
Another convention is that we will also designate the direction of travel: *couterclockwise* is positive movement, and *clockwise* is negative movement. Side note: A central angle of $-\frac{\pi}{3}$ radians puts us at the same place as $+\frac{5\pi}{3}$ radians. (We'll call these two measures *coterminal*.)



Examples C: Convert –55º to radians and convert $-\frac{7\pi}{12}$ radians to degrees.

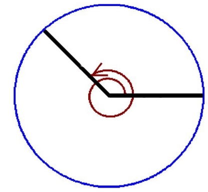
Construct each angle and identify a positive angle (in radians) with which it is coterminal. *answers:* $-\frac{11\pi}{36}$ *and* –105º; $\frac{61\pi}{36}$ *and* $\frac{17\pi}{12}$.

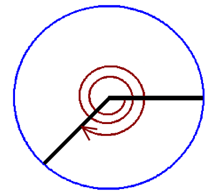




We can also travel more than once around the circle, with each transit taking us 2π radians. So $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$ is coterminal with $\frac{\pi}{6}$, as well as with $\frac{25\pi}{6} = 4\pi + \frac{\pi}{6}$, and so on. (All of these are also coterminal with $-\frac{11\pi}{6} = \frac{\pi}{6} - 2\pi$, $-\frac{23\pi}{6} = -\frac{11\pi}{6} - 2\pi$, etc., however it is very important to note that their direction of travel is opposite.)

Examples D: Construct angles of $\frac{11\pi}{4}$ and $-\frac{19\pi}{4}$ radians. *answers:* →

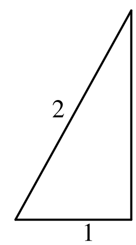




We turn from angles to triangles, specifically right triangles. The first thing to remember is the Pythagorean Theorem (which, by the way, goes back much further in time than Pythagoras), which we usually remember as $a^2+b^2=c^2$. So, that knowing the lengths of any two sides, we can find the length of the third.

Example E: Given two legs of a right triangle with legs of lengths 3 and 4, find the length of the hypotenuse.
answer: 5   *side note: This combination of numbers, in which all three are integers, is called a Pythagorean triple.*

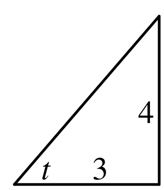
Example F: Given a right triangle with hypotenuse of length 2 and one leg of length 1, find the length of the other leg. *answer:* $\sqrt{3}$



The second attribute of right triangles is a pair of ratios: $\text{sine } t = \dfrac{\text{opposite}}{\text{hypotenuse}}$ and $\text{cosine } t = \dfrac{\text{adjacent}}{\text{hypotenuse}}$.

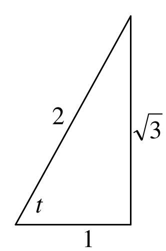
Note: sine is usually abbreviated as sin, and cosine as cos. Second note: A mnemonic sometimes used to remember the trigonometric ratios is SOHCAHTOA. We'll learn about the "TOA" in section 8.4.

Example E revisited: Given the right triangle pictured to the right, find sin *t* and cos *t*.
answers: $\frac{4}{5}$ and $\frac{3}{5}$

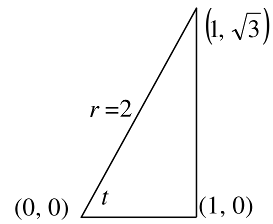


Example F revisited: Given a right triangle with hypotenuse of length 2 and one leg of length 1, find the sin and cos of the larger of the two acute angles.

answers: $\sin t = \frac{\sqrt{3}}{2}, \ \cos t = \frac{1}{2}$

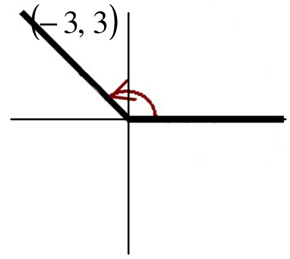


Example F once more: If we place the vertex with angle t at the origin, each vertex can be given Cartesian coordinates, the lengths of the legs become x and y coordinates, and the hypotenuse becomes the radius of a circle centered at the origin. We thus get the trigonometric definition of sin and cos.



$$\sin t = \frac{y}{r} = \frac{\sqrt{3}}{2}, \quad \cos t = \frac{x}{r} = \frac{1}{2}$$

With this way of looking at sin and cos, we can move into Quadrants other than I, and also consider negative as well as positive radian measures.

