Calculus 130, section 3.3 Rates of Change

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Calculus is the mathematics of *change*. Specifically, our first goal will be finding a mathematical way of describing the rate of change of a function. However, even if you've never encountered Calculus before, you have dealt with rates of change. Recall the definition of "slope of a line" from Algebra: $m = \dfrac{\Delta y}{\Delta x} = \dfrac{\text{change in } y}{\text{change in } x}$.

Example A: Find the slope of the line connecting the points $(-1, -6)$ and $(0, -1)$.   *answer*: 5

Example B: When you start a job at a local restaurant, you earn $5.75 per hour. A year later you get a raise to $6.60 per hour. A year and a half after the first raise you get another, to $7.70 per hour. In which space of time did your earnings have a higher rate of change?   *answer*: during the first year

The *average rate of change* provided a way to compare raises given over differing periods of time. More formally: The "average rate of change of a function" is the same as the slope of a secant line connecting two points on the graph of the function. In an algebra class, you may have called this formula the *difference quotient*: $\dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$.

Example C: The function $s(t) = -16t^2 + 10t + 240$ calculates the height of an object, s, after time, t, thrown upward at 10 feet per second from a bridge which is 240 feet above the river below.

Starting from the formula $d = rt$, we can derive $r = \dfrac{d}{t}$. The distance traveled is a change in position, $d_2 - d_1$.

Time traveled, like the time passing in this Example is also a calculation of change, $t_2 - t_1$. Velocity is the rate of change of position with respect to time, $\dfrac{\Delta d}{\Delta t}$.

a) What is the height of the rock after 2 seconds? b) How much height did the rock gain in the first 2 seconds?
c) What is the average velocity during the first 2 seconds? *answers*: 196 ft; loss of 44 ft; falling at 22 fps

Example D: Given $f(x) = x^3 - 8x + 2$, find the *average rate of change* over the interval $[-2, -1]$. *answer*: -1
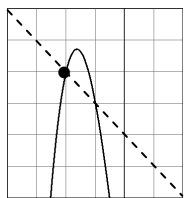

Example E: Given the function $f(x) = \dfrac{1}{x+1}$, find the *average rate of change* over the interval $[1, 3]$.

answer: $-\dfrac{1}{8}$

Example F: Given the function $f(x) = \sqrt{x+1}$, find the *average rate of change* over the interval $[3, 8]$.

answer: $\dfrac{1}{5}$

Until now, there has been some distance between the two points used to calculate average rate of change. If we could move the second point ever closer to the first, in fact so close that we can hardly distinguish between the two points, we'd have an idea of how the curve is changing at that point. That is, we'd be able to calculate the *instantaneous rate of change* of the function at that point.
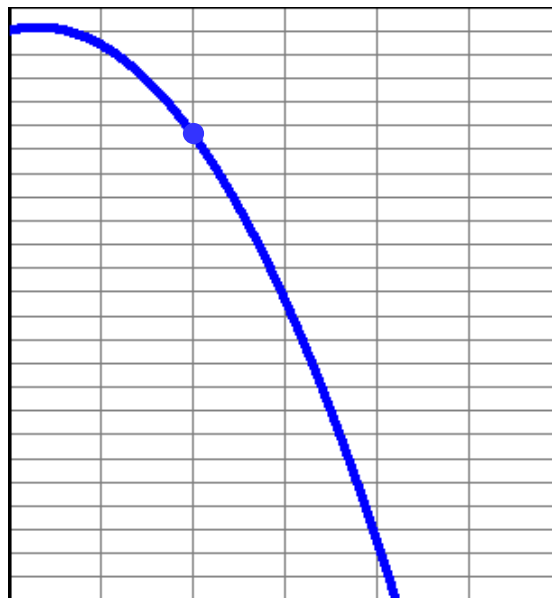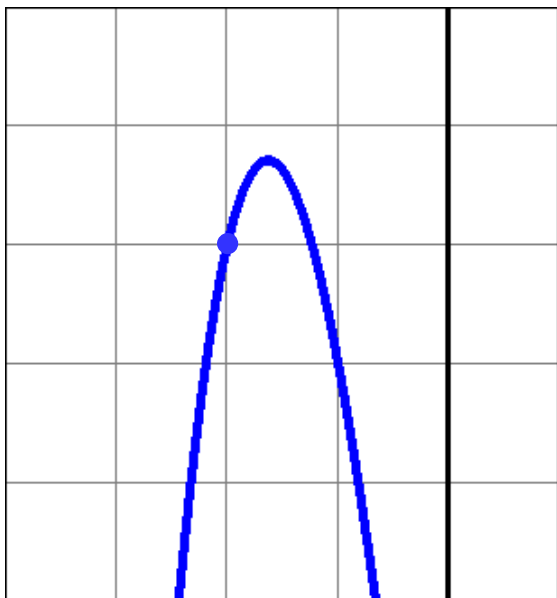
Symbolically, [instantaneous rate of change of a function $f(x)$] $= \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{(x+h) - x} = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

Example C revisited: The function $s(t) = -16t^2 + 10t + 240$ calculates the height of an object, s, after time, t, thrown upward at 10 feet per second from a bridge which is 240 feet above the river below.

d) What is the instantaneous rate of change (velocity) at $t = 2$ seconds? *answer*: falling at 54 fps

Example D revisited: Given $f(x) = x^3 - 8x + 2$, find the *instantaneous rate of change* at $x = -2$. *answer*: 4