Calculus 130, section 4.3 The Chain Rule

notes prepared by Tim Pilachowski

Composition of functions is taking one function's formula and inserting it into another's. Vocabulary and notation varies: "f composition g" = "f of g" = $f \circ g = (f \circ g)(x) = f[g(x)]$.

Example A: Given $f(x) = x^2 - x$ and $g(x) = 3x + 1$, find the algebraic rule for $(f \circ g)(x)$.

Answer: $f \circ g = 9x^2 + 3x$

To find the derivative of a composition of functions, we have the *chain rule*: $\dfrac{d}{dx} f[g(x)] = f'[g(x)] * g'(x)$.

Example A extended: Given $f(x) = x^2 - x$ and $g(x) = 3x + 1$, find $\left(f[g(x)]\right)'$. *Answer*: $18x + 3$

Example B: Given $f(x) = \sqrt{4x^3 - 3}$ and $g(x) = 2x^2 - x$, find $\left(f[g(x)]\right)'$. *answer*: $\dfrac{d}{dx} f[g(x)] = \dfrac{6\left(2x^2 - x\right)^2 (4x - 1)}{\sqrt{4\left(2x^2 - x\right)^3 - 3}}$

For the time being, the only functions we'll be working with are power functions, but we'll soon move on to exponential, logarithm and trigonometric functions, for which the chain rule is not only important but necessary.

One way to think of the chain rule is "the derivative of the outside applied to the inside, times the derivative of the inside". Possibly the most important task in using the chain rule is correctly identifying the "outside and "inside" functions, i.e. which one is "f" and which one is "g". Section 4.3 is practice in doing precisely this.

Example C: Given $h(x) = \sqrt[3]{x + \sqrt{2x}}$ determine all values of x for which $h'(x) = 0$.

Answer: none
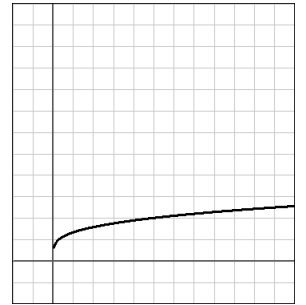
Note that $h'(x) > 0$, for all values of x in the domain, which means that $h(x)$ is increasing for all values of x in its domain. Since the domain of $h = [0, \infty)$, we conclude that h has an absolute minimum at $x = 0$, and that it has no maximum.

4.2 Example B revisited: Given $n(x) = (5x^4 - 1)^2$, find $n'(x)$. *Answer*: $40x^3(5x^4 - 1)$

Example D. Given $f(x) = (2x+1)^4 (3x-1)^5$, determine where f has horizontal tangents.

Multiplying these out would be time-intensive and fraught with possible error. The product rule and chain rule provide an easier method.

Answer: $x = -\frac{1}{2}$, $x = -\frac{7}{54}$, $x = \frac{1}{3}$

We have an alternate way of writing the chain rule using Leibnitz notation: for $y = f(u)$ and $u = g(x)$,

$$\frac{dy}{dx} = \frac{dy}{du} * \frac{du}{dx}.$$

The text has no verification of the chain rule. Here is one way to look at the chain rule, using the definition of derivative and properties of fractions:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x} * \frac{\Delta u}{\Delta u}\right) = \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta u} * \frac{\Delta u}{\Delta x}\right) = \left(\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta u}\right)\left(\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x}\right)$$

Since u is a function of x, so as $\Delta x \to 0$, Δu also approaches 0. So,

$$\frac{dy}{dx} = \left(\lim_{\Delta x \to 0}\frac{\Delta y}{\Delta u}\right)\left(\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x}\right) = \left(\lim_{\Delta u \to 0}\frac{\Delta y}{\Delta u}\right)\left(\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x}\right) = \frac{dy}{du} * \frac{du}{dx}$$

There are a couple of caveats. 1) Any process proving the chain rule must assume $\Delta u \neq 0$. The case in which $\Delta u = 0$ requires a good bit of high-powered finagling, which you will not need to know about. 2) As noted earlier in the course, the differentials above are not fractions, although the notation used can make it appear that way. In this case, however, the form provides a way to remember the chain rule.

Example E: Given $y = \left(3x^2 - \frac{2x+1}{x^2}\right)^9$, find $\frac{dy}{dx}$. *Answer:* $\frac{dy}{dx} = 9\left(3x^2 - \frac{2x+1}{x^2}\right)^8\left(6x + \frac{2}{x^2} + \frac{2}{x^3}\right)$

Example F: An environmental study determined that the level of carbon monoxide (in parts per million) in the air surrounding a small city was a function of the number of people living there (p = population in thousands): $C(p) = \sqrt{0.5p^2 + 17}$. Population is, in turn, a function of time (in years): $p(t) = 3.1 + 0.1t^2$.

a) Explain what $\frac{dC}{dp}$, $\frac{dp}{dt}$, and $\frac{dC}{dt}$ represent. b) Find the equation to express the level of CO as a function of time. c) Use the chain rule to find $\frac{dC}{dt}\Big|_{t=3}$. *Answer* b) $C(t) = \sqrt{0.5(3.1 + 0.1t^2)^2 + 17}$; c) 0.24