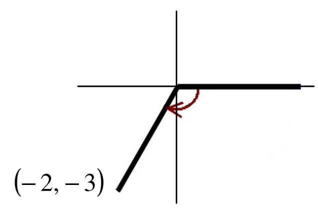


Example G: Find the values of sin t and cos t in the angle t (measured in radians) pictured to the right.

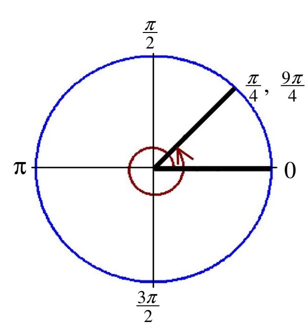
answers: $\sin t = \frac{1}{\sqrt{2}}, \quad \cos t = -\frac{1}{\sqrt{2}}$

Example H: Find the values of sin t and cos t in the angle t (measured in radians) pictured to the right.

answers: $\sin t = -\frac{3}{\sqrt{13}}, \quad \cos t = -\frac{2}{\sqrt{13}}$



We now move to some properties of sin and cos. If we first of all specify that $r = 1$ (the unit circle), then $\sin t = \frac{y}{1} = y$ and $\cos t = \frac{x}{1} = x$, and we can define sin and cos as coordinates on the unit circle. This provides a means to identify sin and cos for angles that do not form a triangle, e.g. 0, $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$.

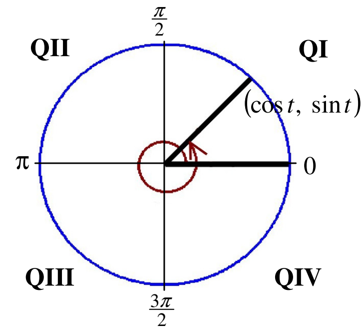


t	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
cos t (x-coordinate)				
sin t (y-coordinate)				

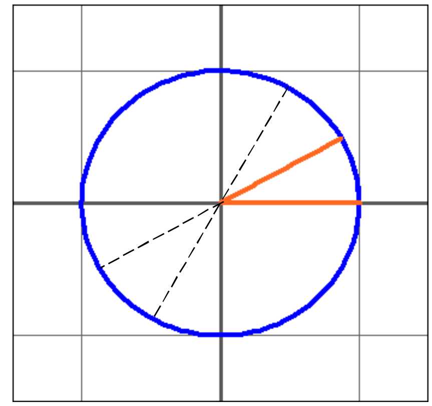
Also note that coterminal angles, such as $\frac{\pi}{4}$ and $\frac{9\pi}{4}$, will have the same sin and cos.

Secondly, we can use these unit-circle coordinates and the Pythagorean formula to show that $\sin^2 t + \cos^2 t = 1$ for all angles t, since each point on the unit circle forms a triangle with hypotenuse 1.

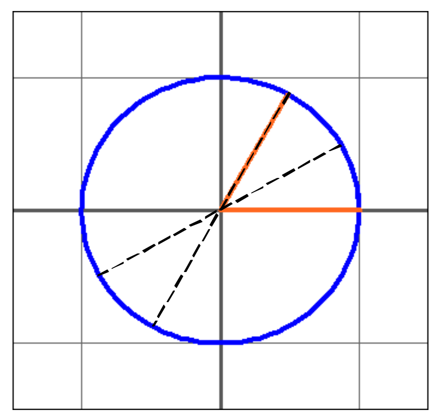
We continue with the unit circle, with $r = 1$, and the definition of sine and cosine as coordinates on the circle. Our first task is to derive a few important and useful trig values.
(After the derivations I'll give you an easy way to remember the values that we derived.)



