Calculus 130, section 3.1b Limits, section 3.2 Continuity

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1) The notation $\lim_{x \to a^+} f(x)$ is read "the limit of $f(x)$ as x approaches a from the right".

2) The notation $\lim_{x \to a^-} f(x)$ is read "the limit of $f(x)$ as x approaches a from the left".

3) If there is some identifiable real number L such that $\lim_{x \to a^+} f(x) = L$ and also $\lim_{x \to a^-} f(x) = L$, then we will be able to write $\lim_{x \to a} f(x) = L$, which is read "the limit of $f(x)$ as x approaches a". It is implicit in this statement that the limit is being taken as "x approaches a from either side".

In some of the examples done last time, we had $\lim_{x \to a} f(x) = \infty$. Now we turn to consideration of $\lim_{x \to \infty} f(x)$. (This is the concept underlying the determination of horizontal and slant asymptotes for rational and exponential functions.)

Examples D: For the following functions, determine whether $\lim_{x \to \infty} f(x) = L$ exists, and if so, the value of L.

a) $f(x) = 3$ b) $f(x) = \frac{3x^2 - 1}{2}$ c) $f(x) = \frac{3x^2 - 1}{2x - 1}$ d) $f(x) = \frac{3x^2 - 1}{2x^2 - 1}$ e) $f(x) = \frac{3x^2 - 1}{2x^3 - 1}$

answers: 3, ∞, ∞, $\frac{3}{2}$, 0

Examples E: For the following functions, determine whether $\lim_{x \to \infty} f(x) = L$ exists, and if so, the value of L.

a) $f(x) = \ln(x - 1)$ b) $f(x) = e^{x-1}$ c) $f(x) = e^{1-x}$ d) $f(x) = \dfrac{3}{2 + e^{-x}}$ e) $f(x) = \cos(x - 1)$

answers: ∞, ∞, 0, $\frac{3}{2}$, DNE

A pair of basic but very important principles are illustrated in the Examples above: As a denominator → ∞, a fraction → 0, while any constants → themselves.

With respect to power functions, for any real number $n > 0$, $\lim_{x \to \infty} x^n = \infty$ and $\lim_{x \to \infty} \dfrac{1}{x^n} = 0$.

With respect to exponential functions, for any real number $k > 0$, $\lim_{x \to \infty} b^{kx} = \infty$ and $\lim_{x \to \infty} \dfrac{1}{b^{kx}} = 0$.

Finally, for a given constant a, $\lim_{x \to \infty} a = a$.

Here begins section 3.2

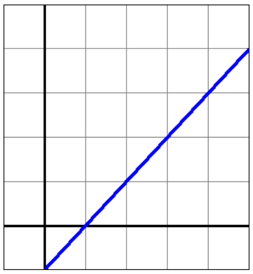
An extension of limits is the concept of **continuity of a function**. Specifically, a function is continuous at a value $x = c$ if and only if

1) $f(c)$ is defined 2) $\lim\limits_{x \to c} f(x)$ exists 3) $\lim\limits_{x \to c} f(x) = f(c)$.

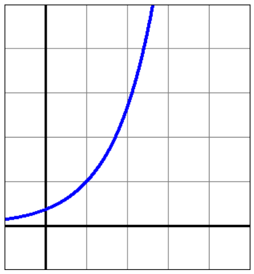
If a function is not continuous at $x = c$, then it is **discontinuous**.

Examples A revisited: Determine whether the following functions are continuous at $x = 1$. Explain why.

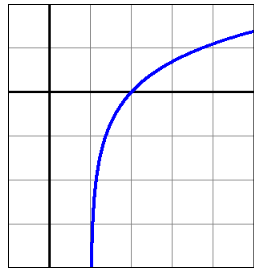
a) $f(x) = x - 1$



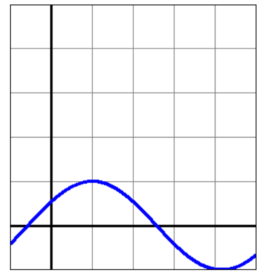
b) $f(x) = e^{x-1}$



c) $f(x) = \ln(x - 1)$



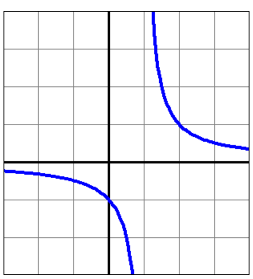
d) $f(x) = \cos(x - 1)$



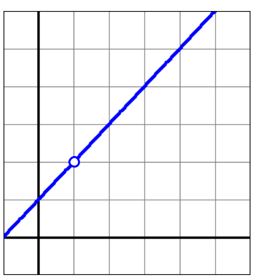
answers: yes, yes, no, yes

Examples B revisited: Determine whether the following functions are continuous at $x = 1$. Explain why.

