Calculus 130, section 3.4 Definition of the Derivative

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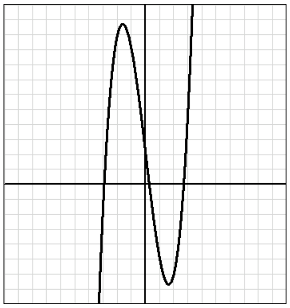
This section formalizes the process begun in section 3.3.



Consider the function $f(x)=x^3-8x+2$, pictured to the left. (To reproduce this on your graphing calculator set your window to [–10, 10] by [8, 12].) The curve's "slope" is far from constant. From $-\infty$ until someplace between $x=-2$ and $x=-1$ the function is increasing (has a positive slope). The curve then levels off (slope = 0, a *relative maximum*), changes direction, and is decreasing (slope is negative) until someplace between $x = 1$ and $x = 2$. On the other side of this *relative minimum* (toward ∞) the function is once again increasing.

The task at hand is to find a way to describe the rate of change of $f(x)$ in a way that not only makes sense intuitively, but also complements all of the other information we know about functions and their graphs.

In section 3.3, we began by calculating *average rate of change* as the slope of the secant line connecting two points. We then calculated the *instantaneous rate of change* of the function at a point by finding

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

We were, in effect, moving a second point so close to the first we couldn't distinguish between the two points. The resulting "line through the two points" was no longer a secant line but a *tangent line.* (A tangent line intersects a curve at exactly one point.)

The slope of the tangent line to a curve at a point
is equal to the slope of the curve at that point
is equal to the instantaneous rate of change of the function at that point.

A formula that provides the slope/rate of change of a function is called the *first derivative*, and is formally defined $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. (Eventually we'll introduce the Leibnitz notation $\frac{df}{dx}$ for the derivative.)

Example A: Find a formula for first derivative of the function $f(x) = 5x - 1$. *answer*: $f'(x)=5$

Example C: The function $s(t)=-16t^2+10t+240$ calculates the height of a rock, s, after time, t, thrown upward at 10 feet per second from a bridge which is 240 feet above the river below. Use the definition of the first derivative to find $s'(t)$, then evaluate $s'(0)$ and $s'(2)$.

Example D: Given $f(x) = x^3 - 8x + 2$, first find $f'(x)$, then find the equation of the line tangent to the curve at $x = -2$.

Example E: Given the function $f(x) = \frac{1}{x+1}$, first find $f'(x)$. Then find the instantaneous rate of change of the function at $x = -1$ and at $x = 0$.

Example F: Given the function $f(x) = \sqrt{x+1}$, first find $f'(x)$, then find the slope of the curve at $x = -1$, then find the slope of the curve at $x = 0$.