Calculus 130, section 3.1a Limits

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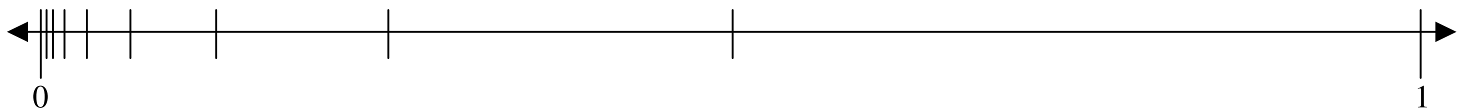
We now move to an examination of the concept of **limits** in mathematics. Non-technically, taking a limit is moving constantly toward something without ever getting there. Finding $\lim\limits_{x \to \infty}$ is akin to walking toward the horizon: even though you keep moving, there is always more horizon off in the distance.

Here's another perspective: Consider decreasing your distance away from an object by half, then half again, then half again, etc. This is like being on a number line at 1, and moving toward 0.



First you'd go to $\frac{1}{2}$, then $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots, \frac{1}{2^{100}}, \ldots, \frac{1}{2^{1000}}, \ldots, \frac{1}{2^{1,000,000}}, \ldots, \frac{1}{2^{1,000,000,000,000,000}}, \ldots,$

You'd be always getting closer to 0, but never actually reaching 0. In mathematical parlance this would be finding $\lim\limits_{x \to 0^+}$, "the limit as x approaches 0 from the right".

You've already encountered limits, albeit in a more limited fashion, when you identified the asymptotes of rational function graphs in an Algebra II or PreCalculus class. Even though you may or may not have called it "taking a limit" at that time, the process was the same. We'll be a little more formal and rigorous in this class.

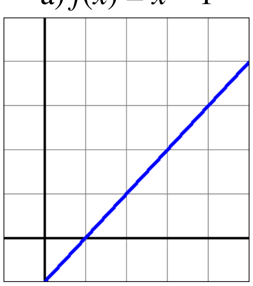
1) The notation $\lim\limits_{x \to a^+} f(x)$ is read "the limit of $f(x)$ as x approaches a from the right".

2) The notation $\lim\limits_{x \to a^-} f(x)$ is read "the limit of $f(x)$ as x approaches a from the left".

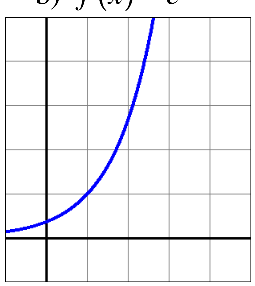
3) If there is some identifiable real number L such that $\lim\limits_{x \to a^+} f(x) = L$ and also $\lim\limits_{x \to a^-} f(x) = L$, then we will be able to write $\lim\limits_{x \to a} f(x) = L$, which is read "the limit of $f(x)$ as x approaches a". It is implicit in this statement that the limit is being taken as "x approaches a from either side".

Examples A: For the following functions, determine whether $\lim\limits_{x \to 1} f(x) = L$ exists, and if so, the value of L.

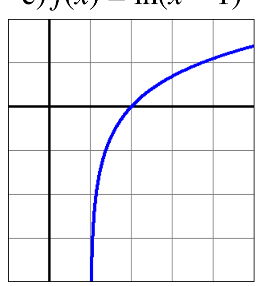
a) $f(x) = x - 1$



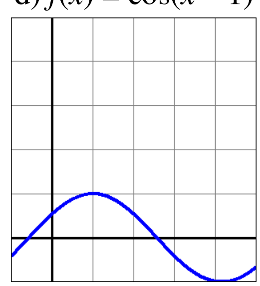
b) $f(x) = e^{x-1}$



c) $f(x) = \ln(x - 1)$



d) $f(x) = \cos(x - 1)$



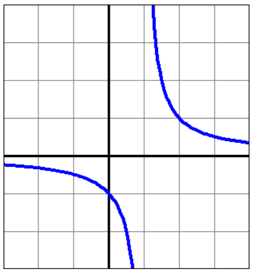
answers: 0, 1, DNE, 1

Of the examples above, the only one for which $x = 1$ is not in the domain is $f(x) = \ln(x - 1)$, so it is the only one for which $\lim_{x\to 1} f(x) = L$ does not exist.

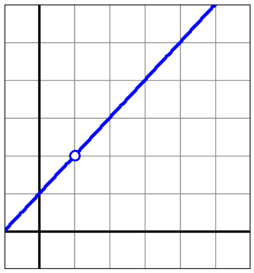
A similar situation exists in the case of rational functions, for which the denominator $\neq 0$.

Examples B: For the following functions, determine whether $\lim_{x\to 1} f(x) = L$ exists, and if so, the value of L.

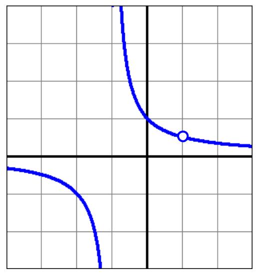
a) $f(x) = \dfrac{1}{x-1}$



b) $f(x) = \dfrac{x^2-1}{x-1}$

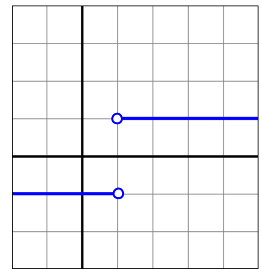


c) $f(x) = \dfrac{x-1}{x^2-1}$



d) $f(x) = \dfrac{x-1}{x^2+1}$

e) $f(x) = \dfrac{|x-1|}{x-1}$



answers: DNE, 2, $\frac{1}{2}$, 0, DNE

Be generally familiar with the properties of limits given in your text. Bottom lines:

The limit of a sum/difference/product is the sum/difference/product of the limits.

For the most part, the limit of a quotient is the quotient of the limits, except when the limit of the denominator = 0 (as in examples B-a and B-c above).

Example B-d revisited: Use the properties of limits to find $\lim_{x\to 1} \dfrac{x-1}{x^2+1}$. *answer*: 0

Examples C: Use the properties of limits to find a) $\lim_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$ and b) $\lim_{x\to 0} \dfrac{[1/(x+1)]-1}{x}$.

answers: $\frac{1}{2}$, -1