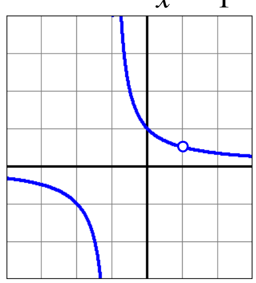
a) $f(x) = \dfrac{1}{x-1}$



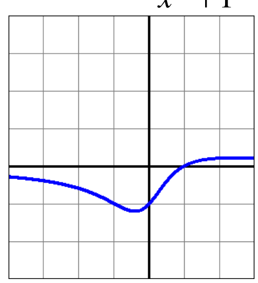
b) $f(x) = \dfrac{x^2 - 1}{x-1}$



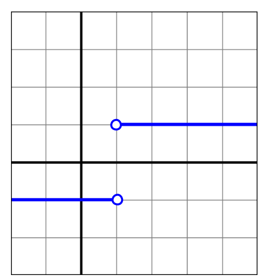
c) $f(x) = \dfrac{x-1}{x^2 - 1}$



d) $f(x) = \dfrac{x-1}{x^2 + 1}$



e) $f(x) = \dfrac{|x-1|}{x-1}$

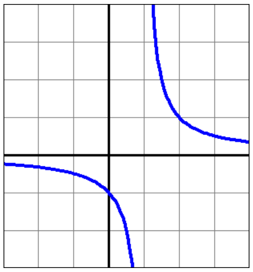
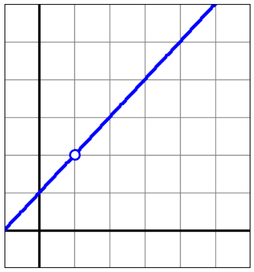
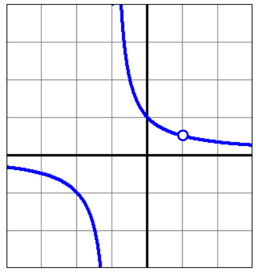
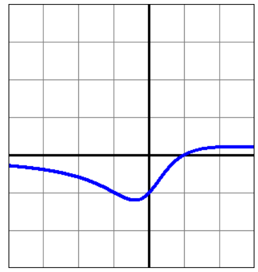
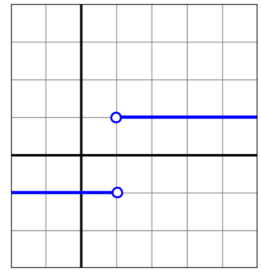


answers: no, no, no, yes, no

Definition: A function is **continuous on an open interval** (a, b) [i.e. $a < x < b$] if it is continuous at every x-value in that interval.

Examples B again: Determine all open intervals on which each of the following functions is continuous.

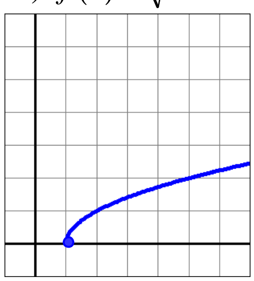
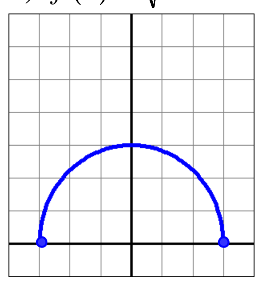
a) $f(x) = \frac{1}{x-1}$ b) $f(x) = \frac{x^2-1}{x-1}$ c) $f(x) = \frac{x-1}{x^2-1}$ d) $f(x) = \frac{x-1}{x^2+1}$ e) $f(x) = \frac{|x-1|}{x-1}$

answers: $(-\infty, 1)$ & $(1, \infty)$, $(-\infty, 1)$ & $(1, \infty)$, $(-\infty, -1)$, $(-1, 1)$ & $(1, \infty)$, $(-\infty, \infty)$, $(-\infty, 1)$ & $(1, \infty)$

Examples F: Determine all intervals on which each of the following functions is continuous.

a) $f(x) = \sqrt{x-1}$ b) $f(x) = \sqrt{9-x^2}$

Neither of these functions satisfies the definition of "continuous" at their endpoints: for a) at $(1, 0)$, and for b) at $(-3, 0)$ and $(3, 0)$. Is there a way that we can think about continuity so that we can include such endpoints?

Definition: A function is **continuous on a closed interval** $[a, b]$ if and only if

1) it is continuous on (a, b)
2) it is continuous from the right at $x = a$, i.e. $\lim_{x \to a^+} f(x) = f(a)$
3) it is continuous from the left at $x = b$, i.e. $\lim_{x \to b^-} f(x) = f(b)$

Examples F: Determine all intervals on which each of the following functions is continuous.

a) $f(x) = \sqrt{x-1}$ b) $f(x) = \sqrt{9-x^2}$

answers: $[1, \infty)$, $[-3, 3]$