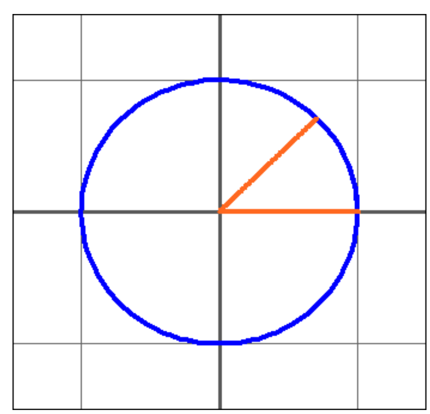
For the angle $t = \dfrac{\pi}{6}$, we can construct an equilateral triangle to show that the y-coordinate on the unit circle $= \sin\dfrac{\pi}{6} = \dfrac{1}{2}$. The basic trigonometric identity $\cos^2 t + \sin^2 t = 1$ leads us to x-coordinate on the unit circle $= \cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$.



For the angle $t = \dfrac{\pi}{3}$, we can construct a $\dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{\pi}{2}$ triangle to show that the x-coordinate on the unit circle $= \cos\dfrac{\pi}{3} = \dfrac{1}{2}$ and the y-coordinate on the unit circle $= \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$.

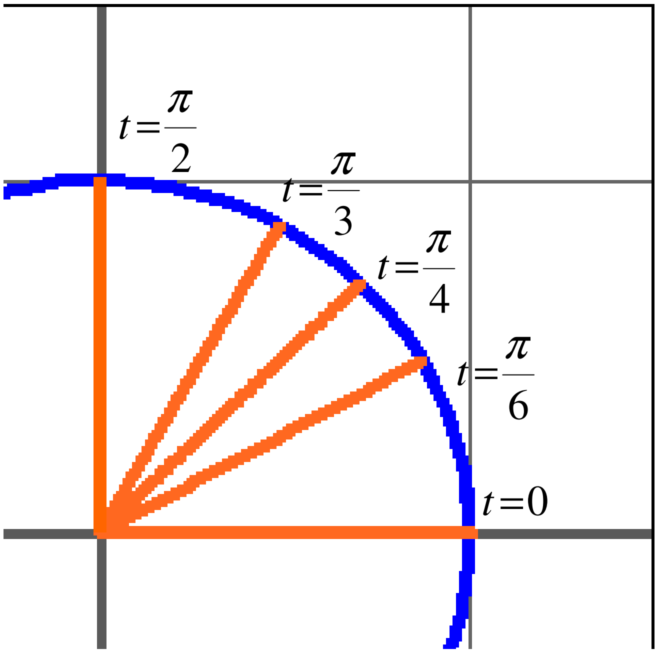


For the angle $t = \dfrac{\pi}{4}$, we can construct a triangle and show that the x-coordinate and the y-coordinate must equal each other, then use the basic trigonometric identity $\cos^2 t + \sin^2 t = 1$ to show that $\cos\dfrac{\pi}{4} = \sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$.



Here’s a mnemonic device for remembering these basic trigonometric values.
Horizontal from the origin, we have $t = 0$, and $\cos 0 = 1$, $\sin 0 = 0$.

We’re going to work our way down, then up from $t = 0$ to $t = \frac{\pi}{2}$, where $\cos\frac{\pi}{2} = 0$, $\sin\frac{\pi}{2} = 1$.



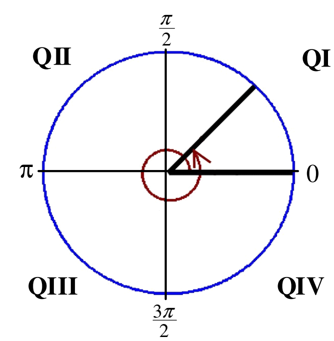
t = angle	$\cos t$	$\sin t$

If we were to continue moving around the unit circle, we’d derive the following table of values:

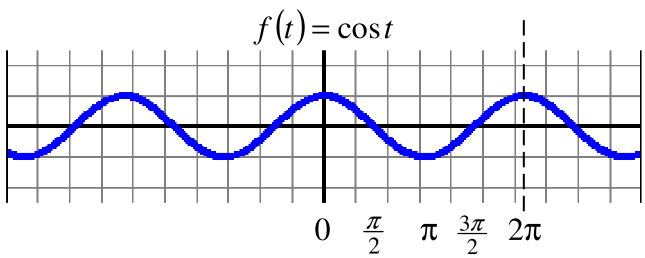
t	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	QIII	$\frac{3\pi}{2}$	QIV	2π
$\cos t$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1	$(-)$	0	$(+)$	1
$\sin t$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$(-)$	-1	$(-)$	0

