Calculus 130, section 4.1 Techniques for Finding Derivatives
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We now take the ideas of section 3.4 (the slope of a curve at a point P = the slope of the tangent at point P = the value of the first derivative at point P) and develop some of the shortcuts. (Note that the derivative of a function is itself a function.) The process of finding a derivative is called **differentiation**. The first derivative of f has several notations: f', $f'(x)$, $\dfrac{dy}{dx}$, $\dfrac{d}{dx}[f(x)]$ and $D_x[f(x)]$. (The last one is used in the text, so you need to be familiar with it, but we won't be using it in lectures or on the Exam.)

Examples A: Use the limit definition of the derivative to find the first derivative of $m(x) = x^2$, $n(x) = x^3$, $p(x) = x^{-1}$ and $q(x) = x^{\frac{1}{2}}$.

$m'(x) = ?$

$n'(x) = ?$

$p'(x) = ?$

$q'(x) = ?$

We can go through a similar process for any function of the form $f(x) = x^r$ where r could be any number. Specifically, we'd find that $f'(x) = r\,x^{r-1}$. Such a function is called a **power function**, and this property of derivatives is called the **power rule**.

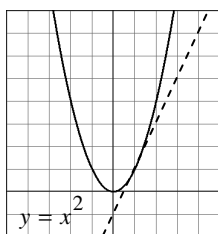Example B: Given $y = x$, find $\dfrac{dy}{dx}$. *Answer*: 1

Note that the answer to Example B fits with what we already know about lines in slope-intercept form.

Example C: Given $f(x) = 5$, find $f'(x)$. *Answer*: 0

Note that the answer to Example C fits with what we already know about constant functions and the slopes of horizontal lines.

Example D: Given $f(x) = \dfrac{1}{x^3}$, find $\dfrac{d}{dx}[f(x)]$. *Answer*: $-\dfrac{3}{x^4}$

Given a function $y = k * f(x)$, where k is a constant coefficient of a variable function, what would be its first derivative? Think in terms of what you know about transformations and what you've learned about slope of the tangent line. The constant will either stretch the graph (when $|k| > 1$) or shrink the graph (when $|k| < 1$). What effect will this stretch/shrink have on the slope of the tangent line? Will it stretch/shrink at the same rate as the curve? Consider the quadratics pictured to the left. For $f(x) = x^2$, $f(1) = 1$ and the slope of the tangent line = 2. For $f(x) = 2x^2$, $f(1) = 2$ and the slope of the tangent line = 4. We might begin to suspect that as a function undergoes a stretch/shrink, the tangent line stretches/shrinks at the same rate. Indeed, we already know, from the properties of limits, that

$$\lim_{h \to 0} \frac{kf(x+h) - kf(x)}{h} = \lim_{h \to 0} \frac{k[f(x+h) - f(x)]}{h} = k * \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
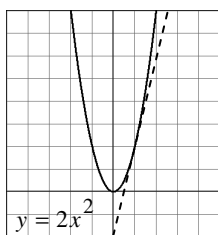
which is to say that given a function $y = k * f(x)$, $\dfrac{dy}{dx} = k * f'(x)$ [**constant-multiple rule**].

Example E: Given $f(x) = 7x^3$, find the first derivative. *Answer*: $21x^2$

Can we do the same if we add two functions together? That is, Given $p(x) = f(x) + g(x)$, does $p'(x) = f'(x) + g'(x)$? Since $\lim_{h \to 0} \dfrac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h}$

$= \lim_{h \to 0} \dfrac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h} = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \dfrac{g(x+h) - g(x)}{h}$, the answer is "Yes."

The derivative of a sum is the sum of the derivatives [**sum rule**]. Note that the constant-multiple rule and sum rule work together to give us the derivative of a subtraction, since $f(x) - g(x) = f(x) + (-1 * g(x))$.

Example F: Given $f(x) = x^3 - 8x + 2$, find the first derivative. *Answer*: $3x^2 - 8$

Example G: Given $g(x) = 2x^5 - \dfrac{x^4}{4} + 3\sqrt[3]{x} - \dfrac{7}{x^2}$, find the first derivative. *Answer*: $10x^4 - x^3 + \dfrac{1}{x^{2/3}} + \dfrac{14}{x^3}$

Caution, Be careful, Warning, Warning! Danger, Will Robinson! There is no similar easy process for the derivative of a product, nor is there a similar easy process for the derivative of a quotient. We'll need to work a good bit for those.

Some final notions about derivatives and differentiability:

In line with the concept of slope of the curve at a point being equal to the slope of the tangent line at that point, $f'(x) > 0$ implies that f is increasing. Likewise $f'(x) < 0$ implies that f is decreasing. However, the implication does *not* go the other way. The fact that a function is increasing does *not* imply that $f'(x) > 0$. The simplest example is $y = x^3$. Although x^3 is increasing everywhere on its domain, its derivative is not always positive. The curve momentarily "levels out" precisely when $x = 0$, then continues upward. At this point $\dfrac{d}{dx}(x^3) = 0$. A similar limitation exists for decreasing functions.

Another important concept for differentiability of functions is the idea of being *continuous*. Specifically, the existence of a derivative at a point implies that the function is continuous at that point, i.e. its graph is not disconnected pieces. However, the implication does *not* go the other way. A function may be continuous, but may not be differentiable at all points. Two examples are shown below. On the left, $f(x) = \sqrt{25 - x^2}$ is defined and has values at $x = -5$ and $x = 5$, but the derivative is undefined since the tangent is a vertical line whose slope is undefined. On the right, $f(x) = -|x| + 5$ is defined and continuous for all real numbers, but the derivative does not exist at $x = 0$ because the tangent line on the left has a positive slope, while the tangent line on the right has a negative slope, i.e $\lim\limits_{x \to 0^-} \dfrac{f(x+h) - f(x)}{h} \neq \lim\limits_{x \to 0^+} \dfrac{f(x+h) - f(x)}{h}$.