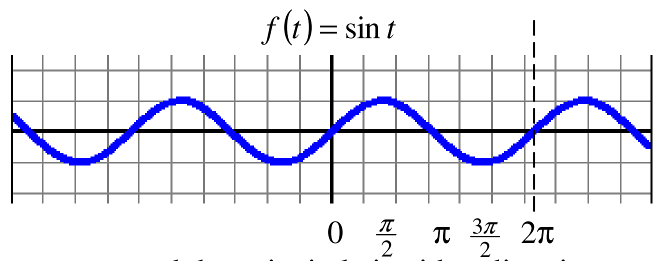
The pattern continues, with the appropriate + or – sign, through Quadrants III and IV, then begins all over again, repeating from 2π, then again at 4π, then again at 6π, etc.

Examples I: Use symmetry of the unit circle to find
$\cos\frac{2\pi}{3}$, $\sin\frac{2\pi}{3}$, $\cos\left(-\frac{\pi}{6}\right)$, $\sin\left(-\frac{\pi}{6}\right)$.



Both $f(t) = \cos t$ and $f(t) = \sin t$ are *periodic functions*, with period equal to 2π. The graphs are pictured below. (More on the graphs later.)





The domain of both functions is all real numbers, since we can go around the unit circle in either direction as many times as we want. The range of each is $-1 \le y \le 1$.

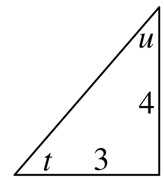
Examples of applications which sometimes use sine and cosine to model periodic behavior include: temperature fluctuations, tides, seasonal sales, regular breathing, blood pressure (systolic and diastolic), circadian rhythms, and populations of migratory animals.

Now we move on to the next trigonometric function tangent, or tan t:

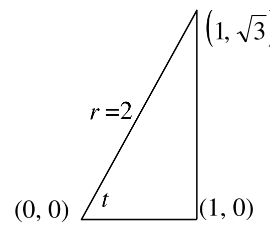
$$\tan t = \frac{\sin t}{\cos t} = \frac{\text{opposite}}{\text{hypotenuse}} \div \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\text{opposite}}{\text{adjacent}}.$$

The last three letters in the mnemonic SOHCAHTOA refer to tangent.

Example E again: Given the right triangle pictured to the right, find tan t.
answers: $\tan t = \frac{4}{3}$

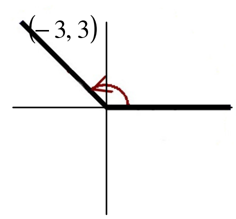


Example F again: If we place the vertex of a triangle with angle t at the origin, the lengths of the legs become x and y coordinates. We thus get the trigonometric definition of tangent.
answer: $\sqrt{3}$

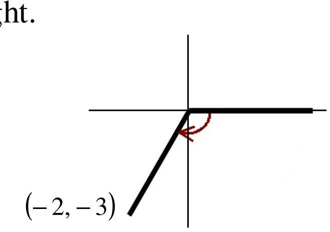


As with sine and cosine, we can move into Quadrants other than I, and also consider negative as well as positive radian measures.

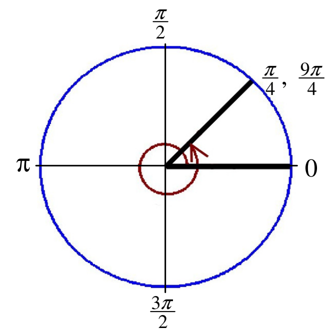
Example G again: Find the value of tan t in the angle t (in radians) pictured to the right.
answer: -1



Example H again: Find the value of tan t in the angle t (in radians) pictured to the right.
answer: $\frac{3}{2}$



Going back to the unit circle (in which $r = 1$), then using $\tan t = \dfrac{y}{x}$ allows us to consider tan t for angles that do not form a triangle, e.g. $0,\ \dfrac{\pi}{2},\ \pi$ and $\dfrac{3\pi}{2}$.



T	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$
sin t (y-coordinate)				
cos t (x-coordinate)				
tan t (y/x)				

Also note that coterminal angles, such as $-\dfrac{\pi}{2}, \dfrac{3\pi}{2}$ and $\dfrac{7\pi}{2}$, have the same tangent.

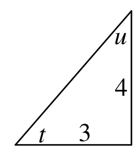
A graph of tan t is pictured in your text, in the middle of section 2.4

It's time to introduce the rest of the trig functions: secant, cosecant and cotangent. By definition:

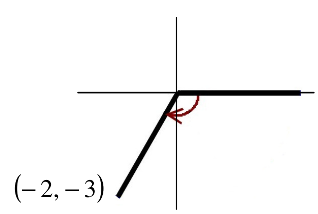
$$\sec t = \frac{1}{\cos t}, \quad \csc t = \frac{1}{\sin t}, \quad \text{and} \quad \cot t = \frac{\cos t}{\sin t} = \frac{1}{\tan t}$$

You'll need these only as a means to rewrite everything as sine, cosine or tangent.

Example E déjà vu: Given the right triangle pictured to the right, find sec t, csc t and cot t.
answers: $\frac{5}{3}, \frac{5}{4}, \frac{3}{4}$

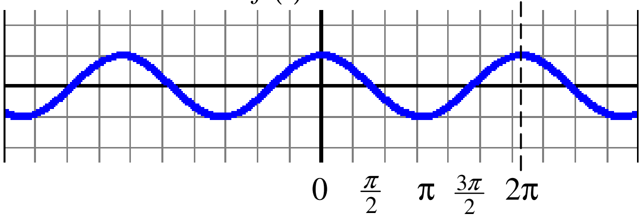


Example H déjà vu: Find the value of sec t, csc t and cot t in the angle t pictured to the right. *answers*: $-\frac{\sqrt{13}}{2},\ -\frac{\sqrt{13}}{3},\ \frac{2}{3}$

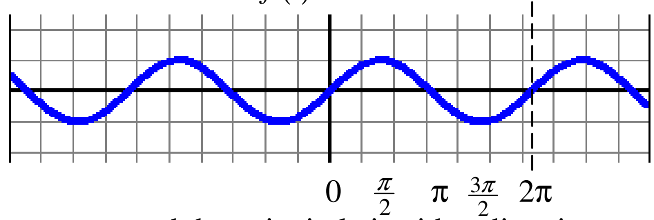


Both $f(t) = \cos t$ and $f(t) = \sin t$ are *periodic functions*. The graphs are pictured below.

$f(t) = \cos t$



$f(t) = \sin t$

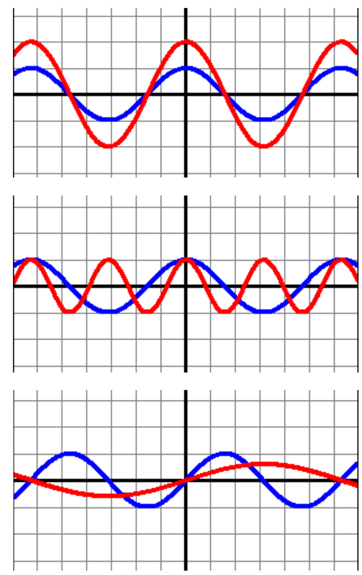


The domain of both functions is all real numbers, since we can go around the unit circle in either direction as many times as we want. The range of each is $-1 \le y \le 1$.

We won't be asking you to sketch the graphs, but we will ask you to determine the *amplitude* and *period* of various sine and cosine functions. Both $f(t) = \cos t$ and $f(t) = \sin t$ have an amplitude of 1, meaning that the graphs go up 1 and down 1 unit from the middle. Both $f(t) = \cos t$ and $f(t) = \sin t$ have a period equal to 2π.

Example J. Determine the amplitude and period of $f(t) = 2\cos t$, $f(t) = \sin 2t$ and $f(t) = \frac{3}{5}\cos\frac{t}{2}$.

answers: 2, 2π; 1, π; $\frac{3}{5}$, 4π



For functions $y = a\cos(bx - c)$ and $y = a\sin(bx - c)$, the amplitude = a, and the period = $\frac{2\pi}{b}$.

Example K: Temperature (F°) during a 24-hour period can be modeled by $T = 72 + 18\sin\left(\frac{\pi(t-8)}{12}\right)$, $t \ge 0$, where $t = 0$ corresponds to midnight. a) Approximate the temperature at 6 am. b) What are the amplitude and period of this function? c) What are the highest and lowest temperatures according to this model?

answers: 63° F; 18, 24 ; 90° F, 54° F